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GEOMETRIC IDENTITIES

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This is a survey of so-called geometric identities. In addition to the results mentioned for open surfaces below we will mention recent progress on work for closed surfaces[5],[6].

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus g with n punctures or geodesic boundary components. We will suppose that $3g-3+n \geq 1$ so that Σ admits a Riemannian metric of constant curvature -1, a hyperbolic structure of finite area, which, by Gauss-Bonnet, satisfies Area $\Sigma = 2\pi|\chi(\Sigma)| = 2\pi(2g-2+n)$.

For simplicity, we suppose that $n = 1$ so that Σ has a single boundary component or possibly a cusp, of length $\ell(\delta) \geq 0$ where a boundary component of length 0 is a cusp. A *geometric identity* is a relation between the lengths of the closed simple geodesics on the surface Σ. The known geometric identities fall into 3 groups:

(1) Basmajian Identities
(2) McShane Identities
(3) Bridgeman Identities

We will explain briefly how each of these groups of identities is proven. The proofs follow from the existence of a decomposition of some geometric object related to the surface into two parts one of which is neglible and the other which further decomposes into pieces which can be classified and their "size" computed.

Common to the proof all these identities is the following well known result which allows us to conclude that one of the two parts is negligible hence makes no contribution to the identity.

Theorem 1.1 (Ahlfors). *Let Γ be a finitely generated fuchsian group and $\Lambda \subset \partial\mathbb{H}$ its limit set. Then either Λ is $\partial\mathbb{H}$ or Λ has measure zero.*

1.1. **Unified approach.** We present here a unified approach to the Basmajian and McShane identities. The Basmajian identity is proved using the standard decomposition of the ideal boundary $\partial\mathbb{H}$ into the limit set Λ and the regular set, classifying the components of the regular set and associating a "size" to each of them and finally applying Theorem 1.1 to deduce the identity. Likewise, we sketch a proof of the McShane identity using a decomposition of the ideal boundary into a subset of the limit set Λ_x and its complement, we classify the components of the complement and associating a "size" to each of them and finally apply Theorem 1.1 to deduce the identity.

To state the Basmajian and Bridgeman Identities it is necessary to define orthogeodesics. Let $\hat{\delta}, \hat{\delta}'$ be a pair of disjoint geodesics in $\mathbb{H} \cup \partial\mathbb{H}$ lifts of some, not necessarily distinct, geodesics δ, δ' on the surface Σ. Under the hypothesis $\hat{\delta}, \hat{\delta}'$ admit a common perpendicular and the projection of this to the surface is an orthogeodesic α^*.

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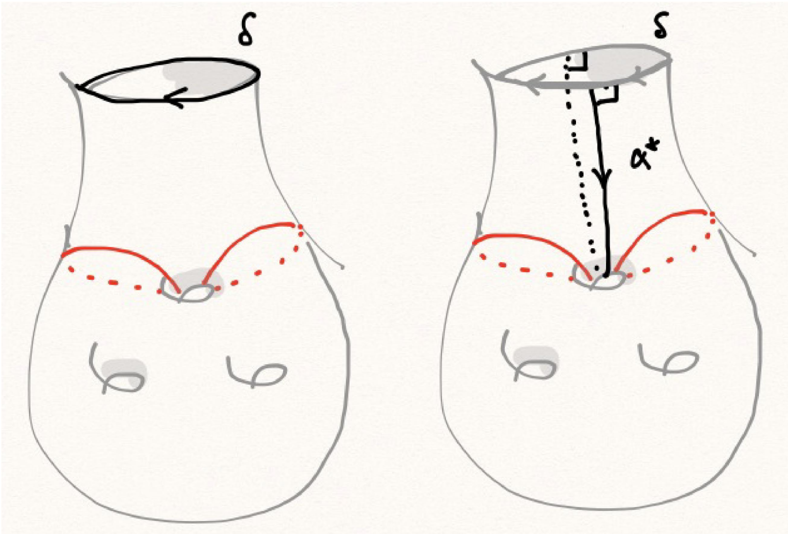

FIGURE 1. An orthogeodesic in an embedded pair of pants

McShane's identities are usually stated in terms of embedded pairs of pants. However, in the context of this article, it is useful to bear in mind that an embedded pair of pants contains a unique unoriented orthogeodesic (Figure 1).

2. Limit Set

A *Fuchsian group* Γ is a discrete subgroup of isom$^+(\mathbb{H})$. If Γ is torsion free then the quotient of $\mathbb{H}$ by the action of Γ is a surface $\Sigma = \mathbb{H}/\Gamma$. and $\pi_1(\Sigma) \simeq \Gamma$. The limit set $\Lambda(\Gamma)$ of Γ is the smallest closed Γ-invariant subset and, provided Γ is not virtually abelian, this is a perfect set. The complement of the limit set $\Omega(\Gamma)$ is called the *regular set* it is a (possibly empty) Γ-invariant open set. Further, if Γ is finitely generated and Σ does not have finite area then $\Omega(\Gamma)$ is dense and consists of countably many open intervals. If Γ contains no parabolic elements then the orbits of the action of Γ on $\Omega(\Gamma)$ are in 1-1 correspondence with the ends of Σ. Thus, we have a Γ-invariant decomposition of the ideal boundary of $\mathbb{H}$ as

$$\partial\mathbb{H} = \Lambda(\Gamma) \sqcup \Omega(\Gamma).$$

We shall denote $\partial\Omega$ the set of all the points a, b such that the intersection of the interval $[a, b] \subset \partial\mathbb{H}$ with the limit set Λ is $\{a, b\}$.

Given Γ finitely generated and Σ of infinite area there is a canonical way to associate a subsurface $C(\Sigma) \subset \Sigma$ of finite area with totally geodesic boundary called the *convex core*. Let $C(\Lambda) \subset \mathbb{H}$ be the convex hull of the limit set, this is a closed, Γ-invariant subset whose frontier consists of countably many complete geodesics. The quotient $C(\Sigma) := C(\Lambda)/\Gamma$ embeds naturally into $\Sigma = \mathbb{H}/\Gamma$. By construction, $C(\Lambda)$ is the universal cover of Σ the embedding induces an isomorphism between $\pi_1(\Sigma) \simeq \Gamma$ and $\pi_1(C(\Sigma))$. In particular :

Proposition 2.1. *The components of the regular set, i.e. the maximal intervals in the complement of* Λ*, are in 1-1 correspondence with lifts of the boundary geodesics of* Σ*.*

Note that if $\gamma \subset \mathbb{H}$ is a geodesic with an endpoint in $\partial\Omega$ then the corresponding geodesic in the surface Σ contains some boundary geodesic in its closure. In what follows we will think of $C(\Lambda)$ as a generalized polygon and refer to the geodesics of $C(\Lambda)$ as *sides*. We

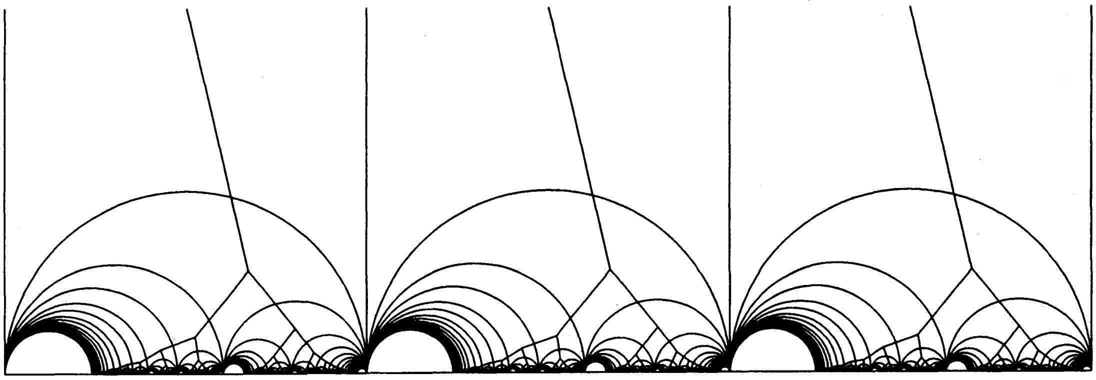

FIGURE 2. Convex hull of the limit set for a 2 generator Fuchsian group

associate to pairs of distinct sides an *orthogeodesic* $\hat{\alpha}^*$ this is just the unique common perpendicular joining the sides. The image of $\hat{\alpha}^*$ is an geodesic arc α^* which meets the boundary of $C(\Sigma)$ perpendicularly which is an *orthogeodesic on the surface.* By definition, the lengths of $\hat{\alpha}^*$ and α^* are the same and clearly the length of $\hat{\alpha}^*$ can be computed as a cross ratio of the endpoints of the associated sides of $C(\Lambda)$.

Finally, recall that an action is *minimal* iff the orbit of any point is dense.

Proposition 2.2. *The action of* Γ *on* $\Lambda(\Gamma)$ *is minimal.*

See for example [2] for a proof.

2.1. **Remarks.** In the next section we state and sketch proof of the analogues of Propositions (2.1) and 2.2 for the action of the mapping class group. We record the following useful observations:

(1) From the above we have

$$\Lambda \subset \bigcap_z \overline{\Gamma.z}.$$

(2) In addition we have

$$\forall z \in \Gamma, \overline{\Gamma.z} \subset \Lambda,$$

so the fact that the set there is a dense orbit Λ_x means that there is no hope of decomposing the orbit structure further. Note that if $\Gamma_1 \subset \Gamma$ is a non trivial normal subgroup then, by minimality,

$$\Lambda(\Gamma_1) = \Lambda(\Gamma)$$

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3. NIELSEN ACTION OF THE MAPPING CLASS GROUP

Let Σ be a compact surface with non empty totally geodesic boundary as above. In this paragraph we describe a construction due to Nielsen of an action of the mapping class group of Σ on the limit set of Γ. Recall that *mapping class group* is the set of isotopy classes of diffeomorphisms that fix the boundary of Σ pointwise.

3.1. **Construction of the action.** We now construct an action of the mapping class group on the limit set Λ following [9]. Consider the closure of the convex hull $\overline{C(\Lambda)}$ with respect to the induced topology on $\mathbb{H} \sqcup \partial\mathbb{H}$ considered as a subset of the Riemann sphere. The frontier of this set $\partial\overline{C(\Lambda)} \subset \mathbb{H} \sqcup \partial\mathbb{H}$ consists of lifts of the boundary of Σ together with the closure in $\partial\mathbb{H}$ of the set of their ideal endpoints. The intersection $\overline{\partial C(\Lambda)} \cap \partial\mathbb{H}$ is non empty and, by minimality of the action of Γ on the limit set, $\overline{\partial C(\Lambda)} \cap \partial\mathbb{H} = \Lambda$.

Fix a basepoint $p \in \partial C(\Lambda)$ and we denote $\delta^{\sim}$ the unique geodesic in $\partial C(\Lambda)$ containing p. Let $z \in \overline{\partial C(\Lambda)}$ and $[p, z]$ denote the geodesic ray starting at p and with (possibly ideal) endpoint z.

Proposition 3.1. (1) *There is a natural action of the mapping class group of* Σ *on* $\overline{\partial C(\Lambda)}$.

(2) *The set* $\Lambda_0 := \{z \in \Lambda : [p, z]$ *is the lift of a simple ray on* $\Sigma\}$ *is invariant.*

(3) *The limit set* Λ *is invariant under this action but is not minimal.*

The argument to prove (1) is standard negatively curved spaces (compare [?]). Let ϕ be a diffeomorphism of Σ that fixes $\partial\Sigma$ pointwise then there is a unique lift $\phi^{\sim}$ to $C(\Sigma)$ that fixes p. This map extends to a homeomorphism of $\overline{C(\Sigma)}$ which we continue to denote $\phi^{\sim}$. If ψ is a diffeomorphism isotopic to ϕ through isotopies that fix the boundary pointwies then the corresponding extension $\psi^{\sim}$ coincides with $\phi^{\sim}$ on $\partial\overline{C(\Sigma)}$. In other words the restriction of the extension only depends on the mapping class of ϕ. To prove this one must see that if z is an ideal endpoint of one of the support geodesics of $C(\Sigma)$ then $\phi^{\sim}(z) = \psi^{\sim}(z)$. It is easy to see that it suffices to show that the geodesic rays $[p, \phi^{\sim}(z)]$ and $[p, \psi^{\sim}(z)]$ are the same. Since the surface is compact the and ψ, ϕ the image of $[p, z]$ under $\psi^{\sim}$ (resp. $\phi^{\sim}$) remains at bounded distance from $[p, \phi^{\sim}(z)]$ (resp. $[p, \psi^{\sim}(z)]$). Further ψ, ϕ are homotopic so the images remain at bounded distance from each other and so the result follows.

To prove (2) let $z \in \Lambda$ such that $[p, z]$ is the lift of a simple ray $\gamma \subset \Sigma$ and $\phi^{\sim}$ the extension of a lift of a diffeomorphism of Σ as before. The curve $\phi(\gamma)$ is simple since ϕ is a injective and the image of $[p, z]$ under $\phi^{\sim}$ is a lift of $\phi(\gamma)$ at bounded distance fom $[p, \phi(z)]$. It is not difficult to see that $[p, \phi(z)]$ projects to a simple curve on Σ.

The invariance of Λ under the action is proved in a similar way. Finally, if $z \in \Lambda_0$ the the closure of its orbit is contained in Γ_0. Suppose $w \in \Lambda \setminus \Lambda_0$ then there is a non trivial element $g \in \Gamma$ such that $g([p, w]) \cap [p, w] = x$ where x is evidently the lift of a self intersection point of the projection of $[p, z]$ to Σ. By continuity of g, if w' is sufficiently close to w then $g([p, w']) \cap [p, w'] \neq \emptyset$ so that $[p, w']$ is the lift of a curve with at least one self intersection. Thus $w' \notin \Lambda_0$ and so Λ_0, contains the orbit of $z \in \Lambda_0$, but is not dense in Λ

□

3.2. **Orbit decomposition of the limit set.** The limit set decomposes into orbits under the $\mathcal{MCG}$-action described in the previous paragraph.

Suppose that $z \in \Gamma_0$ and let γ denote the geodesic determined by $[p, z]$. The point z is the fixed point of a hyperbolic elements of Γ if and only if there is a closed simple geodesic ω in the closure of γ and for brevity we say γ *spirals to* ω. Define

$$\Lambda_h := \Lambda_0 \cap \{\text{set of fixed points of hyperbolic elements of } \Gamma\}$$

Lemma 3.2. *The set Λ_h decomposes into finitely many $\mathcal{MCG}$ orbits.*

If $z \in \Lambda_h$ then $[p, z]$ determines a geodesic γ_z. The geodesic γ_z determines a unique embedded pair of pants in $P \subset \Sigma$ which has δ as one boundary component and γ, the closed simple geodesic in the closure of γ_z. By the classification of surfaces the complement of P fall into finitely many homeomorphisms types. Therefore, there are only finitely many possibilities for γ_z up to the action of the group of homeomorphisms of Σ.

□

Corollary 3.3. *If $w, z \in \Lambda_h$ and and the closed geodesic ω determined by z is not a boundary component (i.e. it is essential) then*

$$z \in \overline{\mathcal{MCG}.w}$$

so that

$$\overline{\mathcal{MCG}.w} \subset \overline{\mathcal{MCG}.z}.$$

In particular if both w and z both determine essential closed simple geodesics in Σ then

$$\overline{\mathcal{MCG}.w} = \overline{\mathcal{MCG}.z}.$$

Proof. The inclusion follows trivially from the first part since orbit closures are $\mathcal{MCG}$-invariant.

To show that z is an accumulation point of w's orbit we begin by noting that there are finitely many mapping classes ϕ_k such that the closed simple geodesics determined by the images of $\phi_k(\omega)$ fill the surface. Let β denote the geodesic determined by $[p, w]$ then β meets one of the $\phi_k(\omega)$. The images of β by iterates of a Dehn twist round $\phi_k(\omega)$ provide a sequence of geodesics that converge to $\phi_k(\gamma)$. Lifting to $\mathbb{H}$ one sees that the corresponding sequence of of images of w converge to z.

□

Now we define $\Lambda_x \subset \Lambda_0$ to be the set of z such that the geodesic on Σ determined by $[p, z]$ does not spiral to a boundary component.

Theorem 3.4. (1) *The set Λ_x is contained in the closure of the orbit of any point z.*
(2) *Moreover Λ_x is a minimal set for the action of the mapping class group.*

Proof. The proof of (1) is exactly the same as Corollary 3.3. The key to showing minimality is Lemma 3.2 above. It suffices to show that given $w \in \Lambda_x$ there is some sequence of points $z_n \in \Lambda_h$ that converges to w. Since there are only finitely many $\mathcal{MCG}$-orbits we can suppose that all the z_n belong to the same orbit, $\mathcal{MCG}.z$ say, so that w is an accumulation point of this orbit so $\overline{\mathcal{MCG}.z} = \Gamma_x$. But by (1)

$$\overline{\mathcal{MCG}.z} \subset \overline{\mathcal{MCG}.w} \subset \Gamma_x$$

so all the orbits are dense.

Let $w \in \Gamma_x$ and β denote the geodesic determined by $[p, w]$ then $w \in \overline{\mathcal{MCG}.z}$ for a point z as above. The techniques introduced in [7] apply and one can construct a sequence γ_n of simple common perpendiculars to the boundary that converge to β. Each of these arcs determines a pair of pants and a corresponding a gap on the boundary δ. The gaps are bounded by points in Γ_x such that the corresponding orthogeodesics spiral to closed simple geodesics. For each n one can choose a point $z_n \in \Gamma_x$ that an endpoint of the gap determined by γ_n and which lies between the initial point of γ_n and the initial point of β.

□

3.3. **Remarks.** Before continuing to the prove the identities we record the following useful observations:

- The point (1) can be expressed succinctly as:

$$\Lambda_x \subset \bigcap_z \overline{\mathcal{MCG}.z}.$$

- In addition we have

$$\forall z \in \Lambda_x, \overline{\mathcal{MCG}.z} \subset \Lambda_x,$$

so the fact that the set there is a dense orbit Λ_x means that there is no hope, just as before, of decomposing the orbit structure further. With a little more care one can show that this set is minimal for any finite index subgroup of the mapping class and even for non trivial normal subgroups such as the Torelli group.
- In fact, Λ_x is the set of non isolated points of Λ_0.

4. Basmajian

The easiest identity is that of Basmajian and is almost a direct application of Theorem 1.1 and Proposition 2.1.

Theorem 4.1 (Basmajian). *Let Σ be a surface with a single totally geodesic boundary component δ. Then*

$$\sum_{\alpha^*} 2\sinh^{-1}\left(\frac{1}{\sinh(\ell(\alpha^*))}\right) = \ell(\delta)$$

Proof. : Let Ω be the regular set, that is the complement of $\Lambda \subset \partial\mathbb{H}$. Under the hypothesis Ω is a countable union of intervals. The identity is proved by considering the nearest point retraction of Ω onto a geodesic $\delta^\sim$ which is a lift of δ. The geodesic δ decomposes into a negligible piece, i.e. the image of Λ, and the image of Ω. This second part (Proposition 2.1) further decomposes into the images of its connected component each of which is associated to (the lift of) an orthogeodesic α^*.

□

5. McShane identities

McShane's identities provide a relation for the lengths of closed geodesics, in particular, if Σ is a hyperbolic punctured

$$\sum_{\alpha} \frac{1}{1+e^{\ell(\alpha)}} = \frac{1}{2},$$

where the sum is over all closed simple geodesics α. This can be obtained as a limiting case when $\ell(\delta) \to 0$ of the identity for the one holed torus (see [7])

$$\sum_{\alpha} \log\left(\frac{1+e^{\frac{1}{2}(\ell(\alpha)-\ell(\delta))}}{1+e^{\frac{1}{2}(\ell(\alpha)+\ell(\delta))}}\right) = \ell(\delta),$$

This is in turn a special case of the identity for a one-holed surface of genus g (see [8])

$$\sum_{P} \log\left(\frac{1+e^{\frac{1}{2}(\ell(\alpha)+\ell(\beta)-\ell(\delta))}}{1+e^{\frac{1}{2}(\ell(\alpha)+\ell(\beta)+\ell(\delta))}}\right) = \ell(\delta)$$

where P is an embedded pair of pants with waist δ and legs α, β

We remark that the identity for the punctured torus P on a holed torus is just the indentity for a one-holed surface of genus g where an embedded pants has "waist" of length δ and "legs" α, α. So in some senses it is an "happy accident" that the sum over all closed simple geodesics.

5.1. **Proof.** The identity is proved, in a completely analogous fashion to Basmajian's identity, by considering the nearest point retraction of the complement of Λ_x onto a geodesic $\delta^\sim$ which is a lift of δ. The set Λ_x is invariant under the subgroup of Γ that preserves $\delta^\sim$ so this yields a decomposition of δ as a a negligible piece K, namely the image of Γ_x, and its complement. The latter further decomposes into countably many pieces, called *gaps*, in 1-1 correspondence with simple orthogeodesics and hence pairs of pants via :

Theorem 5.1. *The intervals in the complement of Λ_x are in 1-1 correspondence with lifts of embedded pairs of pants P.*

Proof. The follows from the classification as in [7] and [8] □

Corollary 5.2.

$$\sum_P \log\left(\frac{1+e^{\frac{1}{2}(\ell(\alpha)+\ell(\beta)-\ell(\delta))}}{1+e^{\frac{1}{2}(\ell(\alpha)+\ell(\beta)+\ell(\delta))}}\right) = \ell(\delta)$$

Proof. The computation of the size of a gap cna be found in [8] □

6. Bridgeman

The Bridgeman identity is based on a decomposition of the unit tangent bundle of the surface. We denote $p : T\mathbb{H}^n \to \mathbb{H}^n$ the canonical map that associates to a tangent vector its basepoint. If $v \in T\mathbb{H}^n$ is a (non zero) tangent vector then $\gamma_v : \mathbb{R} \to \mathbb{H}^n$ is the unique geodesic parameterised by arclength such that $\dot\gamma_v(0)$ is a positive multiple of v. The geodesic γ_v determines a pair of distinct points $\gamma_v(\pm\infty)$ in the ideal boundary of $\mathbb{H}^n$. Observe that the map

$$\begin{array}{rcl} v & \mapsto & \gamma_v(-\infty) \\ T\mathbb{H}^n & \to & \partial\mathbb{H}^n \end{array}$$

is smooth and, in particular, the preimage of any measurable subset of $\partial\mathbb{H}^n$ is a measurable subset of the tangent bundle. By considering $\gamma_{-v}(\infty)) = \gamma_v(-\infty)$ as well, one obtains a smooth embedding of the unit tangent bundle into the product
$\partial\mathbb{H}^n \times \partial\mathbb{H}^n \times \mathbb{R}$ and we apply Fubini's Theorem to obtain:

Lemma 6.1. *If $K \subset \partial\mathbb{H}^n$ is measure 0 then $K_\infty = \{v, \gamma_v(\infty) \in K\} \subset T\mathbb{H}^n$ is measure 0.*

Bridgeman [3] constructs a decomposition of the unit tangent bundle of $CH(\Lambda)$, the convex hull of Λ. Fix a lift $\hat\delta$ of δ and let Ω be as before. The endpoints of $\hat\delta$ determine a connected component of Ω and, moreover, any other such component shares endpoints with some another lift of δ, $\hat\delta'$ say. Define the *Bridgeman's set* $\mathcal{B}(\hat\delta, \hat\delta')$ for the pair $\hat\delta, \hat\delta'$ to be the set of v in the unit tangent bundle of $CH(\Lambda)$ tangent to a geodesic γ_v meeting both $\hat\delta$ and $\hat\delta'$.

Theorem 6.2 (Bridgeman). (1) *$CH(\Lambda)$ is the disjoint union a negligible part contained in K_∞ above and countably many Bridgeman's sets.*

(2) *The volume of* $\mathcal{B}(\hat{\delta}, \hat{\delta}')$ *is* $\mathcal{L}\left(\frac{4}{\cosh^2(\ell(\alpha^*)/2)}\right)$ *where* $\ell(\alpha^*)$ *is the length of the unique ortho geodesic determined by the pair* $\hat{\delta}, \hat{\delta}'$.

(3) *The volume of the unit tangent bundle of* Σ *is*

$$\sum_{\alpha^*} 8\mathcal{L}\left(\frac{1}{\cosh^2(\ell(\alpha^*)/2)}\right).$$

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