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A FINITE PRESENTATION OF THE LEVEL 2 PRINCIPAL CONGRUENCE SUBGROUP OF $GL(n;\mathbb{Z})$

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ABSTRACT. It is known that the level 2 principal congruence subgroup of $GL(n;\mathbb{Z})$ has a finite generating set (see [6]). In this paper, we give a finite presentation of the level 2 principal congruence subgroup of $GL(n;\mathbb{Z})$.

1. INTRODUCTION

For $n\geq 1$, let $\Gamma_2(n)=\ker(GL(n;\mathbb{Z})\twoheadrightarrow GL(n;\mathbb{Z}_2))$ denote the *level* 2 *principal congruence subgroup* of $GL(n;\mathbb{Z})$. Note that for $A\in\Gamma_2(n)$ the diagonal entries of A are odd and the others are even.

For $1\leq i,j\leq n$ with $i\neq j$, let E_{ij} denote the matrix whose (i,j) entry is 2, diagonal entries are 1 and others are 0, and let F_i denote the matrix whose (i,i) entry is -1, other diagonal entries are 1 and others are 0. It is known that $\Gamma_2(n)$ is generated by E_{ij} and F_i for $1\leq i,j\leq n$ with $i\neq j$ (see [6]).

In this paper, we give a finite presentation of $\Gamma_2(n)$.

Theorem 1.1. *For $n\geq 1$, $\Gamma_2(n)$ has a finite presentation with generators E_{ij} and F_i, for $1\leq i,j\leq n$ with $i\neq j$, and with relators*

(1) F_i^2 *for* $1\leq i\leq n$,
(2) $(E_{ij}F_i)^2$, $(E_{ij}F_j)^2$, $(F_iF_j)^2$ *for* $1\leq i,j\leq n$ *with* $i\neq j$ *(when* $n\geq 2$*)*,
(3) (a) $[E_{ij},E_{ik}]$, $[E_{ij},E_{kj}]$, $[E_{ij},F_k]$, $[E_{ij},E_{ki}]E_{kj}^2$ *for* $1\leq i,j,k\leq n$, *and* i,j,k *are mutually different (when* $n\geq 3$*)*
(b) $[E_{ji}F_jE_{ij}F_iE_{ki}^{-1}E_{kj},E_{ki}F_kE_{ik}F_iE_{ji}^{-1}E_{jk}]$ *for* $1\leq i<j<k\leq n$ *(when* $n\geq 3$*)*,
(4) $[E_{ij},E_{kl}]$ *for* $1\leq i,j,k,l\leq n$, *and* i,j,k,l *are mutually different (when* $n\geq 4$*)*,

where $[X,Y]=X^{-1}Y^{-1}XY$.

We now explain about an application of Theorem 1.1. For $g\geq 1$, let N_g denote a non-orientable closed surface of genus g, that is, N_g is a connected sum of g real projective planes. Let $\cdot : H_1(N_g;R)\times H_1(N_g;R)\rightarrow\mathbb{Z}_2$ denote the mod 2 intersection form, and let $\mathrm{Aut}(H_1(N_g;R),\cdot)$ denote the group of automorphisms over $H_1(N_g;R)$ preserving the mod 2 intersection form $\cdot$, where $R=\mathbb{Z}$ or $\mathbb{Z}_2$. Consider the natural epimorphism

$$\Phi_g:\mathrm{Aut}(H_1(N_g;\mathbb{Z}),\cdot)\rightarrow\mathrm{Aut}(H_1(N_g;\mathbb{Z}_2),\cdot).$$

MacCarthy and Pinkall [6] showed that $\Gamma_2(g-1)$ is isomorphic to $\ker\Phi_g$.

We denote by $\mathcal{M}(N_g)$ the group of isotopy classes of diffeomorphisms over N_g. The group $\mathcal{M}(N_g)$ is called the *mapping class group* of N_g. In [6] and [3], it is shown that the natural homomorphism $\mathcal{M}(N_g)\rightarrow\mathrm{Aut}(H_1(N_g;R),\cdot)$ is surjective, where $R=\mathbb{Z}$ or $\mathbb{Z}_2$. Let $\mathcal{I}(N_g)$ denote the kernel of $\mathcal{M}(N_g)\rightarrow\mathrm{Aut}(H_1(N_g;\mathbb{Z}),\cdot)$. We say $\mathcal{I}(N_g)$ the *Torelli group* of N_g. In [4], Hirose and the author obtained a generating set of $\mathcal{I}(N_g)$ for $g\geq 4$, using Theorem 1.1.

2. PRELIMINARIES

In this section, we explain about some facts for presentations of groups.

2.1. **Basics on presentations of groups.**
Let G_1, G_2 and G_3 be groups with a short exact sequence

$$1 \to G_1 \overset{\phi}{\to} G_2 \overset{\pi}{\to} G_3 \to 1.$$

If G_1 and G_3 are presented then we can obtain a presentation of G_2. In particular, if G_1 and G_3 are finitely presented then G_2 can be finitely presented.

More precisely, a presentation of G_2 is obtained as follows. Let $G_1 = \langle X_1 \mid R_1 \rangle$ and $G_3 = \langle X_3 \mid R_3 \rangle$. For each $x \in X_3$, we choose $\widetilde{x} \in \pi^{-1}(x)$. We put $X_2 = \{\phi(x_1), \widetilde{x_3} \mid x_1 \in X_1, x_3 \in X_3\}$. For $r = a_1^{\varepsilon_1} \cdots a_k^{\varepsilon_k} \in R_3$, let $\widetilde{r} = \widetilde{a_1}^{\varepsilon_1} \cdots \widetilde{a_k}^{\varepsilon_k}$. For $g \in \ker \pi$, let $\overline{g}$ be a word over $\phi(X_1)$ with $g = \overline{g}$. Let $A = \{\phi(r_1) \mid r_1 \in R_1\}$, $B = \{\widetilde{r_3}\overline{\widetilde{r_3}}^{-1} \mid r_3 \in R_3\}$ and $C = \{\widetilde{x_3}\phi(x_1)\widetilde{x_3}^{-1}\overline{\widetilde{x_3}\phi(x_1)\widetilde{x_3}^{-1}}^{-1} \mid x_1 \in X_1, x_3 \in X_3\}$. We put $R_2 = A \cup B \cup C$. Then we have $G_2 = \langle X_2 \mid R_2 \rangle$.

In addition, if there is a homomorphism $\rho : G_3 \to G_2$ such that $\pi \circ \rho = id_{G_3}$, choose $\widetilde{x} = \rho(x) \in \pi(x)^{-1}$ for $x \in X_1$. Then, we have the relation $\widetilde{r} = 1$ in G_2 for $r \in R_3$.

If G_2 is presented then we can examine a presentation of G_1, by the Reidemeister-Schreier method. In particular, if G_3 is a finite group, that is, the index of Imϕ is finite, then G_1 can be finitely presented.

For further information see [5].

2.2. **Presentations of groups acting on a simplicial complex.**
Let X be a simplicial complex, and let G be a group acting on X by isomorphisms as a simplicial map. We suppose that the action of G on X is *without rotation*, that is, for a simplex $\Delta \in X$ and $g \in G$, if $g(\Delta) = \Delta$ then $g(v) = v$ for all vertices $v \in \Delta$. For a simplex $\Delta \in X$, let G_Δ be the stabilizer of Δ. For $k \geq 0$, the *k-skeleton* $X^{(k)}$ is the subcomplex of X consisting of all simplices of dimension at most k.

Consider a homomorphism $\Phi : *_{v \in X^{(0)}} G_v \to G$. For $g \in G$, if g stabilizes a vertex $w \in X^{(0)}$, we denote g by g_w as an element in $G_w < *_{v \in X^{(0)}} G_v$. For a 1-simplex $\{v, w\} \in X$ and $g \in G_v \cap G_w$, we have $g_v g_w^{-1} \in \ker \Phi$. We call this the *edge relator*.

At first, for any 1-simplex $\{v, w\}$, choose an orientation such that orientations are preserved by the action of G. Namely, orientations of $\{v, w\}$ and $g\{v, w\}$ are compatible for all $g \in G$. We denote the oriented 1-simplex $\{v, w\}$ by (v, w). Similarly, choose orders of 2-simplices, and denote the ordered 2-simplex $\{v_1, v_2, v_3\}$ by (v_1, v_2, v_3). For an oriented 1-simplex $e = (v, w)$, let $o(e) = v$ and $t(e) = w$. For an oriented 2-simplex $\tau = (v_1, v_2, v_3)$, we call v_1 the base point of τ.

Next, choose an oriented tree T of X such that a set of vertices of T is a set of representative elements for vertices of the orbit space $G\backslash X$. Let V denote the set of vertices of T. In addition, choose a set E of representative elements for oriented 1-simplices of $G\backslash X$ such that $o(e) \in V$ for $e \in E$ and 1-simplices of T is in E, and a set F of representative elements for ordered 2-simplices of $G\backslash X$ such that the base point of τ is in V for $\tau \in F$. For $e \in E$, let $w(e)$ denote the element in V which is equivalent to $t(e)$ by the action of G, and choose $g_e \in G$ such that $g_e(w(e)) = t(e)$ and $g_e = 1$ if $e \in T$.

For a 1-simplex $\{v, w\}$ with $v \in V$, note that $\{v, w\} = \{o(e), hg_e w(e)\}$ or $\{w(e), hg_e^{-1} o(e)\}$ for some $e \in E$ and $h \in G_v$. Then we define respectively $g_{\{v,w\}} = hg_e$ or hg_e^{-1}. Let α be a loop in X starting at a vertex of V. We denote $\alpha = \{v_i, \{v_i, v_{i+1}\} \mid 1 \leq i \leq k, v_{k+1} = v_1\}$. Note that $v_1, g_1^{-1} v_2 \in V$, where $g_1 = g_{\{v_1, v_2\}}$. For $2 \leq i \leq k$, define $g_i = g_{g_{i-1}^{-1} \cdots g_1^{-1} \{v_i, v_{i+1}\}}$, inductively. Note that for $2 \leq i \leq k$, there exists an oriented 1-simplex e_i such that $o(e_i) \in V$ and $\{v_i, v_{i+1}\} = g_1 \cdots g_{i-1} \{o(e_i), t(e_i)\}$. Let $g_\alpha = g_1 \cdots g_k$. We have $g_\alpha(v_1) = v_1$, namely, $g_\alpha \in G_{v_1}$.

For $e \in E$, put a word $\hat{g}_e$. For a 1-simplex $\{v, w\}$ with $v \in V$, let $\hat{g}_{\{v,w\}} = h\hat{g}_e$ or $h\hat{g}_e^{-1}$ if $g_{\{v,w\}} = hg_e$ or hg_e^{-1}, respectively. For a loop α in X starting at a vertex of V, let $\hat{g}_\alpha = \hat{g}_1 \cdots \hat{g}_k$ if $g_\alpha = g_1 \cdots g_k$. Note that we can define g_τ and $\hat{g}_\tau$ for $\tau \in F$, regarding τ as a loop in X. Let $\widehat{G} = (*_{v\in V} G_v) * (*_{e\in E}\langle \hat{g}_e\rangle)$.

The following theorem is a special case of the result of Brown [1].

Theorem 2.1 ([1]). *Let X be a simply connected simplicial complex, and let G be a group acting without rotation on X by isomorphisms as a simplicial map. Then G is isomorphic to the quotient of $\widehat{G}$ by the normal subgroup generated by followings*

(1) $\hat{g}_e$, *where* $e \in T$,

(2) $\hat{g}_e^{-1} X_{o(e)} \hat{g}_e (g_e^{-1} X g_e)_{w(e)}^{-1}$, *where* $e \in E$ *and* $X \in G_e$,

(3) $\hat{g}_\tau g_\tau^{-1}$, *where* $\tau \in F$.

3. Outline of the proof of Theorem 1.1

We will prove Theorem 1.1 by induction on n. Let $e_1, \ldots, e_n$ be canonical normal vectors in $\mathbb{Z}^n$, and let $\Gamma_2(n)_{e_t}$ denote a subgroup of $\Gamma_2(n)$ which consists of matrices $A \in \Gamma_2(n)$ such that $Ae_t = e_t$. We first prepare the next lemma.

Lemma 3.1. *For $1 \le t \le n$ there is a short exact sequence*

$$0 \to \mathbb{Z}^{n-1} \to \Gamma_2(n)_{e_t} \to \Gamma_2(n-1) \to 1.$$

Proof. For $\mathbb{Z}^{n-1}$ we give the presentation $\mathbb{Z}^{n-1} = \langle x_1, \ldots, x_{n-1} \mid x_i x_j x_i^{-1} x_j^{-1} (1 \le i < j \le n-1)\rangle$. Let $\mathbb{Z}^{n-1} \to \Gamma_2(n)_{e_t}$ be the homomorphism which sends x_i to E_{ti} when $i < t$ and to E_{ti+1} when $i \ge t$. Let $\Gamma_2(n)_{e_t} \to \Gamma_2(n-1)$ be the homomorphism which sends A to A_{tt}, where A_{ij} is the $(n-1)$-submatrix of A obtained by removing the i-row vector and the j-column vector of A. Then, it follows that the sequence $0 \to \mathbb{Z}^{n-1} \to \Gamma_2(n)_{e_t} \to \Gamma_2(n-1) \to 1$ is exact. □

It is clear that Theorem 1.1 is valid in the case $n = 1$. In addition, the case $n = 2$ of Theorem 1.1 is proved by using the Reidemeister-Schreier method. We now prove Theorem 1.1 for $n \ge 3$, using Lemma 3.1.

3.1. The case $n = 3$ of Theorem 1.1.

For $R = \mathbb{Z}$ or $\mathbb{Z}_2$, let $\mathcal{B}_n(R)$ denote the simplicial complex whose $(k-1)$-simplex $\{x_1, \ldots, x_k\}$ is the set of k-vectors $x_i \in R^n$ such that $x_1, \ldots, x_k$ are mutually different column vectors of a matrix $A \in GL(n; R)$. In [2], Day and Putman proved that $\mathcal{B}_n(\mathbb{Z})$ is $(n-2)$-connected. Here, a simplicial complex X is m-connected if its geometric realization $|X|$ is m-connected. In addition, X is -1-connected if X is nonempty. Note that there is the natural left action $\Gamma_2(n) \times \mathcal{B}_n(\mathbb{Z}) \to \mathcal{B}_n(\mathbb{Z})$ defined by $A\{x_1, \ldots, x_k\} = \{Ax_1, \ldots, Ax_k\}$ for $A \in \Gamma_2(n)$ and $\{x_1, \ldots, x_k\} \in \mathcal{B}_n(\mathbb{Z})$, and that the action is without rotation.

Since $GL(n; \mathbb{Z}_2)$ is the quotient of $GL(n; \mathbb{Z})$ by $\Gamma_2(n)$, it follows that the orbit space $\Gamma_2(n)\backslash\mathcal{B}_n(\mathbb{Z})$ is isomorphic to $\mathcal{B}_n(\mathbb{Z}_2)$. Let $\varphi : \mathcal{B}_n(\mathbb{Z}) \to \mathcal{B}_n(\mathbb{Z}_2)$ be a natural surjection induced by the surjection $GL(n; \mathbb{Z}) \twoheadrightarrow GL(n; \mathbb{Z}_2)$. For $1 \le i \le 7$, let v_i be $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, $v_4 = e_1 + e_2$, $v_5 = e_1 + e_3$, $v_6 = e_2 + e_3$ and $v_7 = e_1 + e_2 + e_3$. Then, the vertices of $\mathcal{B}_n(\mathbb{Z}_2)$ are $\varphi(v_i)$, the 1-simplices are $\varphi(\{v_i, v_j\})$, and the 2-simplices are $\varphi(\{v_1, v_2, v_3\})$, $\varphi(\{v_1, v_2, v_5\})$, $\varphi(\{v_1, v_2, v_6\})$, $\varphi(\{v_1, v_2, v_7\})$, $\varphi(\{v_1, v_3, v_4\})$, $\varphi(\{v_1, v_3, v_6\})$, $\varphi(\{v_1, v_3, v_7\})$, $\varphi(\{v_1, v_4, v_5\})$, $\varphi(\{v_1, v_4, v_6\})$, $\varphi(\{v_1, v_4, v_7\})$, $\varphi(\{v_1, v_5, v_6\})$, $\varphi(\{v_1, v_5, v_7\})$, $\varphi(\{v_2, v_3, v_4\})$, $\varphi(\{v_2, v_3, v_5\})$, $\varphi(\{v_2, v_3, v_7\})$, $\varphi(\{v_2, v_4, v_5\})$, $\varphi(\{v_2, v_4, v_6\})$, $\varphi(\{v_2, v_4, v_7\})$, $\varphi(\{v_2, v_5, v_6\})$, $\varphi(\{v_2, v_6, v_7\})$, $\varphi(\{v_3, v_4, v_5\})$, $\varphi(\{v_3, v_4, v_6\})$, $\varphi(\{v_3, v_5, v_6\})$, $\varphi(\{v_3, v_5, v_7\})$, $\varphi(\{v_3, v_6, v_7\})$, $\varphi(\{v_4, v_5, v_7\})$, $\varphi(\{v_4, v_6, v_7\})$ and $\varphi(\{v_5, v_6, v_7\})$. (Note that $\{v_1, v_2, v_4\}$, $\{v_1, v_6, v_7\}$, $\{v_1, v_3, v_5\}$, $\{v_2, v_3, v_6\}$, $\{v_2, v_5, v_7\}$, $\{v_3, v_4, v_7\}$ and $\{v_4, v_5, v_6\}$ are not 2-simplices of $\mathcal{B}_n(\mathbb{Z})$.)

We prove the next lemma.

Lemma 3.2. *$\Gamma_2(3)$ is isomorphic to the quotient of $*_{1\le i\le 7}\Gamma_2(3)_{v_i}$ by the normal subgroup generated by edge relators.*

Proof. We set followings

- $V = \{v_1, \cdots v_7\}$,
- $T = \{(v_1, v_i) \mid 2 \le i \le 7\} \cup V$,
- $E = \{(v_i, v_j) \mid 1 \le i < j \le 7\}$,
- $F = \{(v_i, v_j, v_k) \mid 1 \le i < j < k \le 7, \varphi(\{v_i, v_j, v_k\}) \in \mathcal{B}_n(\mathbb{Z}_2)\}$.

For $e = (v_i, v_j) \in E$, since $w(e) = t(e)$, we choose $g_e = 1$, and write $g_{ij} = g_e$. By Theorem 2.1, $\Gamma_2(3)$ is isomorphic to the quotient of $(*_{1\le i\le 7}\Gamma_2(3)_{v_i}) * (*_{1\le i<j\le 7}\langle \hat{g}_{ij}\rangle)$ by the normal subgroup generated by followings

(1) $\hat{g}_{1i}$, where $2 \le i \le 7$,
(2) $\hat{g}_{ij}^{-1} X_{v_i} \hat{g}_{ij} X_{v_j}^{-1}$, where $1 \le i < j \le 7$ and $X \in \Gamma_2(3)_{(v_i,v_j)}$,
(3) $\hat{g}_\tau g_\tau^{-1}$, where $\tau \in F$.

Note that $g_\tau = g_{ij} g_{jk} g_{ik}^{-1}$ for $\tau = (v_i, v_j, v_k)$. Hence, the relation $\hat{g_\tau} g_\tau^{-1} = 1$ is equivalent to the relation $\hat{g}_{ij}\hat{g}_{jk} = \hat{g}_{ik}$. Since $\hat{g}_{1i} = 1$ for $2 \le i \le 7$, we have the relation $\hat{g}_{ij} = 1$ for $2 \le i < j \le 7$ except $(i,j) = (2,4)$, $(3,5)$ and $(6,7)$. For example, the relation $\hat{g}_{23} = 1$ is obtained from the relation $\hat{g}_{12}\hat{g}_{23} = \hat{g}_{13}$. In addition, relations $\hat{g}_{24} = 1$, $\hat{g}_{35} = 1$ and $\hat{g}_{67} = 1$ are obtained from relations $\hat{g}_{23}\hat{g}_{34} = \hat{g}_{24}$, $\hat{g}_{23}\hat{g}_{35} = \hat{g}_{25}$ and $\hat{g}_{26}\hat{g}_{67} = \hat{g}_{27}$, respectively. Hence, we have the relation $\hat{g}_{ij} = 1$ for $1 \le i < j \le 7$. Therefore, $\Gamma_2(3)$ is isomorphic to the quotient of $*_{1\le i\le 7}\Gamma_2(3)_{v_i}$ by the normal subgroup generated by $A = \{X_{v_i} X_{v_j}^{-1} \mid 1 \le i < j \le 7, X \in \Gamma_2(3)_{(v_i,v_j)}\}$. Since A is the set of edge relators, we obtain the claim. □

From Lemma 3.1 and Lemma 3.2, we obtain the presentation of $\Gamma_2(3)$.

3.2. **The case $n \ge 4$ of Theorem 1.1.**

In this subsection, we introduce a simplicial complex which $\Gamma_2(n)$ acts on.

Let $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ denote the subcomplex of $\mathcal{B}_n(\mathbb{Z})$ whose $(k-1)$-simplex $\{x_1, \dots, x_k\}$ is the set of k-vectors $x_i \in \mathbb{Z}^n$ such that $x_1, \dots, x_k$ are mutually different column vectors of a matrix $A \in \Gamma_2(n)$. Note that for a vertex v, we have $v \equiv e_i \mod 2$ for some $1 \le i \le n$.

We have the following.

Proposition 3.3. *For $n \ge 4$, the simplicial complex $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is simply connected.*

We will prove this proposition in Appendix. We now prove Theorem 1.1.

Lemma 3.4. *For any $n \ge 4$, $\Gamma_2(n)$ is isomorphic to the quotient of $*_{1\le i\le n}\Gamma_2(n)_{e_i}$ by the normal subgroup generated by edge relators.*

Proof. For a $(k-1)$-simplex $\Delta = \{x_1, \dots, x_k\} \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ with $x_j \equiv e_{i(j)} \mod 2$, let $A \in \Gamma_2(n)$ be an extension of Δ. Then we have $A^{-1} \cdot \Delta = \{e_{i(1)}, \dots, e_{i(k)}\}$. Therefore, we have

$$\Gamma_2(n)\backslash\Gamma_2\mathcal{B}_n(\mathbb{Z}) = \{\{e_{i(1)}, \dots, e_{i(k)}\} \mid 1 \le k \le n, 1 \le i(1) < \cdots < i(k) \le n\}.$$

It is clear that $\Gamma_2(n)\backslash\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is contractible. Note that the action of $\Gamma_2(n)$ on $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is without rotation.

We first set followings.

- $T = \{(e_1, e_i) \mid 2 \le i \le n\}$.
- $E = \{(e_i, e_j) \mid 1 \le i < j \le n\}$.
- $F = \{(e_i, e_j, e_k) \mid 1 \le i < j < k \le n\}$.
- For $e \in E$, we choose $g_e = 1$, and write $g_e = g_{ij}$ when $e = (e_i, e_j)$.
- For $\tau = (e_i, e_j, e_k) \in F$, let $g_\tau = g_{ij} g_{jk} g_{ik}^{-1}$.

Then, since $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is simply connected, it follows from Theorem 2.1 that $\Gamma_2(n)$ is isomorphic to the quotient of $((*_{1\leq i\leq n}\Gamma_2(n)_{e_i}) * (*_{1\leq i<j\leq n}\langle \hat{g}_{ij}\rangle))$ by the normal subgroup generated by followings

(1) $\hat{g}_{1i}$, where $2 \leq i \leq n$,
(2) $\hat{g}_{ij}^{-1}X_{e_i}\hat{g}_{ij}X_{e_j}^{-1}$, where $1 \leq i < j \leq n$ and $X \in \Gamma_2(n)_{(e_i,e_j)}$,
(3) $\hat{g}_\tau g_\tau^{-1}$, where $\tau \in F$.

Since $g_\tau = 1$, the relation $\hat{g}_\tau g_\tau^{-1}$ is equivalent to the relation $\hat{g}_{ij}\hat{g}_{jk} = \hat{g}_{ik}$ if $\tau = (e_i, e_j, e_k)$. By relations $\hat{g}_{1i} = 1$, we have the relation $\hat{g}_{ij} = 1$ for $1 \leq i < j \leq n$. Thus, we obtain the claim. □

From Lemma 3.1 and Lemma 3.4, we obtain the presentation of $\Gamma_2(n)$, by induction on n. Thus, we finish the proof of Theorem 1.1.

Appendix A

In this appendix, we prove Proposition 3.3. In a proof of this proposition, we will use their idea for proving that $\mathcal{B}_n(\mathbb{Z})$ is $(n-2)$-connected (see [2]).

A.1. **Preparation.**

Let X be a simplicial complex. Then we define followings.

- For a simplex $\Delta \in X$, $\mathrm{star}_X(\Delta)$ is the subcomplex of X whose simplex $\Delta' \in X$ satisfies that Δ, $\Delta' \subset \Delta''$ for some simplex $\Delta'' \in X$. We also define $\mathrm{star}_X(\emptyset) = X$.
- For a simplex $\Delta \in X$, $\mathrm{link}_X(\Delta)$ is the subcomplex of $\mathrm{star}_X(\Delta)$ whose simplex $\Delta' \in \mathrm{star}_X(\Delta)$ does not intersect Δ. We also define $\mathrm{link}_X(\emptyset) = X$.

For a $(k-1)$-simplex $\Delta = \{x_1, \dots, x_k\}$, $A \in \Gamma_2(n)$ is an *extension* of Δ if each x_i is a column vector of A. Here, we prove followings.

Lemma A.1. *For $n \geq 2$, $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is path connected.*

Proof. We first consider the case $n = 2$. Let $v_0 = v_{01}e_1 + v_{02}e_2 \in \Gamma_2\mathcal{B}_2(\mathbb{Z})$ be a vertex. Then there exist vertices $v_1 = v_{11}e_1 + v_{12}e_2, \dots, v_k = v_{k1}e_1 + v_{k2}e_2 \in \Gamma_2\mathcal{B}_2(\mathbb{Z})$ such that $\{v_i, v_{i+1}\} \in \Gamma_2\mathcal{B}_2(\mathbb{Z})$, $|v_{i1}| > |v_{i+1\,1}|$ for $0 \leq i \leq k-1$ and $v_k = e_1$ or e_2, for some positive integer k. Hence, $\Gamma_2\mathcal{B}_2(\mathbb{Z})$ is path connected.

Next, we suppose $n \geq 3$. Let $v, w \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ be vertices. Without loss of generality, we suppose $v \equiv e_1$ and $w \equiv e_2 \mod 2$. Then there is an extension $A \in \Gamma_2(n)$ of v. We write $A^{-1}w = \sum_{i=i}^n a_ie_i$. Let $S_{A^{-1}w} = \sum_{i=3}^n |a_i|$. For $3 \leq i \leq n$, if $|a_2| < |a_i|$, there is an integer $u \in \mathbb{Z}$ such that $|a_2| > |a_i + 2ua_2|$. Then we have that $S_{E_{i2}^u A^{-1}w} < S_{A^{-1}w}$ and $E_{i2}^u A^{-1}v = e_1$. If $|a_2| > |a_i| \neq 0$, there is an integer $u' \in \mathbb{Z}$ such that $|a_2 + 2u'a_i| < |a_i|$. Then we have that $S_{E_{2i}^{u'} A^{-1}w} < S_{A^{-1}w}$ and $E_{2i}^{u'} A^{-1}v = e_1$. Repeating this operation, we conclude that there exists $B \in \Gamma_2(n)$ such that $S_{Bw} = 0$ and $Bv = e_1$. Note that Bw can be regarded as a vertex in $\Gamma_2\mathcal{B}_2(\mathbb{Z})$. Hence, Bw is joined to e_1 or e_2, that is, Bw is joined to Bv. Therefore, v and w are joined by a path. Thus, $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is path connected. □

Lemma A.2. *Let $\Delta \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ be a $(k-1)$-simplex. Then we have followings.*

- $\mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)$ *is isomorphic to* $\mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1, \dots, e_k\})$ *as a simplicial complex.*
- $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)$ *is isomorphic to* $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1, \dots, e_k\})$ *as a simplicial complex.*

Proof. For $\Delta = \{x_1, \dots, x_k\}$, suppose $x_j \equiv e_{i(j)} \mod 2$. Let $A \in \Gamma_2(n)$ be an extension of Δ. Then restrictions of the action of A^{-1} on $\Gamma_2\mathcal{B}_n(\mathbb{Z})$

$$A^{-1}|_{\mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)} \; : \; \mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta) \to \mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_{i(1)}, \dots, e_{i(k)}\}),$$
$$A^{-1}|_{\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)} \; : \; \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta) \to \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_{i(1)}, \dots, e_{i(k)}\})$$

are isomorphisms as a simplicial map. It is clear that $\mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_{i(1)},\dots,e_{i(k)}\})$ and $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_{i(1)},\dots,e_{i(k)}\})$ are respectively isomorphic to $\mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1,\dots,e_k\})$ and $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1,\dots,e_k\})$. Thus, we obtain the claim. □

Corollary A.3. *Let $\Delta \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ be a $(k-1)$-simplex. If $n-k \geq 2$, then $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)$ is path connected.*

Proof. By a similar argument to the proof of Lemma A.1, we have that $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1,\dots,e_k\})$ is path connected. By Lemma A.2, $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)$ is also path connected. □

A.2. **Proof of Proposition 3.3.**

We suppose $n \geq 4$. Let $\alpha = \{x_i, \{x_i, x_{i+1}\} \mid 1 \leq i \leq k, x_{k+1} = x_1\}$ be a loop on $\Gamma_2\mathcal{B}_n(\mathbb{Z})$. We show that α is null-homotopic.

For $v = \sum_{i=1}^n v_i e_i \in \mathbb{Z}^n$, we define $\mathrm{Rank}(v) = |v_n|$. Let $R_\alpha = \max \mathrm{Rank}(x_i)$.

We first prove the next lemma.

Lemma A.4. *For a 1-simplex $\{v,w\} \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ with $\mathrm{Rank}(v) = \mathrm{Rank}(w) = 0$, we have $\{v,w\} \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$.*

Proof. Note that $v \not\equiv w \mod 2$. Suppose that $v \equiv e_i$, $w \equiv e_j \mod 2$ and $i < j$. Since $\mathrm{Rank}(v) = \mathrm{Rank}(w) = 0$, we have that $v, w \not\equiv e_n \mod 2$. Then there exists an extension $A = (a_1 \cdots a_n) \in \Gamma_2(n)$ of $\{v,w\}$. Let $S_A = \sum_{l=1}^n \mathrm{Rank}(a_l)$. Note that S_A is odd.

First, we consider the case $S_A = 1$. Note that $\mathrm{Rank}(a_l) = 0$ for $1 \leq l \leq n-1$ and $\mathrm{Rank}(a_n) = 1$. Then there exists $B \in \Gamma_2(n)$ such that $BA = (a_1 \cdots a_{n-1} e_n)$. Hence, we have that $\{v,w\} = \{a_i, a_j\} \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$.

Next, we suppose $S_A \geq 3$. Note that there exists $1 \leq l \leq n-1$ except $l = i,j$ such that $\mathrm{Rank}(a_l) \neq 0$. If $\mathrm{Rank}(a_l) > \mathrm{Rank}(a_n)$, there exists an integer $u \in \mathbb{Z}$ such that $\mathrm{Rank}(a_l + 2ua_n) < \mathrm{Rank}(a_n)$. Then we have that AE_{nl}^u is an extension of $\{v,w\}$ and that $S_{AE_{nl}^u} < S_A$. Similarly, if $\mathrm{Rank}(a_l) < \mathrm{Rank}(a_n)$, there exists an integer $u' \in \mathbb{Z}$ such that $\mathrm{Rank}(a_l) > \mathrm{Rank}(a_n + 2u'a_l)$. Then we have that $AE_{ln}^{u'}$ is an extension of $\{v,w\}$ and that $S_{AE_{ln}^{u'}} < S_A$. Repeating this operation, we conclude that there exists an extension $A' \in \Gamma_2(n)$ of $\{v,w\}$ such that $S_{A'} = 1$. Therefore, we have $\{v,w\} \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$. Thus, we obtain the claim. □

When $R_\alpha = 0$, by this lemma, we have $\{x_i, x_{i+1}\} \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$. Namely, the loop α is in $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$. Since $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$ is the subcomplex of $\mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$ and $\mathrm{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$ is contractible, α is null-homotopic. Therefore, we next assume $R_\alpha > 0$.

Suppose that R_α is odd. Then there exists $1 \leq i \leq k$ such that $\mathrm{Rank}(x_i) = R_\alpha$. Since R_α is odd, we have that $x_i \equiv e_n$, $x_{i\pm1} \not\equiv e_n \mod 2$ and $\mathrm{Rank}(x_{i\pm1}) < R_\alpha$. By Corollary A.3, we have that $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(x_i)$ is path connected. Since $x_{i\pm1} \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(x_i)$, there exists a path $\{y_j, y_l, \{y_j, y_{j+1}\} \mid 1 \leq j \leq l-1\}$ on $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(x_i)$ between $x_{i\pm1}$ such that $y_1 = x_{i-1}$ and $y_l = x_{i+1}$ (see Figure 1). Since R_α is odd and $\mathrm{Rank}(y_j)$ is even for each y_j, there exists an integer $s_j \in \mathbb{Z}$ such that $\mathrm{Rank}(y_j') < R_\alpha$, where $y_j' = y_j + 2s_j x_i$. We choose $s_j = 0$ if $\mathrm{Rank}(y_j) < R_\alpha$. Then we have that the path $\{y_j', y_l', \{y_j', y_{j+1}'\} \mid 1 \leq j \leq l-1\}$ between $x_{i\pm1}$ is in $\mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(x_i)$ (see Figure 1). Let $\alpha' = \alpha \cup \{y_j', y_l', \{y_j', y_{j+1}'\} \mid 1 \leq j \leq l-1\} \setminus \{x_i, \{x_i, x_{i\pm1}\}\}$. Then α' is homotopic to α (see Figure 1). For all x_i with $\mathrm{Rank}(x_i) = R_\alpha$, applying the same operation, we conclude that $R_\beta < R_\alpha$, where β is a resulting loop which is homotopic to α.

Next, suppose that R_α is even. Then there exists $1 \leq i \leq k$ such that $\mathrm{Rank}(x_i) = R_\alpha$. Since R_α is even, we have $x_i \not\equiv e_n \mod 2$.

Remark A.5. *Under the assumption $n \geq 4$, we may suppose all of following conditions.*

- $\mathrm{Rank}(x_{i\pm1}) < R_\alpha$,
- $x_{i\pm1} \not\equiv e_n \mod 2$,
- $x_{i-1} \not\equiv x_{i+1} \mod 2$.

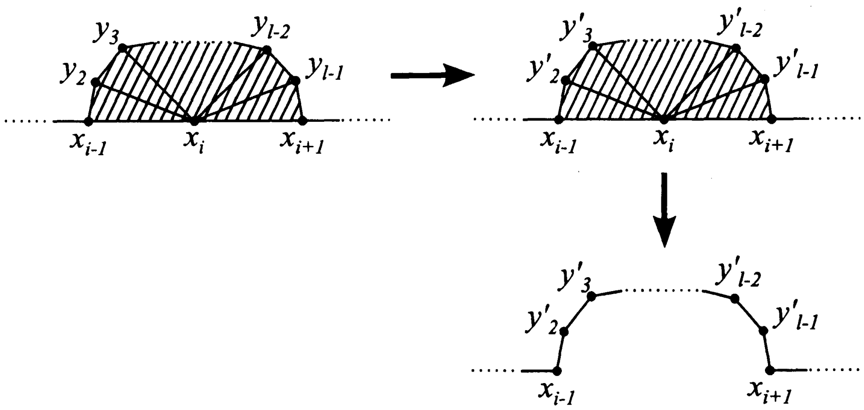

FIGURE 1. The case R_α is odd.

Proof. If $\mathrm{Rank}(x_{i-1}) = R_\alpha$, then there exists a vertex $y \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{x_{i-1}, x_i\})$ such that $y \equiv e_n \mod 2$ and $\mathrm{Rank}(y) < R_\alpha$, since R_α is even and $\mathrm{Rank}(y)$ is odd. Let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α. Hence, considering α' in place of α, we may suppose $\mathrm{Rank}(x_{i-1}) < R_\alpha$. Similarly, we may suppose $\mathrm{Rank}(x_{i+1}) < R_\alpha$.

If $x_{i-1} \equiv e_n \mod 2$, then there exists a vertex $y \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{x_{i-1}, x_i\})$ such that $y \not\equiv e_n \mod 2$ and $\mathrm{Rank}(y) < \mathrm{Rank}(x_{i-1})(< R_\alpha)$, since $\mathrm{Rank}(x_{i-1})$ is odd and $\mathrm{Rank}(y)$ is even. Let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α. Hence, considering α' in place of α, we may suppose $\mathrm{Rank}(x_{i-1}) < R_\alpha$ and $x_{i-1} \not\equiv e_n \mod 2$. Similarly, we may suppose $\mathrm{Rank}(x_{i+1}) < R_\alpha$ and $x_{i+1} \not\equiv e_n \mod 2$.

Suppose that $\mathrm{Rank}(x_{i\pm1}) < R_\alpha$ and $x_{i\pm1} \not\equiv e_n \mod 2$. If $x_{i-1} \equiv x_{i+1} \mod 2$, then there exists a vertex $y \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{x_{i-1}, x_i\})$ such that $y \not\equiv x_{i+1}, e_n \mod 2$ and $\mathrm{Rank}(y) \leq \mathrm{Rank}(x_{i-1})(< R_\alpha)$, since $n \geq 4$. Let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α. Hence, considering α' in place of α, we may suppose that $\mathrm{Rank}(x_{i\pm1}) < R_\alpha$, $x_{i\pm1} \not\equiv e_n \mod 2$ and $x_{i-1} \not\equiv x_{i+1} \mod 2$. □

We now suppose the conditions of the above remark. Suppose that $x_i \equiv e_s$, $x_{i-1} \equiv e_t$ and $x_{i+1} \equiv e_u \mod 2$, where s, t and u are mutually different and not equal to n. Then there exists $A \in \Gamma_2(n)$ such that $Ax_i = e_s$, $Ax_{i-1} = e_t$ and $\mathrm{Rank}(Ax_{i+1}) = 0$. In fact, since $\{x_{i-1}, x_i\}$ is a 1-simplex in $\Gamma_2\mathcal{B}_n(\mathbb{Z})$, there is an extension $B \in \Gamma_2(n)$ of $\{x_{i-1}, x_i\}$. We write $B^{-1}x_{i+1} = \sum_{j=1}^n a_j e_j$. It follows that there exist an even integer b_u and an odd integer b_n such that $a_u b_n - a_n b_u = \gcd(a_u, a_n)$. Then we have that

$$\begin{pmatrix} a_u/\gcd(a_u, a_n) & b_u \\ a_n/\gcd(a_u, a_n) & b_n \end{pmatrix}^{-1} \begin{pmatrix} a_u \\ a_n \end{pmatrix} = \begin{pmatrix} \gcd(a_u, a_n) \\ 0 \end{pmatrix}.$$

Let $C \in \Gamma_2(n)$ be the matrix whose (u, u) entry is $a_u/\gcd(a_u, a_n)$, (n, u) entry is $a_n/\gcd(a_u, a_n)$, (u, n) entry is b_u, (n, n) entry is b_n, other diagonal entries are 1 and other entries are 0. Then it follows that $Ax_i = e_s$, $Ax_{i-1} = e_t$ and $\mathrm{Rank}(Ax_{i+1}) = 0$, where $A = C^{-1}B^{-1}$.

Since $\{e_s, Ax_{i+1}\}$ is a 1-simplex and $\mathrm{Rank}(e_s) = \mathrm{Rank}(Ax_{i+1}) = 0$, by Lemma A.4, we have that $\{e_s, Ax_{i+1}\} \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$. Namely, we have that $e_n \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_s, Ax_{i+1}\})$. In addition, it is clear that $e_n \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_s, e_t\})$. Hence, we have that $A^{-1}e_n \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{x_i, x_{i\pm1}\})$ (see Figure 2). Then, there exists an integer l such that $\mathrm{Rank}(x'_i) < R_\alpha$, where $x'_i = A^{-1}e_n + 2lx_i$. We have also that $x'_i \in \mathrm{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{x_i, x_{i\pm1}\})$ (see Figure 2). Let $\alpha' = \alpha \cup \{\{x'_i\}, \{x'_i, x_{i\pm1}\}\} \setminus \{x_i, \{x_i, x_{i\pm1}\}\}$. Then α' is homotopic to α (see Figure 2). Similar to the case R_α is odd, for all x_i with $\mathrm{Rank}(x_i) = R_\alpha$, applying the same operation, we conclude that $R_\beta < R_\alpha$, where β is a resulting loop which is homotopic to α.

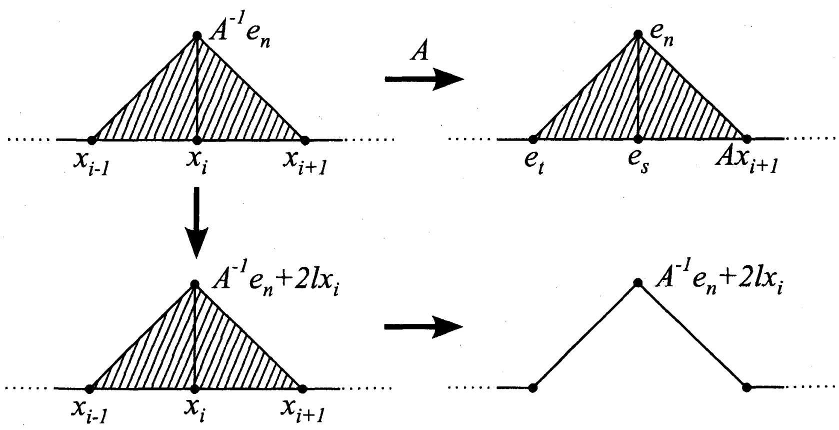

FIGURE 2. The case R_α is even.

Repeating this operation until $R_\alpha = 0$, we conclude that the loop α on $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is null homotopic. Thus, $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is simply connected.

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