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Infinitary Method for Finite Structures

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1 Introduction

Finite Ramsey theorem states that for all numbers $m, n, n_c \in \omega$ there is a number $N \in \omega$ such that

- for every finite coloring $f : [N]^m \to n_c$, we can find $X \in [N]^n$ such that f is constant on $[X]^m$.

The statement above is usually written in symbols as $N \to (n)^m_{n_c}$. This version of Ramsey's theorem follows from the following infinite versions of the theorem:

- (Infinite Ramsey Theorem) $\omega \to (\omega)^m_{n_c}$;
- (Weak Infinite Ramsey Theorem) $\forall n \in \omega[\omega \to (n)^m_{n_c}]$.

In the case of $m = 2$, (finite) Ramsey theorem can be viewed as a statement on the class of finite complete graphs and coloring on the edges. In [1], Jaroslav Nešetřil and Vǒjtěch Rödl showed that (finite) Ramsey theorem can be expanded to the class of linearly ordered graphs and colorings on the subgraphs isomorphic to a given graph.

- Let $A \subset B$ be finite linearly ordered graphs. Then there is a finite linearly ordered graph C such that for every finite coloring f on $\binom{C}{A} = \{A' \subset C : A' \cong A\}$ we can find $X \in \binom{C}{B}$ for which f is constant on $\binom{X}{A}$.

Like the case of original Ramsey's theorem, this statement has an infinite version from which the finite version easily follows.

In this article, we present an infinite version and prove it by using an infinitary method.

2 Preliminaries

Let L be a finite relational language. For each $R \in L$, the arity of R will be denoted by n_R. We assume every R is irreflexive and symmetric. (I don't know this is essential or not.)

A structure $(M, \leq)$ is called a preordered set if $\leq$ is symmetric and transitive. It is clear that the relation $x \approx y$ ($\leftrightarrow x \leq y \leq x$) defines an equivalence relation on M. The induced structure $(M/\approx, \leq)$ clearly becomes an ordered set. A preordered set $(M, \leq)$ will be called a linearly preordered set of width n, if the induced order is linear and $|M/\approx| = n$. For $i = 0, .., n-1$, let $M(i)$ be the i-th smallest $\approx$-class. A subset $A \subset M$ is called a section if $|A \cap M(i)| = 1$ ($\forall i < n$). If $|A \cap M(i)| \leq 1$ ($\forall i < n$), A is called a partial section. The set of all partial sections A with $|A| = k$ will be denoted by $[M]^k$.

Let M be an $(L \cup \{\leq\})$-structure. M will be called a preordered L-structure if (i) $M|\leq$ is a preordered set, and (ii) R^M is a subset of $[M]^{n_k}$. Notice that if M is of width n, then $R^M = \emptyset$ for all R with $n < n_R$.

Definition 1. Let $n \in \omega$ and let M be an $(L \cup \{\leq\})$-structure. We say that M is a preordered random L-structure of width n if the following conditions are satisfied:

1. $M|\leq$ is a linearly preordered set of width n.
2. For any $i < n$, any finite set $I \subset M_i$ and any finite set $\{R_j : j < m\} \subset L$, if $S_j, T_j \subset [M \setminus M(i)]^{n_{R_j}-1}$ $(j < n)$ are sets of partial sections with
$$S_j \cap T_j = \emptyset \ \ (j < n),$$
then there is $d \in M(i) \setminus I$ such that
$$M \models \bigwedge_{j<m} \bigwedge_{A \in S_j} R_j(d, A) \wedge \bigwedge_{j<m} \bigwedge_{B \in T_j} \neg R_j(d, B).$$

If $L = \{R\}$ with a binary R, then a preordered random L-structure will be simply called a preordered random graph.

Remark 2. The conditions 1 and 2 are expressed by a first order $(L \cup \{\leq\})$-axioms, call it $T_{L,n}$. $T_{L,n}$ is an ω-categorical theory admitting elimination of quantifiers. This can be shown by a usual back-and-forth argument.

3 Infinitary Method

Definition 3. Let M be a preordered random L-structure of width n. Let $N \subset \bigcup_{i<l} M(i)$ and $\bar{d} = d_0, ..., d_{m-1} \in \bigcup_{l \leq i < n} M(i)$. We say that N is generic over $\bar{d}$ if there is a set $\hat{N}$ with $\bar{d} \in \hat{N} \subset \bigcup_{l \leq <n} M(i)$ such that $N\hat{N}$ is a preordered random L-structure.

Let M be a preordered L-structure of width n and let $c : [M]^n \to 2$ be a coloring. For a subsection $\bar{a} = \{a_i : i \geq k\}$ of M $(a_i \in M(i))$, we can define a function $c'(X) = c(X, \bar{a})$ for a section X of $M(0) \cup \cdots \cup M(k-1)$. Such a function will be referred as an induced coloring.

Lemma 4. *Let M be a pre-ordered random L-structure of width $n \in \omega$. Let B be a linearly pre-ordered finite L-structure of width n. Then, for any coloring $c : [M]^n \to n_c \in \omega$, we can find $B' \in \binom{M}{B}$ with the following property:*

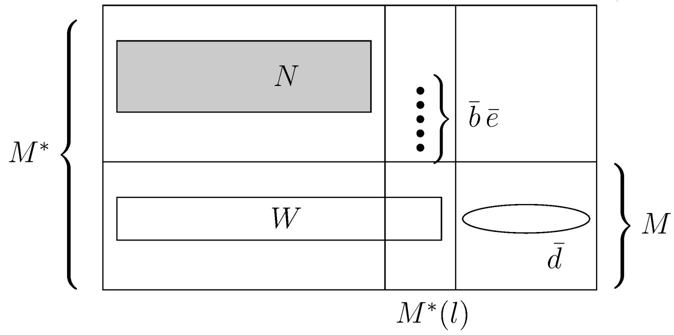
(*) For each section A of B', $\binom{B'}{A}$ is c-monochromatic, i.e., c is constant on $\binom{B'}{A}$.

Proof. The notation $A \cong A'$ will be used to express that A and A' are isomorphic as L-structures. By induction on l, we prove the following statement holds for any M and c:

$(\dagger)_l$ Let $\bar{d} \in \bigcup_{l \leq i < n} M(i)$. Then for any $W \subset \bigcup_{i<l} M(i)$ of width l, we can find $W' \subset \bigcup_{i<l} M(i)$ having the following properties:

- $W' \cong_{\bar{d}} W$;
- Every induced coloring $c'(A) = c(A, \bar{d}_0)$ $(\bar{d}_0 \subset \bar{d})$ is $\bar{d}$-locally constant (in W') in the sense that $A \cong_{\bar{d}} A' \Rightarrow c'(A) = c'(A')$ for all sections A, A' of W'.

Notice that $(\dagger)_n$ proves the lemma. For $l = 1$, $M(0)$ is considered as an infinite set with finitely many random unary predicates. So $(\dagger)_1$ follows from a pigeonhole principle. We assume $(\dagger)_l$ holds for any M and c, and we discuss the case for $l+1$. We are given (M, c), $\bar{d}$ and W of width $l+1$.



We choose an ω_1-saturated elementary extension $(M^*, c) \succ (M, c)$. From now on we work in M^*. For simplifying the presentation, we introduce the notion of a good sequence. Let $N \subset \bigcup_{i<l} M^*(i)$. We say that a sequence $\bar{b}\,\bar{e} = b_0, ..., b_{k_0-1}, e_0, ..., e_{k_1-1} \in M^*(l)$ is a good sequence for N if the following are satisfied:

- N is generic over $\bar{b} \cup \bar{e} \cup \bar{d}$;
- Induced colorings $c(A, b_i, \bar{d}_0)$ $(i < k_0,\ \bar{d}_0 \subset \bar{d})$ are $\{b_j\}_{j \le i} \cup \bar{d}$-locally constant (in N) in the following sense: if $A \cong_{\{b_j\}_{j\le i}, \bar{d}} A'$ then $c(A, b_i, \bar{d}_0) = c(A', b_i, \bar{d}_0)$ $(\forall A, A' \in [N]^l)$;
- Induced colorings $c(A, e_i, \bar{d}_0)$ $(i < k_1,\ \bar{d}_0 \subset \bar{d})$ are $e_i, \bar{d}$-locally constant (in N).

From the definition, for every initial segment $\bar{b}' \subset \bar{b}$ of a good sequence $\bar{b}\,\bar{e}$, $\bar{b}'\,\bar{e}$ is a good sequence (for the same N).

Claim A. *For $k \in \omega$ and qf-types $p_0(x), \ldots, p_{k-1}(x)$ over $\bar{d}$, realized in $M^*(l)$, we can find $N \subset \bigcup_{i<l} M^*(i)$ and a sequence $\bar{b} = b_0, ..., b_k$ (pairwise distinct) such that*

- *$\bar{b}$ (with $\bar{e}$ empty) is a good sequence for N*
- *$b_i \models p_i(x)$ for $i < k$.*

Proof of Claim. Let $b_0 \models p_0$. Let $N_0 \subset \bigcup_{i<l} M^*(i)$ be a structure generic over $b_0\bar{d}$. By preparing a set $Z = \{z_a : a \in N_0\}$ of variables, we define a set $\Gamma(Z)$ of formulas expressing the following:

1. $Z \cong_{b_0, \bar{d}} N_0$ (hence Z is generic over $b_0, \bar{d}$);
2. For all sections X and X' in Z, and for all $\bar{d}_0 \subset \bar{d}$,

$$X \cong_{b_0\bar{d}} X' \Rightarrow c(X, b_0, \bar{d}_0) = c(X', b_0, \bar{d}_0).$$

By the induction hypothesis $(\dagger)_l$ applied to N_0, we see that every finite $V \subset N_0$ has a copy $V' \cong_{b_0\bar{d}} V$ in N_0 such that $c(X, b_0, \bar{d}_0)$ depends only on the qf-type of X over $b_0\bar{d}$. This shows that $\Gamma(Z)$ is finitely satisfiable in N_0. So, by replacing N_0 by a realization of $\Gamma(Z)$, we can assume that every induced coloring $c(X, b_0, \bar{d}_0)$ is $b_0\bar{d}$-locally constant in N_0.

Then choose $b_1 \models p_1$, $b_1 \neq b_0$ such that N_0 is generic over $b_0 b_1 \bar{d}$. Again, by the induction hypothesis, every V has a copy $V' \cong_{b_0 b_1 \bar{d}} V$ in N_0 such that every induced coloring $c'(A) = c(A, b_1, \bar{d}_0)$ ($\bar{d}_0 \subset \bar{d}$) is $b_0 b_1 \bar{d}$-locally constant on W'. So, by the saturation, we can find N_1 (generic over $b_0 b_1 \bar{d}$) with the properties:

3. $X, X' \in [N_1]^l$, $X \cong_{b_0 \bar{d}} X' \;\Rightarrow\; c(X, b_0, \bar{d}_0) = c(X', b_0, \bar{d}_0)$;

4. $X, X' \in [N_1]^l$, $X \cong_{b_0, b_1 \bar{d}} X' \;\Rightarrow\; c(X, b_1, \bar{d}_0) = c(X', b_1, \bar{d}_0)$.

The property 3 can be assumed, since each copy V' is chosen in N_0, and since N_0 has the property 2. Continuing this construction, we have an arbitrarily long good sequence (for some N). (End of Proof of Claim)

Claim B. *For $k \in \omega$, there is a number $k^* = kn^*$ with the following property:*

(**) if $\bar{b}\,\bar{e} = b_0 ... b_{k^*}\, \bar{e}$ is a good sequence for N with $b_{ki} b_{ki+1} ... b_{ki+(k-1)} \cong_{\bar{d}} b_{kj} b_{kj+1} ... b_{kj+(k-1)}$ for every $i, j < n^*$, then there is a structure N_0 and $i^* < n^*$ such that, by letting $\bar{b}' = b_{ki^*} ... b_{ki^*+(k-1)}$, $\bar{e}' = b_{k^*}\, \bar{e}$, the sequence $\bar{b}'\,\bar{e}'$ is good for N_0.

Proof of Claim. n^* is a sufficiently large number so that the following argument is true. Suppose that we are given N and $\bar{b}\,\bar{e}$. We write B_i for the k-sequence $b_{ki}, \dots, b_{ki+(k-1)}$. Let us consider induced colorings $c(A, b_{k^*}, \bar{d}_0)$ ($\bar{d}_0 \subset \bar{d}$). For a section $A = \{a_0, \dots, a_{l-1}\}$ of N ($a_0 \in N(0), \dots, a_{l-1} \in N(l-1)$) and $i < n^*$, let

$$\mathrm{qftp}^*(A, B_i/\bar{d}) = \bigcup_{\substack{R \in L \\ t \in \{0,1\}}} \{R(\bar{x}_I, y_j, \bar{d}')^t : I \subset l, j < k, \bar{d}' \subset \bar{d}, N \models R(A_I, b_{ki+j}, \bar{d}')^t\},$$

where $\bar{x}_I = (x_j)_{j \in I}$ and $A_I = (a_j)_{j \in I}$. By the definition of a good sequence, the value $c(A, b_{k^*}, \bar{d}_0)$ depends only on $\mathrm{qftp}(A/\bar{b}\,\bar{d})$, and the choice of $\bar{d}_0 \subset \bar{d}$. Moreover, by the property of M, $\mathrm{qftp}(A/\bar{b}\,\bar{d})$ is determined by $\mathrm{qftp}(A/b_{k^*}\bar{d})$ and $\bigcup_{i<n^*} \mathrm{qftp}^*(A, B_i/\bar{d})$.

Let Q_i be the set of all qf-types of the form $\mathrm{qftp}^*(A, B_i/\bar{d})$. Notice that Q_i does not depend on i, because of the assumption of B_i. So hereafter we simply write Q for Q_i. Let F be the set of all functions from the pairs of the form $(\bar{d}_0, \mathrm{qftp}(A/b_{k^*}\bar{d}))$ to n_c. Then we can naturally define a function $f : Q^{n^*} \to F$ by

$$f((q_i)_{i<n^*})(\bar{d}_0, q(\bar{x})) = c(A, b_{k^*}, \bar{d}_0),$$

where A is a section satisfying $q(\bar{x}) \cup \bigcup_{i<n^*} q_i(\bar{x}, B_i)$. By applying H-J theorem, we can find a line $\alpha = \alpha(v) \in (Q \cup \{v\})^{n^*} \setminus Q^{n^*}$ such that $\alpha(Q)$ is f-monochromatic. Let i^* be the minimum i such that the i-th element α_i of α is v. For a set $Z = \{z_a : a \in N\}$ of variables, we define a set $\Delta(Z)$ of formulas expressing the following:

1. $Z \cong_{B_{i^*}\bar{e}'\bar{d}} N$ (hence Z is generic over $B_{i^*}\bar{e}'\bar{d}$);

2. For all sections X and X' in Z, and for all $\bar{d}_0 \subset \bar{d}$,

   (a) if $e \in \bar{e}' (= b_{k^*}\, \bar{e})$ and $X \cong_{e\bar{d}} X'$ then $c(X, e, \bar{d}_0) = c(X', e, \bar{d}_0)$,

   (b) if $b_j \in B_{i^*} (= b_{ki^*}, \dots, b_{ki^*+(k-1)})$ and $X \cong_{b_{ki^*}, \dots, b_j \bar{d}} X'$ then $c(X b_j \bar{d}_0) = c(X' b_j \bar{d}_0)$.

We show the finite satisfiability of $\Delta(Z)$ in N. Let $V \subset N$ be any finite subset of width l. By the genericity, there is $V' \subset N$ with $V' \cong_{B_{i^*}, \bar{e}', \bar{d}} V$ satisfying the following: For each $i < n^*$,

- $\mathrm{qftp}^*(V', B_i/\bar{d}) = \mathrm{qftp}^*(V, B_{i^*}/\bar{d})$, if $\alpha_i = v$;
- every section $X \subset V'$ satisfies $\alpha_i(X, B_i)$, if $\alpha_i \in Q$.

By our choice of f and α, the color $c(X, b_{k^*}, \bar{d}_0)$ depends only on qf-type of X over $b_{k^*}\bar{d}$ and the choice of d_0. Therefore V' satisfies (a finite part of) condition (2a) of Δ. Since every section $X \subset V'$ satisfies $\alpha_i(X, B_i)$ for $i < i^*$, the condition (2b) follows from the fact that $\bar{b}\,\bar{e}$ is a good sequence for N and $V' \subset N$. So, by the saturation of M^*, there is N_0 realizing Δ. Then $B_{i^*}\bar{b}'$ is a good sequence for N_0. (End of Proof of Claim)

Claim C. *Let $p_0(x), ..., p_{k-1}(x)$ and $q_0(x), ..., q_{m-1}(x)$ be two sequences of quantifier free 1-types over $\bar{d}$, where x is a variable for an element in $M^*(l)$. Then we can find N and a sequence $b_0, \ldots, b_{k-1}, e_0, \ldots, e_{m-1} \in M^*(l)$ being good for N such that $b_i \models p_i$ $(i < k)$ and $e_i \models q_i$ $(i < m)$.*

Proof of Claim. We prove by induction on m. Since the case of $m = 0$ is trivial by Claim A, we assume the Claim has been proven for $\leq m$. We are given p_i $(i < k)$ and q_i $(i \leq m)$. Choose $k^* = kn^*$ sufficiently large. For $i < k^*$, let $p'_i = p_{(i \bmod k)}$ and apply the induction hypothesis to $p'_0, \ldots, p'_{k^*-1}, q_0$ and $q_1, \ldots, q_m$. With the notation in Claim B, we can find N' and a good sequence $\bar{b}\,\bar{e} = b_0, ...b_{k^*}\bar{e}$ for N' such that $B_i \models p_0(x_0) \cup ... \cup p_{m-1}(x_{m-1})$ $(i < n^*)$, $b_{k^*} \models q_0$, and that $\bar{e} \models q_1(x_1) \cup ... \cup q_s(x_m)$. Then, by Claim B, we can find N and $B_{i^*} \subset \{b_i\}_{i<k^*}$ such that $B_{i^*}\bar{e}'$ is good for N, where $\bar{e}' = b_{k^*}\,\bar{e}$. This sequence is a required one. (End of Proof of Claim)

With choosing $k = 0$ and m large enough in Claim C, (for appropriate q_i's) we can find a copy $W' \subset N_0 \cup \bar{e}$ of W satisfying the conditions in $(\dagger)_{l+1}$. By $(M, c) \prec (M^*, c)$, W' can be chosen in M.

□

Remark 5. The partite lemma follows from our infinite version. Let X be a linearly ordered structure of width n and Y a linearly pre-ordered of the same width. Suppose for a contradiction that there is no finite Z satisfying the required condition. Let $M \models T_{L,n}$ be a countable model and let $\{a_i\}_{i\in\omega}$ be an enumeration of M. We put $Z_n = \{a_i\}_{i\leq n}$. Since Z_n does not satisfy the required condition, we can find a section X_n and a coloring $c_n : \binom{Z_n}{X_n} \to 2$ such that no $\binom{Y'}{X_n}$ with $Y \cong Y' \subset Z_n$ is c_n-monochromatic. By König's lemma, there is an infinite sequence $n_0 < n_1 < n_2 \cdots$ such that $c_{n_0} \subset c_{n_1} \subset \cdots$. We can also assume that all the X_{n_i} are the same, say X. Let $c^* = \bigcup_{i\in\omega} c_{n_i}$. Then, no $\binom{Y'}{X}$ with $Y \cong Y' \subset Z$ would be c^*-monochromatic. This contradicts our infinite version of partite lemma.

4 Ordered Random Structures

Theorem 6. *Let $A \subset B$ be two linearly ordered L-structures of finite width. Then there is a number m^* such that if G is a linearly pre-ordered random L-structure of width m^* then, for every coloring c on $\binom{G}{A}$, we can find a copy $B' \subset G$ of B such that $\binom{B'}{A}$ is c-monochromatic.*

Proof. Let $n = |A|$ and $m = |B|$. Let m^* be a sufficiently large number compared with n and m. We assume c is defined on the subsections of size n. Let (G^*, c) be an ω_1-saturated elementary extension of (G, c). It is sufficient to find a desired copy B' in G^*. Let $\{I_i : i < k\}$ be an enumeration of all n-element subsets of m^*. Starting from $G_0 = G^*$, we inductively choose linearly pre-ordered random L-structures $G_i \subset G^*$ $(i < k)$ of width m^* such that, for every $j < i$, $G_i|I_j = \bigcup_{l \in I_j} G_i(l)$ is locally monochromatic. Suppose we have already defined G_i. We define c^* on $[G_i]^{m^*}$ by

$$c^*(Y) = c(Y|I_i).$$

Then, by the partite lemma, each finite substructure of G_i (of width m^*) has a copy $Z \subset G_i$ such that c^* is locally constant on Z. Let X, X' be sections of $Z|I_i$ with $X \cong X'$. Then there are isomorphic sections $Y \supset X$ and $Y' \supset X'$ of G_i. By the definition of c^*, we have $c(X) = c(Y|I_i) = c^*(Y) = c^*(Y') = c(X')$. (For $j < i$, we already have that c is locally constant on $Z|I_j$.) By compactness, we have G_{i+1} satisfying the required condition. Finally let $H = G_k$. c is locally constant on $[H|I_i]^n$, for each i. Let n_i be the constant value of $c(A')$, where $A' \cong A$ is a section of $H|I_i$. Since m^* is very large, by Ramsey's theorem, we can choose a set $J \subset n^*$ with $|J| = m$ such that if $I_i, I_{i'} \subset J$ then $n_i = n_{i'}$. It is clear that $H|I$ is a random L-structure. So, we can find a copy $B' \subset H|J$ of B. B' has the desired property. □

Example 7. Let G be a countable random graph and $c : G \to 2$ be a coloring for vertexes. Then there is a random subgraph $H \subset G$ such that c is constant on H: First notice that every R-definable infinite subset of G is a random graph. We may assume that no R-definable infinite subset is c-monochromatic. Let $\{g_i : i \in \omega\}$ be an enumeration of G and let $G_n = \{g_i : i < n\}$. We will define h_i's such that, for each n, $G_n \cong \{h_i : i < n\} \subset G$ and that $c(h_n) = 0$. Suppose that we have defined $\{h_i : i < n\}$. Let $X = \{a \in G : G_n, g_n \cong \{h_i : i < n\}, a\}$. X is an infinite definable subset, so by our assumption, there is $a \in X$ such that $c(a) = 0$. Let $g_n = a$ then we are done.

References

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