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Some Integral Equations Related to A Branching Model

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分枝モデルに関連する積分方程式

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The present study has been carried out based upon the motivation to clarify the mathematical mechanism usually hidden in the background of biological systems, mainly by making use of mathematical models including stochastic models. In particular, we are interested in branching models, and we are very eager to characterize their fundamental properties by formulating an aspect of random branching in the viewpoint of functional equations. As an exciting case study, when a certain class of integral equations is given, then we are going to introduce in this article a method to construct its solution in a probabilistic manner by using branching models. To put things in a distinct way, this implies that the above-mentioned integral equations themselves are nothing but a characterization of the mathematical model that is constructed by a branching process arising in the description of biological systems. Paying attention to the tree structure that a proper branching process determines, we would introduce a space of marked trees and construct a "tree-based functional" in terms of non-commutative star-product. It is proven that if a certain tree-based ordinary multiplication functional satisfies the integrability condition, then there exists a proper weighted tree-based star-product functional such that the function determined by expectation of the functional gives a unique solution to the original deterministic integral equations.

本研究は確率モデルなどを含むいわゆる数理モデルを主な道具として用いることにより、生命系の背後に隠された数理メカニズムの解明を主な動機として始められた．とくに分枝モデルにおいて，ランダムに枝分かれしていく様相を方程式的に定式化して，その性質を特徴付けることを目指す．スタデイ・ケースとして，あるクラスに属する確定的な積分方程式が与えられたとき，その解を分枝モデルを用いて確率論的に構成する手法について紹介する．このことは視点を変えてみると，生命系の記述に現れる分枝過程を用いた数理モデルの数学的特徴付けとして，上述の積分方程式が出現するという構図になっている．適当な分枝過程の定める樹形構造に着目して，マーク付き樹木の空間を導入し，非可換なスター積に基づいた「樹状汎関数」を構成する．ある樹状通常積汎関数が可積分性の条件をみたせば，適当な重み付き樹状スター積汎関数が存在して，その期待値によって定められる関数は元の積分方程式の一意解であることが示される．

1 Introduction

Let $D_0 := \mathbb{R}^3 \setminus \{0\}$ and $\mathbb{R}_+ := [0, \infty)$. For every $\alpha, \beta \in \mathbb{C}^3$, we use the symbol $\alpha \cdot \beta$ for inner product, and we put $e_x := x/|x|$ for every $x \in D_0$. In this article we consider the deterministic nonlinear integral equation of the type

$$e^{\lambda t|x|^2} u(t,x) = u_0(x) + \frac{\lambda}{2}\int_0^t \mathrm{d}s\ e^{\lambda s|x|^2} \int p(s,x,y;u) n(x,y) \mathrm{d}y + \frac{\lambda}{2}\int_0^t e^{\lambda s|x|^2} f(s,x)\mathrm{d}s, \qquad \text{for} \quad \forall (t,x) \in \mathbb{R}_+ \times D_0, \tag{1}$$

where u is an unknown function : $\mathbb{R}_+ \times D_0 \to \mathbb{C}^3$, $\lambda > 0$, and $u_0 : D_0 \to \mathbb{C}^3$ is the initial data. Moreover, $f : \mathbb{R}_+ \times D_0 \to \mathbb{C}^3$ is a given function satisfying $f(t,x)/|x|^2 =: \tilde{f} \in L^1(\mathbb{R}_+)$ for each $x \in D_0$. The term p in (1) is given by

$$p(t,x,y;u) = u(t,y) \cdot e_x \{u(t,x-y) - e_x(u(t,x-y) \cdot e_x)\}. \tag{2}$$

Suppose that the integral kernel $n(x,y)$ is bounded and measurable with respect to $\mathrm{d}x \times \mathrm{d}y$. While, we consider a Markov kernel $K : D_0 \to D_0 \times D_0$, namely, for every $z \in D_0$, $K_z(\mathrm{d}x, \mathrm{d}y)$ lies in the space $\mathcal{P}(D_0 \times D_0)$ of all probability measures on $D_0 \times D_0$. When the kernel k is given by $k(x,y) = i|x|^{-2}n(x,y)$, then we define K_z as a Markov kernel satisfying that for any positive measurable function $h = h(x,y)$ on $D_0 \times D_0$,

$$\iint h(x,y)K_z(\mathrm{d}x, \mathrm{d}y) = \int h(x, z-x)k(x,z)\mathrm{d}x. \tag{3}$$

Moreover, we assume that for every measurable functions $f, g > 0$ on $\mathbb{R}^+$,

$$\int h(|z|)\nu(\mathrm{d}z) \int g(|x|)K_z(\mathrm{d}x, \mathrm{d}y) = \int g(|z|)\nu(\mathrm{d}z) \int h(|y|)K_z(\mathrm{d}x, \mathrm{d}y) \tag{4}$$

holds, where the measure ν is given by $\nu(dz) = |z|^{-3}\mathrm{d}z$.

2 Main results

In this section we shall state the main results on the existence and uniqueness of solutions to the nonlinear integral equation (1). That is to say, we derive a probabilistic representation of the solutions to (1) by employing the star-product functional. As a matter of fact, the solution $u(t,x)$ is nothing but a probabilistic solution. Let

$$M_\bigstar^{\langle u_0, f\rangle}(\omega) = \prod \bigstar_{[x_{\bar{m}}]} \Xi^{m_1}_{m_2 . m_3}[u_0, f](\omega), \tag{5}$$

be a random quantity in terms of tree-based star-product functional with weight functions (u_0, f). On the other hand, $M_*^{\langle U,F\rangle}(\omega)$ denotes the associated $*$-product functional with weight (U, F). In fact, in a similar manner as (5) we can construct a (U,F)-weighted tree-based $*$-product functional $M_*^{\langle U,F\rangle}(\omega)$. This quantity is indexed by the nodes (x_m) of a binary tree. We suppose that U (resp. F) is a non-negative measurable function on D_0 (resp. $\mathbb{R}_+ \times D_0$) respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each x. Indeed, ordinary multiplication $*$ is taken in construction of the $*$-product functional, instead of the star-product $\bigstar$ in (5).

THEOREM 1. *Suppose that $|u_0(x)| \leqslant U(x)$ for $\forall x$ and $|\tilde{f}(t,x)| \leqslant F(t,x)$ for $\forall t, x$, and also that for some $T > 0$ ($T >> 1$ sufficiently large),*

$$E_{T,x}[M_*^{\langle U,F\rangle}(\omega)] < \infty, \quad \text{a.e.} - x \tag{6}$$

Then there exists a (u_0, f)-weighted tree-based star $\bigstar$-product functional $M_\bigstar^{\langle u_0,f\rangle}(\omega)$, indexed by a set of node labels accordingly to the tree structure which a binary critical branching process $Z^{K_x}(t)$ determines. Furthermore, the function

$$u(t,x) = E_{t,x}[M_\bigstar^{\langle u_0,f\rangle}(\omega)] \tag{7}$$

gives a unique solution to the integral equation (1). Here $E_{t,x}$ denotes the expectation with respect to a probability measure $P_{t,x}$ as the time-reversed law of $Z^{K_x}(t)$.

3 Construction of branching model and tree-like structure

In this section we consider a continuous time binary critical branching process $Z^{K_x}(t)$ on D_0, whose branching rate is given by a parameter $\lambda|x|^2$, whose branching mechanism is binary with equi-probability (see Figure 1), and whose descendant branching particle behavior (or distribution) is determined by the kernel K_x. Next, taking notice of the tree structure by the process $Z^{K_x}(t)$, we denote the space of marked trees

$$\omega = (t, (t_m), (x_m), (\eta_m), m \in \mathcal{V}) \tag{8}$$

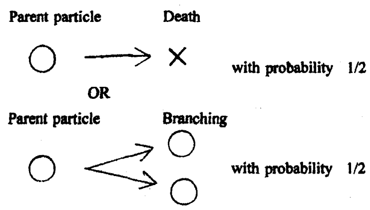

図 1 Binary Branching

by Ω. Furthermore, the time-reversed law of $Z^{K_x}(t)$ on Ω is written as $P_{t,x}$. Here t denotes the birth time of common ancestor, and the particle x_m dies when $\eta_m = 0$, while it generates two descendants x_{m1}, x_{m2} when $\eta_m = 1$. On the other hand,

$$\mathcal{V} = \bigcup_{\ell \geq 0} \{1,2\}^{\ell}$$

is a set of all labels, namely, finite sequences of symbols with length ℓ. For $\omega \in \Omega$ we denote by $\mathcal{N}(\omega)$ the totality of nodes being branching points of tree, and let $N_+(\omega)$ be the set of all nodes m being a member of $\mathcal{V} \setminus \mathcal{N}(\omega)$, whose direct predecessor lies in $\mathcal{N}(\omega)$ and which satisfies the condition $t_m(\omega) > 0$, and let $N_-(\omega)$ be the same set as described above, but satisfying $t_m(\omega) \leqslant 0$. Finally we put

$$N(\omega) = N_+(\omega) \cup N_-(\omega). \tag{9}$$

4 Star ★-product functional and ∗-product functional

Let us now introduce a tree-based star-product functional. First of all, we denote by the symbol $\mathrm{Proj}^z(\cdot)$ a projection of the objective element onto its orthogonal part of the z component in $\mathbb{C}^3$, and we define a ★-product of β, γ for $z \in D_0$ as

$$\beta \bigstar_{[z]} \gamma = -i(\beta \cdot e_z)\mathrm{Proj}^z(\gamma). \tag{10}$$

We shall define $\Theta^m(\omega)$ for each $\omega \in \Omega$ realized as follows. When $m \in N_+(\omega)$, then $\Theta^m(\omega) = \tilde{f}(t_m(\omega), x_m(\omega))$, while $\Theta^m(\omega) = u_0(x_m(\omega))$ if $m \in N_-(\omega)$. Then we define

$$\Xi^{m_1}_{m_2.m_3}(\omega) \equiv \Xi^{m_1}_{m_2,m_3}[u_0, f](\omega) := \Theta^{m_2}(\omega) \bigstar_{[x_{m_1}]} \Theta^{m_3}(\omega), \tag{11}$$

where as for the product order in the star-product ★, when we write $m \prec m'$ lexicographically with respect to the natural order $\prec$, the term Θ^m labelled by m necessarily occupies the left-hand side and the other $\Theta^{m'}$ labelled by m' occupies the right-hand side. And besides, we write

$$\Xi^{\emptyset}_{m,\emptyset}(\omega) \equiv \Xi^{\emptyset}_{m,\emptyset}[u_0, f](\omega) := \Theta^m(\omega), \tag{12}$$

when $m \in \mathcal{V}$ is a label of single terminal point.

Under these circumstances, we consider a random quantity which obtained by executing the star-product ★ inductively at each node in $\mathcal{N}(\omega)$, and we call it a tree-based ★-product functional, and we express it symbolically as

$$M^{\langle u_0, f \rangle}_{\bigstar}(\omega) = \prod \bigstar_{[x_{\tilde{m}}]} \Xi^{m_1}_{m_2.m_3}[u_0, f](\omega), \tag{13}$$

where $m_1 \in \mathcal{N}(\omega)$ and $m_2, m_3 \in N(\omega)$, and by the symbol $\prod \bigstar$ (as a product relative to the star-product) we mean that the star-products ★'s should be succeedingly executed in a lexicographical manner with respect to $x_{\tilde{m}}$ such that $\tilde{m} \in \mathcal{N}(\omega) \cap \{|\tilde{m}| = \ell - 1\}$ when $|m_1| = \ell$.

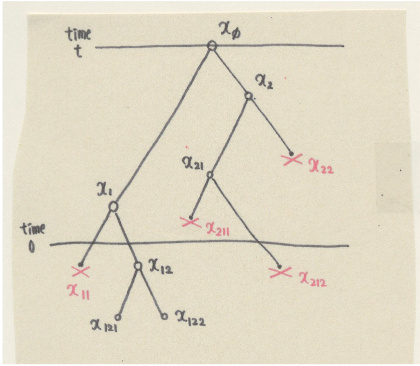

図 2　Example of a Realized Tree ω_1

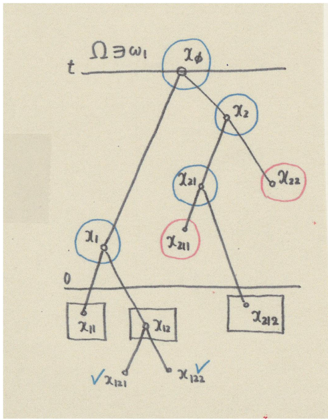

図 3　Classification of Nodes for ω_1

EXAMPLE 2. Suppose that a tree structure ω_1 ($\in \Omega$) has been realized here (see Figure 2). Clearly we have $\mathcal{N}(\omega_1) = \{\phi, 1, 2, 21\}$, $N_+(\omega_1) = \{22, 211\}$, and $N_-(\omega_1) = \{11, 12, 212\}$. However, for this $\omega_1 \in \Omega$, unfortunately labels $\{121\}$, $\{122\}$ are not included in any $\mathcal{N}(\omega_1)$, $N_+(\omega_1)$, nor $N_-(\omega_1)$. As a matter of fact, we can construct

$$\Xi^1_{11,12}(\omega_1) = \Theta^{11}(\omega_1)\bigstar_{[x_1]}\Theta^{12}(\omega_1)$$

by a star-product $u_0(x_{11}(\omega_1))\bigstar_{[X_1]}u_0(x_{12}(\omega_1))$ in accordance with the rule, because both $m_1 = 11$ and $m_2 = 12$ lie in $N_-(\omega)$. As to the node x_{21}, it goes similarly. Hence $\Xi^{21}_{211,212}(\omega_1)$ is given by $\tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1))\bigstar_{[x_{21}]}u_0(x_{212}(\omega_1))$, see Figure 3. Consequently, we obtain finally an explicit representation of the star-product functional

$$M_{\bigstar}^{\langle u_0, f\rangle}(\omega_1) = \left(u_0(x_{11})\bigstar_{[x_1]}u_0(x_{12})\right)\bigstar_{[x_\phi]} \left\{\left(\tilde{f}(t_{211}, x_{211})\bigstar_{[x_{21}]}u_0(x_{212})\right)\bigstar_{[x_2]}\tilde{f}(t_{22}, x_{22})\right\}. \tag{14}$$

□

5 A sketch of the proof of existence result

In this section we shall first construct a (U, F)-weighted tree-based $*$-product functional $M_*^{\langle U,F\rangle}(\omega)$, which is indexed by the nodes (x_m) of a binary tree. Moreover, in construction of the functional, the product is taken as ordinary multiplication $*$ instead of the star-product $\bigstar$. We need the following technical lemma.

LEMMA 3. *For* $0 \leqslant t \leqslant T$ *and* $x \in D_0$, *the function* $V(t,x) = E_{t,x}[M_*^{\langle U,F\rangle}(\omega)]$ *satisfies*

$$e^{\lambda t|x|^2}V(t,x) = U(x) + \int_0^t \mathrm{d}s \frac{\lambda|x|^2}{2} e^{\lambda s|x|^2}\Big\{ F(s,x) + \int V(s,y)V(s,z)K_x(\mathrm{d}y,\mathrm{d}z)\Big\}. \tag{15}$$

Proof of lemma 3. By making use of the conditional expectation we can get

$$\begin{aligned} V(t,x) &= E_{t,x}[M_*^{\langle U,F\rangle}(\omega)] \\ &= E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi \leqslant 0] + E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi > 0] \\ &= E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi \leqslant 0] + E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi > 0,\ \eta_\phi = 0] \\ &\quad + E_{t,x}[M_*^{\langle U,F\rangle}(\omega),\ t_\phi > 0,\ \eta_\phi = 1]. \end{aligned} \tag{16}$$

As to the first term in (16), the $*$-product functional is allowed to have a simple representation:

$$\begin{aligned} E_{t,x}[M_*^{\langle U,F\rangle}, t_\phi \leqslant 0] &= E_{t,x}[M_*^{\langle U,F\rangle}\cdot 1_{\{t_\phi \leqslant 0\}}] = U(x)\cdot P_{t,x}(t_\phi \leqslant 0) \\ &= U(x)\int_t^\infty f_T(s)\mathrm{d}s = U(x)\int_t^\infty \lambda|x|^2 e^{-\lambda s|x|^2}\mathrm{d}s \\ &= U(x)\cdot \exp\{-\lambda t|x|^2\}. \end{aligned} \tag{17}$$

As to the third term, the Markov property guarantees that the lower tree structure below the first generation branching node point (or location) x_1 is independent of that below the location x_2 with realized $\omega \in \Omega$, hence a $*$-product functional branched after time s is also probabilistically independent of the other $*$-product functional branched after time s. Therefore, an easy computation provides with

$$E_{t,x}[M_*^{\langle U,F\rangle}, t_\phi > 0, \eta_\phi = 1] = \frac{1}{2}\int_0^t \mathrm{d}s\lambda|x|^2 e^{-\lambda|x|^2(t-s)} \times \iint E_{s,x_1}[M_*]\cdot E_{s,x_2}[M_*]K_x(\mathrm{d}x_1,\mathrm{d}x_2).$$

Note that as for the second term, it goes almost similarly. Finally, summing up we obtain

$$\begin{aligned} V(t,x) &= E_{t,x}[M_*^{\langle U,F\rangle}(\omega)] \\ &= U(x)r^{-\lambda t|x|^2} + \int_0^t \frac{\lambda|x|^2}{2}e^{-\lambda|x|^2(t-s)}F(s,x)\mathrm{d}s \\ &\quad + \int_0^t \frac{\lambda|x|^2}{2}e^{-\lambda|x|^2(t-s)}\iint V(s,y)V(s,z)K_x(\mathrm{d}y,\mathrm{d}z)\mathrm{d}s. \end{aligned} \tag{18}$$

This completes the proof. □

Next notice that

$$E_{t,x}[M_*^{\langle U,F\rangle}(\omega)] < \infty \tag{19}$$

holds for $\forall t \in [0,T]$ and $x \in E_c$, where a measurable set E_c denotes the totality of all the elements x in D_0 such that $E_{T,x}[M_*^{\langle U,F\rangle}] < \infty$ holds for a.e.-x. Another important aspect for the proof consists in establishment of the M_*-control inequality.

LEMMA 4. (M_*-control inequality) *The following inequality*

$$|M_\bigstar^{\langle u_0,f\rangle}(\omega)| \leqslant M_*^{\langle U,F\rangle}(\omega) \tag{20}$$

holds $P_{t,x}$-a.s.

In fact, the M_*-control inequality yields immediately from a simple inequality

$$|w \bigstar_{[x]} v| \leqslant |w| \cdot |v| \quad \text{for every} \quad w, v \in \mathbb{C}^3 \quad \text{and every} \quad x \in D_0.$$

If we define

$$u(t,x) := \begin{cases} E_{t,x}[M_\bigstar^{\langle u_0, f\rangle}(\omega)], & \text{on} \quad E_c, \\ 0, & \text{otherwise}, \end{cases}$$

then $u(t,x)$ is well-defined on the whole space D_0 under the assumptions of the main theorem (Theorem 1). Moreover, it follows from the M_*-control inequality (20) that

$$|u(t,x)| \leqslant V(t,x) \quad \text{on} \quad [0,T] \times D_0. \tag{21}$$

On this account, it is easy to see from (15) that

$$\int_0^T \mathrm{d}s \int |u(s,y)| \cdot |u(s,z)| K_x(\mathrm{d}y, \mathrm{d}z) < \infty \qquad \text{for} \quad x \in E_c. \tag{22}$$

Hence, taking (22) into consideration we define the space $\mathcal{D}$ of solutions to (1) as

$$\begin{aligned} \mathcal{D} := \{\varphi : \mathbb{R}_+ \times D_0 \to \mathbb{C}^3;\ \varphi \quad &\text{is continuous in} \quad t \quad \text{and measurable such that} \\ \int_0^\infty \mathrm{d}s \int e^{\lambda |x|^2 s} |\varphi(s,y)| \cdot |\varphi(s,z)| K_x(\mathrm{d}y, \mathrm{d}z) &< \infty \quad \text{holds a.e.} - x\}. \end{aligned} \tag{23}$$

By employing the Markov property with respect to time t_ϕ and by a similar technique as in the proof of Lemma 3, we may proceed in rewriting and calculating the expectation: for $\forall t > 0$ and $x \in E_c$

$$\begin{aligned} u(t,x) &= E_{t,x}[M_\bigstar^{\langle u_0, f\rangle}(\omega)] \\ &= e^{-\lambda t|x|^2} u_0(x) + \int_0^t \mathrm{d}s\ \lambda |x|^2 e^{-\lambda(t-s)|x|^2} \times \\ &\times \frac{1}{2} \left\{ \tilde{f}(s,x) + \iint E_{s,x_1}[M_\bigstar] \bigstar_{[x]} E_{s,x_2}[M_\bigstar] K_x(\mathrm{d}x_1, \mathrm{d}x_2) \right\}. \end{aligned} \tag{24}$$

Furthermore, we may apply the integral equality (3) in the assumption on the Markov kernel for (24) to obtain

$$\begin{aligned} E_{t,x}[M_\bigstar^{\langle u_0, f\rangle}(\omega)] = e^{-\lambda t|x|^2} &\left\{ u_0(x) + \frac{\lambda}{2} \int_0^t e^{\lambda s|x|^2} f(s,x) \mathrm{d}s \right. \\ &\left. + \frac{\lambda}{2} \int_0^t \mathrm{d}s \int e^{\lambda s|x|^2} p(s,x,y;u) n(x,y) \mathrm{d}y \right\}. \end{aligned} \tag{25}$$

Finally we attain that $u(t,x) = E_{t,x}[M_\bigstar^{\langle u_0, f\rangle}(\omega)]$ satisfies the integral equation (1), and this $u(t,x)$ is a solution lying in the space $\mathcal{D}$. This completes the proof of the existence.

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