数理解析研究所講究録
第 1938 巻 2015 年 26-32

A brief survey of recent results on NTP_2 and dense codense predicate expansions

Hyeungjoon Kim

Department of Mathematics, Yonsei University

1 Introduction

The aim of this article is to review/summarize some of the recent results on NTP_2 theories and dense codense predicate expansions due to Chernikov [5], Berenstein and Vassiliev [4], [3], and Berenstein and H. Kim [2]. We do not include all the proofs of these results, and the interested reader is referred to the aforementioned papers for the complete proofs. However, we will sketch the proof of Theorem 3.14, a main result from [2]. This article is organized as follows: In Section 2, we review the notions of burden, NTP_2 and $\kappa_{inp}^{n}(T)$ as well as some of the key facts about them due to several authors (notably the "submultiplicativity of burden" by Chernikov [5]). In Section 3, we review the notion of dense codense predicate expansions and some of the key results due to [4], [3], [2]. We will conclude by sketching the proof of Theorem 3.14, a main result from [2].

2 NTP_2, burden and κ_{inp}^{n}

Throughout this section, we work inside a fixed, sufficiently saturated model of an arbitrary theory T.

The following definition is due to Adler [1].

Definition 2.1. For a partial type $p(\bar{x})$, an *inp-pattern in* $p(\bar{x})$ is a set of formulas $\{\varphi_i(\bar{x},\bar{a}_{i,j}) \mid i<\kappa,\ j<\omega\}$ where κ is a cardinal, satisfying the following:

1. For each $i<\kappa$, $\{\varphi_i(\bar{x},\bar{a}_{i,j}) \mid j<\omega\}$ is k_i-inconsistent for some integer $k_i \geq 2$.

2. For every function $f\colon \kappa \to \omega$, $p(\bar{x}) \cup \{\varphi_i(\bar{x},\bar{a}_{i,f(i)}) \mid i<\kappa\}$ is consistent.

The cardinal κ is called the *depth* of the inp-pattern. The supremum of depths of all inp-patterns in $p(\bar{x})$ is called the *burden* of $p(\bar{x})$, denoted by $\mathrm{bdn}(p(\bar{x}))$. $\mathrm{bdn}(\mathrm{tp}(\bar{a}/A))$ is abbreviated as $\mathrm{bdn}(\bar{a}/A)$.

Remark 2.2. *The notion of inp-patterns (where "inp" stands for "independent partitions") was introduced by Shelah in [7] where he considered inp-patterns in trivial type $\bar{x}=\bar{x}$ in order to define certain cardinals κ_{inp}^{n} associated with a given theory. (We shall recall the definition of κ_{inp}^{n} later.)*

Observation 2.3. The following is clear from the definition of burden.

1. $\mathrm{bdn}(p(\bar{x}))=0$ iff $p(\bar{x})$ is an algebraic type.

2. For any inp-pattern in $\mathrm{tp}(\bar{a}/\bar{b})$, there exists an inp-pattern of the same depth in $\mathrm{tp}(\bar{a})$.

3. For any inp-pattern in $\mathrm{tp}(\bar{a})$ and any tuple $\bar{b}$, there exists an inp-pattern of the same depth in $\mathrm{tp}(\bar{a}\bar{b})$.

4. $\mathrm{bdn}(\bar{a}/\bar{b}) \leq \mathrm{bdn}(\bar{a}) \leq \mathrm{bdn}(\bar{a}\bar{b})$ for any tuples $\bar{a}$ and $\bar{b}$.

It has been observed by many researchers (for example, [8], [5], [1]) that the parameters $\{\bar{a}_{i,j}\}_{i,j}$ in an inp-pattern may be assumed to be 'indiscernible' in a certain sense. More precisely:

Proposition 2.4 ([8], [1], [5]). If there exists an inp-pattern $\{\varphi_i(\bar{x}, \bar{a}_{i,j}) \mid i < \kappa,\ j < \omega\}$ in a type $p(\bar{x})$, we may assume that $\{\bar{a}_{i,j} \mid i < \kappa,\ j < \omega\}$ is a mutually indiscernible array, that is, for each $i < \kappa$, $\{\bar{a}_{i,j} \mid j < \omega\}$ is an indiscernible sequence over $\{\bar{a}_{k,j} \mid k \neq i,\ j < \omega\}$.

Proof. A compactness argument together with repeated applications of Ramsey's theorem. See [8] or [5] for details. □

Theorem 2.5 (Chernikov [5]). *If there exists an inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\mathrm{tp}(\bar{a}\bar{b})$, then either there exists an inp-pattern of depth κ_1 in $\mathrm{tp}(\bar{a})$ or there exists an inp-pattern of depth κ_2 in $\mathrm{tp}(\bar{b}/\bar{a})$.*

Corollary 2.6 ([5], the "submultiplicativity" of burden). *For any tuples $\bar{a}_1, \cdots, \bar{a}_n$ and any cardinals $\kappa_1, \cdots, \kappa_n$,*

$$\mathrm{bdn}(\bar{a}_i) < \kappa_i \ \textit{for each } i \ \Rightarrow \ \mathrm{bdn}(\bar{a}_1 \cdots \bar{a}_n) < \prod_{i=1}^{n} \kappa_i.$$

Proof. We may assume $n = 2$ and that all the κ_i's are nonzero cardinals. We will prove the contrapositive. Assume $\mathrm{bdn}(\bar{a}_1\bar{a}_2) = \kappa_1 \times \kappa_2$. If $\kappa_1 \times \kappa_2$ is a *successor* cardinal, then clearly there exists an inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\mathrm{tp}(\bar{a}_1\bar{a}_2)$, so either $\mathrm{bdn}(\bar{a}_1) \geq \kappa_1$ or $\mathrm{bdn}(\bar{a}_2) \geq \kappa_2$ by Theorem 2.5 and Observation 2.3(2), and we are done. On the other hand, if $\kappa_1 \times \kappa_2$ is a *limit* cardinal, then Theorem 2.5 implies that either $\mathrm{bdn}(\bar{a}_1) \geq \kappa_1 \times \kappa_2$ or $\mathrm{bdn}(\bar{a}_2) \geq \kappa_1 \times \kappa_2$. Hence, either $\mathrm{bdn}(\bar{a}_1) \geq \kappa_1$ or $\mathrm{bdn}(\bar{a}_2) \geq \kappa_2$. □

Next, we recall the notion of κ^n_{inp} introduced by Shelah [7, Section III.7].

Definition 2.7. For any theory T and any integer $n \geq 1$, $\kappa^n_{inp}(T)$ denotes the least cardinal τ such that there does not exist any inp-pattern of depth τ in the type $\{\bar{x} = \bar{x}\}$ where $\bar{x}$ has arity n. And $\kappa^n_{inp}(T) := \infty$ if such τ does not exist.

Remark 2.8. *We use the convention that $\kappa < \infty$ for every cardinal κ.*

Observation 2.9. The following is clear:

1. $n < \kappa^n_{inp}(T)$ (due to the equality symbol in every language).

2. $n \leq m \Rightarrow \kappa^n_{inp}(T) \leq \kappa^m_{inp}(T)$.

3. $\sup_{1 \leq n < \omega} \kappa^n_{inp}(T) \geq \aleph_0$.

Another important consequence of Theorem 2.5 is the following:

Theorem 2.10 (Chernikov [5]). *For any theory T, either $\kappa^n_{inp}(T) < \aleph_0$ for all n, or $\kappa^1_{inp}(T) = \kappa^n_{inp}(T)$ for all n.*

Proof. Suppose that $\kappa^n_{inp}(T) \geq \aleph_0$ for some n. Let N be the least such n.

Claim. $N = 1$.

Proof of Claim. Suppose $N > 1$. Then $C := \kappa^{N-1}_{inp}(T)$ is a natural number, and $\kappa^i_{inp}(T) \leq C$ for all $1 \leq i < N$. Since $C \times C < \kappa^N_{inp}(T)$, there exists an inp-pattern $\{\varphi_i(\bar{x}, \bar{a}_{i,j}) \mid i < C \times C,\ j < \omega\}$ in $\bar{x} = \bar{x}$ where $|\bar{x}| = N$. By Proposition 2.4, we may assume $\{\bar{a}_{i,j}\}_{i,j}$ is a mutually indiscernible array. Hence, if $\bar{b}$ is any tuple realizing $\bigwedge_{i<C\times C} \varphi_i(\bar{x}, \bar{a}_{i,0})$, then $\{\varphi_i(\bar{x}, \bar{a}_{i,j}) \mid i < C \times C,\ j < \omega\}$ is an inp-pattern in $\mathrm{tp}(\bar{b})$. Let $\bar{b}$ be in the form $\bar{c}\bar{d}$ where $\bar{c}$ and $\bar{d}$ have nonzero arities. Then, by Theorem 2.5, either there exists an inp-pattern of depth C in $\operatorname{tp}(\bar{c})$, or there exists an inp-pattern of depth C in $\bar{d}$. This contradicts that $\kappa_{inp}^i(T) \leq C$ for all $1 \leq i < N$. This completes the proof of Claim.

Now we are ready to prove that $\kappa_{inp}^1(T) = \kappa_{inp}^n(T)$ for all n. In the case $\kappa_{inp}^1(T) = \infty$, the assertion is clear. So assume $\kappa_{inp}^1(T) < \infty$. Then $\kappa_{inp}^1(T)$ is an infinite cardinal by Claim above. Let $\tau := \kappa_{inp}^1(T)$. Suppose $\kappa_{inp}^n(T) \neq \tau$ for some n. Let N be the least such n. Then $\kappa_{inp}^i(T) = \tau$ for all $1 \leq i < N$. But, since $\tau = \tau \times \tau < \kappa_{inp}^N(T)$, there exists an inp-pattern $\{\varphi_i(\bar{x}, \bar{a}_{i,j}) \mid i < \tau \times \tau,\ j < \omega\}$ in $\bar{x} = \bar{x}$ where $|\bar{x}| = N$. Now we may repeat the same argument in the latter part of the proof of Claim, and derive a contradiction. □

Definition 2.11 ([1]). A theory T is said to be *strong* if all inp-patterns in $\bar{x} = \bar{x}$ have finite depths, for all arities $|\bar{x}|$.

Remark 2.12. *Clearly, a theory T is strong iff $\kappa_{inp}^n(T) \leq \aleph_0$ for all $n < \omega$. In fact, by Theorem 2.10, T is strong iff $\kappa_{inp}^1(T) \leq \aleph_0$.*

Next, we recall the definition of NTP_2 theories.

Definition 2.13. A theory T is said to have k-TP_2 (where $k \geq 2$ is an integer) if there exists a formula $\varphi(\bar{x}, \bar{y})$ witnessing it, i.e., there exists a set of tuples $\{\bar{a}_{i,j} \mid i, j < \omega\}$ such that:

1. $\{\varphi(\bar{x}, \bar{a}_{i,j}) \mid j < \omega\}$ is k-inconsistent for every $i < \omega$,
2. $\{\varphi(\bar{x}, \bar{a}_{i,f(i)}) \mid i < \omega\}$ is consistent for every function $f\colon \omega \to \omega$.

We say T has NTP_2 if it does not have k-TP_2 for any $k \geq 2$.

Remark 2.14. *The notion of (2-)TP_2 (called the 'tree property of the second kind') was introduced by Shelah [7].*

Observation 2.15. A theory T has k-TP_2 for some k iff $\kappa_{inp}^n(T) = \infty$ for all n. In particular, every strong theory has NTP_2.

Proof. ($\Rightarrow$) is by compactness and Theorem 2.10, and ($\Leftarrow$) is by the pigeon hole principle. □

Suppose that there exists $\{\varphi(\bar{x}, \bar{a}_{i,j}) \mid i, j < \omega\}$ witnessing k-TP_2. Since this is a form of inp-pattern, we know that $\{\bar{a}_{i,j}\}_{i,j<\omega}$ can be chosen to be mutually indiscernible (by Proposition 2.4). In fact, we can even require that the 'rows' of $\{\bar{a}_{i,j}\}_{i,j<\omega}$ form an indiscernible sequence. More precisely: if we let $\bar{\beta}_i := \{\bar{a}_{i,j} \mid j < \omega\}$ for each $i < \omega$, then $\{\bar{\beta}_i \mid i < \omega\}$ is an indiscernible sequence. This can be proved by a routine application of Ramsey's theorem together with compactness. Alternatively, this can be derived from a more general 'tree-indiscernibility' theorem. The interested reader is referred to [6, Lemma 5.6] for details.

We call $\{\bar{a}_{i,j}\}_{i,j<\omega}$ with the indiscernibility condition described above an *indiscernible array*.

Proposition 2.16. If a formula $\varphi(\bar{x}, \bar{y})$ witnesses k-TP_2 for some $k \geq 2$, then some finite conjunction $\psi(\bar{x}, \bar{y}_1 \cdots \bar{y}_N) := \bigwedge_{i=1}^N \varphi(\bar{x}, \bar{y}_i)$ witnesses 2-TP_2. Hence, a theory T has NTP_2 iff it does not have 2-TP_2.

Proof. Use the fact that such $\varphi(\bar{x}, \bar{y})$ can witness k-TP_2 with an indiscernible array. See [6] or [1] for details. □

Remark 2.17. *Hence, there is no ambiguity in saying that a theory T has TP_2 (without specifying 'k' in k-TP_2). However, by convention, TP_2 usually refers to 2-TP_2.*

Theorem 2.18 (Chernikov [5]). *If a theory T has k-TP_2, then it can be witnessed by a formula $\varphi(x, \bar{y})$ where x is a single variable.*

Proof. An immediate consequence of Observation 2.15 and Proposition 2.16. □

3 Dense codense expansions

First, let us recall the definition of *geometric theories.*

Definition 3.1. Let T be a theory.

1. T is said to *eliminate* $^\infty\exists$ if, for any formula $\varphi(x,\bar{y})$, there exists some natural number n such that, for any parameters $\bar{a}$, $\varphi(x,\bar{a})$ has infinitely many realizations iff it has more than n realizations.

2. T is said to satisfy the *exchange property* if, for any model $M \models T$, any subset $A \subseteq M$ and any $b, c \in M \setminus \mathrm{acl}(A)$, $b \in \mathrm{acl}(Ac)$ iff $c \in \mathrm{acl}(Ab)$.

3. T is said to be *geometric* if it eliminates $^\infty\exists$ and satisfies the exchange property.

Definition 3.2. Let T be a geometric complete theory in a language $\mathcal{L}$, and let $\mathcal{L}_H := \mathcal{L} \cup \{H\}$ be the language obtained by adding a new unary predicate symbol H. Given any $\mathcal{L}$-model $M \models T$, let $(M, H(M))$ denote an expansion of M to $\mathcal{L}_H$, where $H(M) := \{x \in M \mid H(x)\}$. $(M, H(M))$ is called a *dense codense expansion of* M if every non-algebraic $\mathcal{L}$-formula $\varphi(x,\bar{a})$ (where x is a single variable) has realizations both in $H(M)$ and in $M \setminus \mathrm{acl}(\bar{a} \cup H(M))$. A dense codense expansion $(M, H(M))$ is called:

1. a *lovely pair* if $H(M) \prec M$ (as $\mathcal{L}$-models),

2. an *H-structure* if $H(M)$ is an $\mathcal{L}$-algebraically independent subset of $\mathcal{M}$.

Remark 3.3. *The condition that every non-algebraic $\mathcal{L}$-formula $\varphi(x,\bar{a})$ has realizations in $H(M)$ is called the* density condition *of H, and the condition that it always has realizations in $M \setminus \mathrm{acl}(\bar{a} \cup H(M))$ is called the* codensity condition *of H.*

Theorem 3.4 ([3], [4]). *Given any geometric complete theory T, all the lovely pairs (resp. H-structures) associated with T are elementarily equivalent to one another.*

Notation. Throughout the rest of this section, we fix a geometric complete theory T in a language $\mathcal{L}$, and let T^* denote the complete $\mathcal{L}_H$-theory of either the lovely pairs or the H-structures associated with T. And we will work inside an arbitrary $\bar{\kappa}$-saturated model $(M, H(M)) \models T^*$ for some sufficiently large cardinal $\bar{\kappa}$. When we say A is a subset of M, we shall mean (unless explicitly stated otherwise) that A is a subset of size $< \bar{\kappa}$. H-subscripts in $\mathrm{acl}_H(\bar{a})$, $\mathrm{tp}_H(\bar{a})$ indicate that they are defined in the language $\mathcal{L}_H$. (On the other hand, T-subscripts will indicate that they are defined in the language $\mathcal{L}$.)

Remark 3.5. *1. It remains open whether T^* eliminates $^\infty\exists$ although [4] provides a partial answer by proving that every $\mathcal{L}_H$-formula in the form $\varphi(x,\bar{y}) \wedge H(\bar{y})$ does eliminate $^\infty\exists$.*

2. Not all models of T^* may be H-structures (resp. lovely pairs) although sufficiently saturated ones are. (See [4, Examples 2.11, 2.12].)

Definition 3.6 ([4], [3]). A subset $A \subseteq M$ is called *H-independent* if $A \mathop{\smile\!\!\!\!\!\!\,|}_{H(A)} H(M)$ (where $\mathop{\smile\!\!\!\!\!\!\,|}$ denotes the algebraic independence relation).

Observation 3.7. If $\bar{a} \in M$ is H-independent, so is $\bar{a}\bar{h}$ for any $\bar{h} \in H(M)$.

Proposition 3.8 ([2]). If there exists $\{\varphi(\bar{x}, \bar{a}_{i,j}) \mid i,j < \omega\}$ witnessing k-TP_2, we may assume that $\{\bar{a}_{i,j}\}_{i,j<\omega}$ is an indiscernible array and that every $\bar{a}_{i,j}$ is H-independent.

Lemma 3.9 ([3], [4]). *For any H-independent tuples $\bar{a}$ and $\bar{b}$,*

$$\mathrm{tp}_H(\bar{a}) = \mathrm{tp}_H(\bar{b}) \Leftrightarrow \mathrm{tp}_T(\bar{a}H(\bar{a})) = \mathrm{tp}_T(\bar{b}H(\bar{b})).$$

Definition 3.10. For any subset $A \subseteq M$, $\mathrm{acl}_T(A \cup H(M))$ is called the *small closure of* A, and is denoted by $\mathrm{scl}(A)$. Any subset $B \subseteq \mathrm{scl}(A)$ is said to be A*-small.*

Remark 3.11. $M \setminus \mathrm{scl}(A)$ *is* $\mathcal{L}_H$*-type definable over* A.

Proposition 3.12 ([2]). For any $\mathcal{L}_H$-formula $\varphi(\bar{x}, \bar{a})$ where $\bar{a}$ is H-independent, there exists some $\mathcal{L}$-formula $\psi(\bar{x}, \bar{a})$ such that

$$\models \varphi(\bar{x}, \bar{a}) \wedge H(\bar{x}) \leftrightarrow \psi(\bar{x}, \bar{a}) \wedge H(\bar{x}).$$

Proof. (Essentially a compactness argument.) Let $X \subseteq M^n$ be the set defined by $\varphi(\bar{x}, \bar{a})$. We may assume that $H(M)^n \cap X$ and $H(M)^n \setminus X$ are both nonempty.

Claim. For any $\bar{h}_1 \in H(M)^n \cap X$ and any $\bar{h}_2 \in H(M)^n \setminus X$, there exists some $\mathcal{L}$-formula $\theta_{\bar{h}_1\bar{h}_2}(\bar{x}, \bar{a})$ such that $\bar{h}_1 \models \theta(\bar{x}, \bar{a})$ and $\bar{h}_2 \models \neg\theta(\bar{x}, \bar{a})$.

Proof of Claim. Given such $\bar{h}_1$ and $\bar{h}_2$, clearly $\mathrm{tp}_H(\bar{h}_1\bar{a}) \neq \mathrm{tp}_H(\bar{h}_2\bar{a})$. Moreover, $\bar{h}_1\bar{a}$ and $\bar{h}_2\bar{a}$ are both H-independent by Observation 3.7. Hence Lemma 3.9 implies $\mathrm{tp}_T(\bar{h}_1\bar{a}) \neq \mathrm{tp}_T(\bar{h}_2\bar{a})$. This completes the proof of Claim.

For each $\bar{h}_2 \in H(M)^n \setminus X$, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma_{\bar{h}_2}(\bar{x}) := \{H(\bar{x}) \wedge \varphi(\bar{x}, \bar{a})\} \cup \{\neg\theta_{\bar{h}_1\bar{h}_2}(\bar{x}, \bar{a}) \mid \bar{h}_1 \in H(M)^n \cap X\}$$

which is clearly inconsistent. Since $(M, H(M))$ is saturated, there exist finitely many tuples $\bar{h}_1^1, \cdots, \bar{h}_1^k$ in $H(M)^n \cap X$ such that the $\mathcal{L}$-formula

$$\psi_{\bar{h}_2}(\bar{x}, \bar{a}) := \bigvee_{i=1}^{k} \theta_{\bar{h}_1^i\bar{h}_2}(\bar{x}, \bar{a})$$

is satisfied by every tuple in $H(M)^n \cap X$. Note $\bar{h}_2 \not\models \psi_{\bar{h}_2}(\bar{x}, \bar{a})$. Next, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma(\bar{x}) := \{H(\bar{x}) \wedge \neg\varphi(\bar{x}, \bar{a})\} \cup \{\psi_{\bar{h}_2}(\bar{x}, \bar{a}) \mid \bar{h}_2 \in H(M)^n \setminus X\}$$

which is clearly inconsistent. Again, since $(M, H(M))$ is saturated, there exist finitely many tuples $\bar{h}_2^1, \cdots, \bar{h}_2^m$ in $H(M)^n \setminus X$ such that the $\mathcal{L}$-formula

$$\psi(\bar{x}, \bar{a}) := \bigwedge_{i=1}^{m} \psi_{\bar{h}_2^i}(\bar{x}, \bar{a})$$

is *not* satisfied by any tuple in $H(M)^n \setminus X$. But $\psi(\bar{x}, \bar{a})$ is satisfied by every tuple in $H(M)^n \cap X$, and hence $\psi(\bar{x}, \bar{a})$ is a desired $\mathcal{L}$-formula. □

Proposition 3.13 ([2]). For any $\mathcal{L}_H$-formula $\varphi(x, \bar{a})$ where x is a single variable and $\bar{a}$ is H-independent, there exists some $\mathcal{L}$-formula $\psi(x, \bar{a})$ such that the symmetric difference $\varphi(x, \bar{a}) \triangle \psi(x, \bar{a})$ defines an $\bar{a}$-small set.

Proof. (Essentially a compactness argument.) Let $X \subseteq M$ be the set defined by $\varphi(x, \bar{a})$. Consider

$$Y_1 := \{x \in X \mid x \notin \mathrm{scl}(\bar{a})\} \quad \text{and} \quad Y_2 := \{x \in M \setminus X \mid x \notin \mathrm{scl}(\bar{a})\}.$$

If Y_1 or Y_2 is empty, then the assertion is clear. So assume that Y_1, Y_2 are both nonempty.

Claim. For any $c_1 \in Y_1$ and any $c_2 \in Y_2$, there exists some $\mathcal{L}$-formula $\theta_{c_1c_2}(x, \bar{a})$ such that $c_1 \models \theta_{c_1c_2}(x, \bar{a})$ and $c_2 \models \neg\theta_{c_1c_2}(x, \bar{a})$.

Proof of Claim. Given such c_1 and c_2, clearly $\mathrm{tp}_H(c_1\bar{a}) \neq \mathrm{tp}_H(c_2\bar{a})$. Moreover, $c_1\bar{a}$ and $c_2\bar{a}$ are clearly both H-independent. Hence Lemma 3.9 implies $\mathrm{tp}_T(c_1\bar{a}) \neq \mathrm{tp}_T(c_2\bar{a})$. This completes the proof of Claim.

Note that Y_1 and Y_2 are both $\mathcal{L}_H$-type definable over $\bar{a}$ (by Remark 3.11). Let $\Sigma_1(x)$ and $\Sigma_2(x)$ be $\mathcal{L}_H$-types over $\bar{a}$ defining Y_1 and Y_2, respectively. For each $c_2 \in Y_2$, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma_1(x) \cup \{\neg\theta_{c_1c_2}(x,\bar{a}) \mid c_1 \in Y_1\}$$

which is clearly inconsistent. Since $(M, H(M))$ is saturated, there exist finitely many $c_1^1, \cdots, c_1^k \in Y_1$ such that the $\mathcal{L}$-formula

$$\psi_{c_2}(x,\bar{a}) := \bigvee_{i=1}^{k} \theta_{c_1^i c_2}(x,\bar{a})$$

is satisfied by every element in Y_1. Note $c_2 \not\models \psi_{c_2}(x,\bar{a})$. Next, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma_2(x) \cup \{\psi_{c_2}(x,\bar{a}) \mid c_2 \in Y_2\}$$

which is clearly inconsistent. Again, since $(M, H(M))$ is saturated, there exist finitely many $c_2^1, \cdots, c_2^m \in Y_2$ such that the $\mathcal{L}$-formula

$$\psi(x,\bar{a}) := \bigwedge_{i=1}^{m} \psi_{c_2^i}(x,\bar{a})$$

is *not* satisfied by any element of Y_2. But $\psi(x,\bar{a})$ is satisfied by every element of Y_1. We conclude that $\varphi(x,\bar{a})\triangle\psi(x,\bar{a})$ defines an $\bar{a}$-small set. □

Theorem 3.14 (Main result in [2]). *Let T be any geometric complete theory.*

1. *If T has NTP_2, so does T^*.*
2. *If T is strong, so is T^*.*

Proof. (Sketch) The proof of 2 is largely a generalization of that of 1, so we only sketch the proof of 1. Propositions 2.16 is routinely used throughout the proof. We start by assuming that T^* has TP_2 witnessed by some $\mathcal{L}_H$-formula $\varphi(x,\bar{y})$ (where x is a single variable due to Theorem 2.18) and an indiscernible array $\mathcal{A} := \{\bar{a}_{i,j}\}_{i,j<\omega}$ where each $\bar{a}_{i,j}$ is H-independent (due to Proposition 3.8). Next, we consider two cases. First consider the case where all the realizations of $\bigwedge_{i<\omega}\varphi(x,\bar{a}_{i,0})$ are in $M \setminus \mathrm{scl}(\mathcal{A})$. In this case, it is relatively straightforward to show (by applying Proposition 3.13 together with the codensity condition of H) that T has TP_2. The other case is more complicated. Basically we reduce it to the case where some $\mathcal{L}_H$-formula in the form $\phi(\bar{x},\bar{w}) \wedge H(\bar{x})$ witnesses TP_2. In fact, we may assume that $\bar{x}$ here is a single variable due to the submultiplicativity of burden (Corollary 2.6). Finally we apply Proposition 3.12 together with the density condition of H to show that T has TP_2. The interested reader is referred to [2] for full details. □

References

[1] H. Adler, *Strong theories, burden and weight*, (2007, preprint available at `http://www.logic.univie.ac.at/~adler/docs/strong.pdf`).

[2] A. Berenstein and H. Kim, *Dense codense predicates and NTP$_2$*, (submitted, preprint available at `http://www.logique.jussieu.fr/modnet/Publications/Preprint%20server/papers/725/725.pdf`).

[3] A. Berenstein and E. Vassiliev, *On lovely pairs of geometric structures*, Annals of Pure and Applied Logic **161** (2010), no. 7, 866–878.

[4] ———, *Geometric structures with a dense independent subset*, (submitted, preprint available at `http://pentagono.uniandes.edu.co/~aberenst/publications.html`).

[5] A. Chernikov, *Theories without the tree property of the second kind*, Annals of Pure and Applied Logic **165** (2014), no. 2, 695–723.

[6] B. Kim, H. Kim, and L. Scow, *Tree indiscernibilities, revisited*, Archive for Mathematical Logic **53** (2014), no. 1-2, 211–232.

[7] S. Shelah, *Classification theory and the number of non-isomorphic models (revised edition)*, North-Holland, Amsterdam-New York, 1990.

[8] A. Tsuboi, *On the number of independent partitions*, The Journal of Symbolic Logic **50** (1985), no. 3, 809–814.

Department of Mathematics
Yonsei University
Seoul, South Korea

E-mail address: `joon9754@yonsei.ac.kr`