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Near model completeness of generic structures

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A theory T is said to be nearly model complete, if every formula is equivalent in T to a Boolean combination of Σ_1-formulas. This notion is a generalization of model completeness. It is known that

Fact Hrushovski's strongly minimal structure is nearly model complete.

On the other hand, Baldwin and Shelah [4] proved the following:

Theorem Shelah-Spencer's random graph is nearly model complete.

The proof is a little complicated. Pourmahdian [7] gave a new proof for this theorem, by adding countable predicates to the language. Both of Hrushovski's strongly minimal structure and Shelah-Spencer's random graph are well-known examples of generic structures.

In this short note, we give a more direct proof for a theorem of Baldwin and Shelah, and moreover generalize both of the above fact and theorem:

Theorem Let M be a generic structure. If $\mathrm{Th}(M)$ is ultra-homogeneous over finite closed sets, then it is nearly model complete.

1 Generic structures

It is assumed that the reader is familiar with the basics of generic structures. In particular, this paper was influenced by papers of Baldwin-Shi [3] and Wagner [8].

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Let L be a language which consists of finite relations with irreflexivity and symmetricity. Let $A, B, C, \ldots$ be L-structures or (hyper-)graphs. A *predimension* $\delta(A)$ of a finite structure A is defined as follows:

$$\delta(A) = |A| - \sum_{R \in L} \alpha_R |R^A|,$$

where $\alpha_R \in (0, 1]$ for each $R \in L$. We denote $\delta(B/A) = \delta(B \cup A) - \delta(A)$.

For finite $A \subset B$, A is said to be *closed* in B (in symbol, $A \leq B$), if $\delta(X/A) \geq 0$ for any $X \subset B - A$. When A, B are not necessarily finite, $A \leq B$ is defined by $A \cap X \leq X$ for any finite $X \subset B$.

For $A \subset B$, there is a smallest set $C \leq B$ containing A. Such a C is denoted by $\mathrm{cl}_B(A)$.

Let $\mathbf{K}^*$ be the class of finite L-structures A with $\delta(A') \geq 0$ for all $A' \subset A$.

Definition 1.1 Let $\mathbf{K} \subset \mathbf{K}^*$. Then a countable L-structure M is said to be $(\mathbf{K}, \leq)$*-generic*, if it satisfies the following:

1. $A \in \mathbf{K}$ for any finite $A \subset M$;
2. M is *rich*, i.e., if $A \leq B \in \mathbf{K}$ and $A \leq M$, then there is a $B'(\cong_A B)$ with $B' \leq M$;
3. M has *finite closures*, i.e., $\mathrm{cl}_M(A)$ is finite for any finite $A \subset M$.

Clearly a generic structure M has finite closures, but any model of $\mathrm{Th}(M)$ does not always have finite closures.

Definition 1.2 Let M be a generic structure. Then we say that $\mathrm{Th}(M)$ has *finite closures*, if any model has finite closures.

By the back-and-forth method, if M, N are $(\mathbf{K}, \leq)$-generic then $M \cong N$. Also, we can see that a generic structure M is *ultra-homogeneous over finite closed sets*, i.e., if A, B are finite with $A \cong B$ and $A, B \leq M$, then $\mathrm{tp}(A) = \mathrm{tp}(B)$.

Definition 1.3 Let M be a generic structure. Then we say that $\mathrm{Th}(M)$ is *ultra-homogeneous over finite closed sets*, if any model is ultra-homogeneous over finite closed sets.

Table 1: Examples of generic structures

	ultra-homogeneous	saturated
Hrushovski's strongly minimal structure	○	○
Hrushovski's stable pseudoplane	○	○
Baldwin's projective plane	○	○
Spencer-Shelah's random graph	○	×

Note 1.4 It is easily checked that M is saturated if and only if $\mathrm{Th}(M)$ has finite closures and is ultra-homogeneous over finite closed sets.

The following are well-known examples of generic structures.

Example 1.5

1. (Hrushovski [5]) A new strongly minimal structure
2. (Hrushovski [6]) An ω-categorical stable pseudoplane
3. (Baldwin [1]) An $\aleph_1$-categorical projective plane
4. (Baldwin-Shelah [4]) Spencer-Shelah's random graph

For examples of generic structures, almost all theories are ultra-homogeneous over finite closed sets: Each of 1,2 and 3 is saturated, and hence, by Note 1.4, the theory is ultra-homogeneous over finite closed sets. 4 is not saturated, because the theory does not have finite closures, however it can be seen that the theory is ultra-homogeneous over finite closed set. (See Table 1)

2 Nearly model complete theories

Definition 2.1 Let T be a theory.

1. T is said to be *model complete*, if whenever $M, N \models T$ and $M \subset N$, then $M \prec N$.

Table 2: Examples of generic structures

	model complete	nearly model complete
Hrushovski's strongly minimal structure	○	○
Hrushovski's stable pseudoplane	?	○
Baldwin's projective plane	?	○
Spencer-Shelah's random graph	×	○

2. It is known that T is model complete if and only if every formula is equivalent in T to some Σ_1-formula.

3. T is said to be *nearly model complete*, if every formula is equivalent in T to a Boolean combination of Σ_1-formulas.

For model completeness, it is known that 1 of Example 1.5 is model complete ([2]) but 4 of Example 1.5 is not model complete ([4]). However, it is not known whether 2 and 3 of Example 1.5 is model complete or not. On the other hand, for near model completeness, it is proved that 1 and 4 of Example 1.5 are nearly model complete. (See Table 2)

Fact 2.2 Hrushovski's strongly minimal structure is nearly model complete.

Theorem 2.3 (Baldwin-Shelah [4], Pourmahdian [7]) Shelah-Spencer's random graph is nearly model complete.

Baldwin and Shelah prove that the theory of a semi-generic structure is nearly model complete. As a corollary, it is obtained that Shelah-Spencer's random graph is nearly model complete. After that, Pourmahdian gives a new proof for this theorem. In both proofs, the notion of a semi-generic structure is used to get near model completeness of Shelah-Spencer's random graph. Then we want to give a more direct proof for a theorem of Baldwin and Shelah, and moreover to generalize Fact 2.2 and Theorem 2.3:

Theorem 2.4 Let M be a generic structure. If $\mathrm{Th}(M)$ is ultra-homogeneous over finite closed sets, then it is nearly model complete.

Proof. Let M $\mathcal{M}$ be a big model. We write $\mathrm{cl}(A) = \mathrm{cl}_{\mathcal{M}}(A)$. For $n \in \omega$, $B \leq_n C$ is defined by $\delta(X/B) \geq 0$ for any $X \subset C - B$ with $|X| \leq n$. We write

$$\mathrm{cl}^n(A) = \bigcap\{B : A \subset B \leq_n \mathcal{M}\}.$$

Note that $\mathrm{cl}(A) = \bigcup_n \mathrm{cl}^n(A)$, and moreover that if A is finite then so is $\mathrm{cl}^n(A)$.

Take any finite $A \subset \mathcal{M}$. It is enough to show that there is some set Σ of a Boolean combination of Σ_1-formulas with $\Sigma \vdash \mathrm{tp}(A)$. Let $B = \mathrm{cl}(A)$. For each $n \in \omega$, let $B_n = \mathrm{cl}^n(A)$. Note that each B_n is finite and $B = \bigcup_n B_n$. Let $\Sigma(X)$ be

$$\{(\exists Y_n)(XY_n \cong AB_n) : n \in \omega\}$$
$$\cup\{\neg(\exists Y_n)(\exists Z)(XY_nZ \cong AB_nC) : B_n \subset C \in \mathbf{K}, B_n \not\leq_n C, n \in \omega\}.$$

Since $A \models \Sigma$, Σ is consistent. Take any $A' \models \Sigma$. Then, for each n, there is a $B'_n \subset \mathcal{M}$ with $A'B'_n \cong AB_n$. By compactness, we can assume that $B'_nB'_{n+1} \cong B_nB_{n+1}$ for any $n \in \omega$. Let $B' = \bigcup_n B'_n$. Clearly $B' \cong B$. Since $A' \models \Sigma$, we have $B'_n \leq_n \mathcal{M}$ for each n, and hence $B' \leq \mathcal{M}$. By ultra-homogeneity, we have $\mathrm{tp}(B') = \mathrm{tp}(B)$. Hence we have $\mathrm{tp}(A') = \mathrm{tp}(A)$.

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