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Equivalence Between Two Models of the Full-Information Duration Problem with Random Horizon

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1 Introduction

In Tamaki (2013), we generalized the classical fixed horizon full-information duration problem by allowing the horizon to be random. Let N denote the length of horizon bounded by n (given positive integer). The random horizon duration problem can be distinguished into two models, MODEL 1 and MODEL 2, according to whether the final stage of the planning horizon is N or n. That is, if the chosen object is the last relative maximum, we can hold it until stage N in MODEL 1, whereas until stage n in MODEL 2. The optimal rule heavily depends on the prior $\mathbf{p}=(p_1,p_2,\ldots,p_n)$ assumed on N. Define, for a given prior $\mathbf{p}$,

$$\pi_k=p_k+p_{k+1}+\cdots+p_n$$

and

$$\sigma_k=\pi_k+(n-k)p_k$$

for $k\leq n$. Then a sufficient condition for the optimal rule to be monotone is that π_{k+j}/π_k (σ_{k+j}/σ_k) is non-increasing in k for each possible value of j for MODEL 1 (MODEL 2). See Tamaki (2013) for this sufficient condition and the monotone rule.

We confine our attention to the class of priors given by, for $1\leq k\leq n$,

$$p_k=\left(\frac{n-k+1}{n}\right)^m-\left(\frac{n-k}{n}\right)^m, \tag{1}$$

or

$$p_k=\frac{\binom{n+m-k}{m}-\binom{n+m-1-k}{m}}{\binom{n+m-1}{m}}=\frac{\binom{n+m-1-k}{m-1}}{\binom{n+m-1}{m}}, \tag{2}$$

where $m=1,2,3,\ldots$ is a given parameter. By straightforward calculation, the optimal rule for these priors is shown to satisfy the above sufficient condition for both MODELs (see the Appendix A).

Remark 1. It is noted that the prior distributions given in (1) and (2) are related to the following urn sampling models with replacement and without replacement. Suppose that there exists an urn containing n balls numbered $1, 2, \ldots, n$. We draw m balls (and observe their numbers) one at a time randomly from the urn with replacement in such a way that, at each stage, drawing a specific ball is equally likely for all n balls. Then p_k in (1) is the probability that the smallest of the m numbers drawn is k. For (2), suppose that there exists an urn containing $n+m-1$ balls numbered $1, 2, \ldots, n+m-1$. We draw m balls randomly from the urn without replacement. Then p_k is the probability that the smallest of the m numbers drawn is k.

2 MODEL 1

Of interest is to derive the limiting optimal payoff $v_m^{(1)}$, as $n \to \infty$, for MODEL 1 when the priors are given by (1) and (2) for a given parameter m. To do so, we use a planar Poisson process (PPP) model which is known to facilitate the derivation of the asymptotic values for some full-information problems (see, e.g., Gnedin (1996), (2004), and Samuels (2004)). A link to the finite problems can be established by embedding suitably the finite independent and identically distributed sequences in the PPP in a similar manner as given to the Gilbert and Mosteller full-information best-choice problem by Gnedin (1996, Section 3).

As a preliminary, in addition to

$$I(c) = \int_c^\infty \frac{e^{-x}}{x} dx, \quad J(c) = \int_0^c \frac{e^x - 1}{x} dx,$$

we introduce the following notations

$$\begin{aligned} I_m(c) &= \int_c^\infty \frac{m! e^{-x}}{x^{m+1}} dx, \\ K_m(c) &= \int_0^c \frac{x^m e^x}{m!} dx, \\ L_m(c) &= \int_0^c \frac{m! e^{-x}}{x^{m+1}} K_m(x) dx, \\ M_m(c) &= \int_0^c \frac{x^m e^x}{m!} L_m(x) dx, \end{aligned}$$

for $m = 0, 1, 2, \ldots$. We can list some properties of these functions.

Corollary 1.
(i) $I_m(c)$ have the following expressions with $I_0(c) = I(c)$: For $m \geq 1$,

$$I_m(c) = (-1)^m \left[I(c) - \frac{e^{-c}}{c} \sum_{k=0}^{m-1} \frac{k!}{(-c)^k} \right].$$

(ii) $K_m(c)$ *have the following expressions with* $K_0(c) = e^c - 1$*: For* $m \geq 1$,

$$K_m(c) = (-1)^m \left[e^c \sum_{k=0}^{m} \frac{(-c)^k}{k!} - 1 \right]. \tag{3}$$

(iii) $L_m(c)$ *have the following expressions with* $L_0(c) = -J(-c)$*: For* $m \geq 1$,

$$L_m(c) = \frac{e^{-c}}{c} \sum_{k=0}^{m-1} \frac{k!}{c^k} K_k(c) - J(-c) - h_m.$$

(iv) $M_m(c)$ *have the following expressions with* $M_0(c) = e^c L_0(c) - J(c)$*: For* $m \geq 1$,

$$M_m(c) = (-1)^m \left[e^c \sum_{k=0}^{m} \frac{(-c)^k}{k!} L_k(c) - J(c) \right]. \tag{4}$$

Now we have the following results.

Theorem 1.: *Let* c_m *be the unique root* c *of the equation*

$$\sum_{k=0}^{m} \frac{(-c)^k}{k!} (1 - L_k(c)) = e^{-c} (1 - J(c)). \tag{5}$$

Then the optimal limiting payoff $v_m^{(1)}$ *is given by*

$$v_m^{(1)} = \frac{L_m(c)}{m+c+1} + \left(\frac{m! K_m(c)}{c^m} - \frac{c e^c L_m(c)}{m+c+1} \right) \times \left(\frac{c^m I_m(c)}{m!} - \frac{c^{m+1} I_{m+1}(c)}{(m+1)!} \right), \tag{6}$$

where, for easier reading, c_m *is abbreviated to* c.

Proof. As mentioned before, we use a PPP model to derive (5) and (6). According to Samuels (2004, Sections 9 and 10), we use a Poisson process with unit rate on the semi-infinite strip $[0,1] \times [0,\infty)$. This turns the problem upside down, making the 'best' become the 'smallest'. The process is scanned from left to right by shifting a vertical detector and the scanning can be stopped each time a point in the PPP, referred to as an *atom* henceforth, is detected. The random number N of objects whose prior is given by (1) or (2) is now represented in a PPP by a vertical cut V_m on $(0,1)$ whose distribution function is given by $F_{V_m}(v) = 1 - (1-v)^m$, $0 \leq v \leq 1$ (see Remark 1). We have to stop before V_m.

Suppose that an atom is identified as a point (t, y) if the atom appears at time t as a candidate (relatively best atom as in the finite problem) of value y in the PPP. Let $R = R(t, y)$ denote the duration until the first point after t which lies below y, if any, and denote $1-t$ if there is no such point. That is, R is the duration when no vertical cut takes place. Then $P\{R > r\} = e^{-yr}$, $0 \le r \le 1-t$ with mass $P\{R = 1-t\} = e^{-y(1-t)}$, because $R > r$ occurs if and only if there is no point in the box domain $[t, t+r] \times [0, y]$ whose area is yr. Let $V_m(t)$ denote the additional time to V_m from time t onward, provided that $V_m > t$. Hence, we have

$$P\{V_m(t) > r\} = \frac{1 - F_{V_m}(t+r)}{1 - F_{V_m}(t)} = \left(\frac{1-t-r}{1-t}\right)^m, \quad 0 < r \le 1-t.$$

Then $D_m(t, y) = \min\{R, V_m(t)\}$ represents the duration related to the point (t, y) when the vertical cut is taken into consideration. Since R and $V_m(t)$ are independent, the expected duration $p_m(t, y) = E[D_m(t, y)]$, when we stop at point (t, y), is calculated as

$$\begin{aligned} p_m(t,y) &= \int_0^{1-t} P\{D_m(t,y) > r\} dr \\ &= \int_0^{1-t} P\{R > r\} P\{V_m(t) > r\} dr \\ &= \int_0^{1-t} e^{-yr} \left(\frac{1-t-r}{1-t}\right)^m dr \qquad (7) \\ &= \frac{m! e^{-(1-t)y}}{y\{(1-t)y\}^m} K_m((1-t)y). \qquad (8) \end{aligned}$$

If we do not choose the point (t, y), but instead choose the atom related to the next candidate, if any, then, since its value is uniformly distributed on $(0, y)$, the duration we can expect to receive is

$$q_m(t,y) = \int_0^{1-t} \left\{ \int_0^y p_m(t+r, z) \frac{1}{y} dz \right\} f_R(r) P\{V_m(t) > r\} dr, \qquad (9)$$

where $f_R(r)$ is the density of R, i.e. $f_R(r) = ye^{-yr}$, $0 \le r \le 1-t$.

Let $c = (1-t)y$, box area of point (t, y). Then, from (8) and (9),

$$\begin{aligned} p_m(t,y) &= \frac{m! e^{-c}}{yc^m} K_m(c), \\ q_m(t,y) &= \frac{m! e^{-c}}{yc^m} M_m(c). \qquad (10) \end{aligned}$$

See the Appendix B for (10). Solving for the locus of point (t, y) at which $p_m(t, y) = q_m(t, y)$, or, equivalently,

$$K_m(c) = M_m(c), \qquad (11)$$

we obtain (5) from (3) and (4). The uniqueness of the solution c of the equation (11) can be ascertained by showing that the function $M_m(c)/K_m(c)$ is increasing in c (see the Appendix C), implying that $p_m(t,y) \geq q_m(t,y)$ means $p_m(t',y') \geq q_m(t',y')$ for $t' > t$ and $y' < y$. So we are in the monotone case of optimal stopping in the infinite problem and can conclude that the optimal rule stops with the first candidate, if any, that lies below the threshold curve $y = c_m/(1-t)$.

In the rest of this proof and in the Appendic D, we write c instead of c_m for easier reading. Let T be the arrival time of the first (leftmost) atom that lies below the optimal threshold curve $y = c/(1-t)$ and S the time when the value of the best (lowest) atom above threshold is equal to the threshold. Then T and S are independent and their densities are given by

$$f_T(t) = c(1-t)^{c-1}, \quad 0 < t < 1, \tag{12}$$

$$f_S(s) = \frac{cs}{(1-s)^{c+2}} e^{-\frac{cs}{1-s}}, \quad 0 < s < 1, \tag{13}$$

respectively (see Sec.10.2 of Samuels (2004)). Let $\bar{F}_{V_m}(v) = 1 - F_{V_m}(v) = (1-v)^m$. Then, from the similar argument in Sec.10.2 of Samuels (2004), combined with his Sec.13.2 given to the best-choice problem with uniform vertical cut, referred to as POR, the optimal payoff can be calculated as

$$\begin{aligned}
v_m^{(1)} &= E\left[p_m\left(S, \frac{c}{1-S}\right) \mathbf{1}_{\{S<T\}} \mathbf{1}_{\{V_m > S\}}\right] \\
&\quad + E\left[\text{average of } p_m\left(T, y : 0 < y < \frac{c}{1-T}\right) \mathbf{1}_{\{T<S\}} \mathbf{1}_{\{V_m > T\}}\right] \\
&= E\left[\bar{F}_{V_m}(S)\, p_m\left(S, \frac{c}{1-S}\right) \mathbf{1}_{\{S<T\}}\right] \\
&\quad + E\left[\text{average of } \bar{F}_{V_m}(T)\, p_m\left(T, y : 0 < y < \frac{c}{1-T}\right) \mathbf{1}_{\{T<S\}}\right] \\
&= \int_0^1 \int_0^t (1-s)^m\, p_m\left(s, \frac{c}{1-s}\right) f_S(s) f_T(t)\, ds\, dt \\
&\quad + \int_0^1 \int_0^s (1-t)^m \left[\int_0^{c/(1-t)} p_m(t,y) \frac{1-t}{c} dy\right] f_T(t) f_S(s)\, dt\, ds. \qquad (14)
\end{aligned}$$

(14) can be simplified to (6) in the Appendix D.

Note that, if the vacuous sum is assumed to be 0, Theorem 1 is still valid for $m = 0$ corresponding to the fixed horizon duration problem.

3 MODEL 2

For the prior given in (1) or (2), there is a simple relation between MODEL 1 and MODEL 2 as shown below in Theorem 2, where we write $v_m^{(i)}$ for the optimal limiting payoff to clarify that the model considered is MODEL $i(=1,2)$.

Theorem 2.: *The optimal limiting payoff for MODEL 2 is* $(m+1)$ *times as large as that for MODEL 1, that is,*

$$v_m^{(2)} = (m+1)v_m^{(1)}. \tag{15}$$

Proof. We here use the same PPP model and the same notations as defined in the proof of Theorem 1 for MODEL 1. However, to distinguish between two MODELs, we denote $p_m(t,y)$ and $q_m(t,y)$ by $p_m^{(i)}(t,y)$ and $q_m^{(i)}(t,y)$ to stand for MODEL i(i=1, 2).

To show (15), it suffices to show that, when we stop at point (t,y), the payoff for MODEL 2 is $(m+1)$ times as large as that for MODEL 1, that is,

$$p_m^{(2)}(t,y) = (m+1)p_m^{(1)}(t,y), \tag{16}$$

because, as a bit of consideration shows, the relations (9) and (14) also hold for MODEL 2 if $p_m(t,y)(=p_m^{(1)}(t,y))$ is replaced by $p_m^{(2)}(t,y)$, thus implying that $q_m^{(2)}(t,y) = (m+1)q_m^{(1)}(t,y)$ and $v_m^{(2)} = (m+1)v_m^{(1)}$ through these relations.

Note that, when we stop at point (t,y) in MODEL 2, the additional payoff, compared to MODEL 1, is $1-(t+V_m(t))$ or 0 depending on whether $V_m(t) < R$ or $V_m(t) \geq R$. Hence, we have

$$p_m^{(2)}(t,y) = p_m^{(1)}(t,y) + E\left[(1-t-V_m(t))\mathbf{1}_{\{V_m(t)<R\}}\right].$$

Coditioning on $V_m(t)$ (via V_m) yields

$$\begin{aligned} E\left[(1-t-V_m(t))\mathbf{1}_{\{V_m(t)<R\}}\right] &= \int_0^{1-t}(1-t-v)e^{-yv}\frac{dF_{V_m}(t+v)}{1-F_{V_m}(t)} \\ &= \int_0^{1-t}(1-t-v)e^{-yv}m\frac{(1-t-v)^{m-1}}{(1-t)^m}dv \\ &= mp_m^{(1)}(t,y), \end{aligned}$$

where the last equality follows from (7). Thus the proof is complete.

Remark 2. Since $E[V_m] = \int_0^1 vf_{V_m}(v)dv = 1/(m+1)$ (see Remark 1), $v_m^{(1)}$ gets small as m gets large, and so $v_m^{(1)}/E[V_m]$ may be considered as a standardized optimal payoff of MODEL 1. Then (15) can be seen as an equivalence relation between the optimal payoff of MODEL 2 and the standardized optimal value of MODEL 1.

Table 1 presents some numerical values of c_m, $v_m^{(1)}$

Table 1

Values of c_m, $v_m^{(1)}$ and $v_m^{(2)}$ for several m, where $v_m^{(2)} = (m+1)v_m^{(1)}$.

	m						
	0	1	2	3	4	5	10
c_m	2.1198	3.6925	5.3520	7.0411	8.7423	10.4495	19.0169
$v_m^{(1)}$	0.4352	0.2022	0.1309	0.0966	0.0765	0.0634	0.0341
$v_m^{(2)}$	0.4352	0.4045	0.3926	0.3865	0.3827	0.3803	0.3746

Appendix A

We show that the priors given by (1) and (2) satisfy the conditionπ_{k+j}/π_k (σ_{k+j}/σ_k) is non-increasing in k for each possible value of j.

The prior given by (1): We have, for $1 \leq k \leq n$,

$$\pi_k = \left(\frac{n-k+1}{n}\right)^m, \quad \sigma_k = n\left[\left(\frac{n-k+1}{n}\right)^{m+1} - \left(\frac{n-k}{n}\right)^{m+1}\right].$$

Hence, it is easy to see that

$$\frac{\pi_{k+j}}{\pi_k} = \left(1 - \frac{j}{n-k+1}\right)^m$$

is decreasing in k for $j + k \leq n$. On the other hand,

$$\frac{\sigma_{k+j}}{\sigma_k} = \frac{(n-k-j+1)^{m+1} - (n-k-j)^{m+1}}{(n-k+1)^{m+1} - (n-k)^{m+1}}.$$

To show that σ_{k+j}/σ_k is non-increasing in k for possible j, it suffices to show that

$$f(x) = \frac{(x-j+1)^{m+1} - (x-j)^{m+1}}{(x+1)^{m+1} - x^{m+1}}$$

is non-decreasing in x for $x \geq j$. This can be done by showing that $df(x)/dx \geq 0$. We have by a straightforward calculation

$$\frac{df(x)}{dx} = \frac{(m+1)g(x)}{[(x+1)^{m+1} - x^{m+1}]^2}$$

where

$$\begin{aligned} g(x) &= j\,[(x+1)^m - x^m]\,[(x-j+1)^m - (x-j)^m] \\ &\quad + \{[(x+1)x - jx]^m - [(x+1)x - j(x+1)]^m\} \\ &> 0, \end{aligned}$$

as desired.

The prior given by (2): We have, for $1 \leq k \leq n$,

$$\pi_k = \frac{\binom{n+m-k}{m}}{\binom{n+m-1}{m}}, \quad \sigma_k = \frac{(n-k+1)^{[m]} + m(n-k)^{[m]}}{n^{[m]}},$$

if the notation $a^{[m]} = a(a+1)\cdots(a+m-1)$ is introduced for a positive integer a. It is easy to see that

$$\frac{\pi_{k+j}}{\pi_k} = \prod_{i=1}^{j}\left(\frac{1}{1 + \frac{m}{n+1-i-k}}\right)$$

is decreasing in k for a fixed j. On the other hand,

$$\frac{\sigma_{k+j}}{\sigma_k} = \frac{(n-k-j+1)^{[m]} + m(n-k-j)^{[m]}}{(n-k+1)^{[m]} + m(n-k)^{[m]}}.$$

Let $A = n-k$, $B = n-k-j$ for which $n-k-j \geq 1$. Then we have by a straightforward calculation

$$\begin{aligned} \frac{\sigma_{k+j}}{\sigma_k} &- \frac{\sigma_{k+1+j}}{\sigma_{k+1}} \\ &= \frac{m(A-B)\left[(m+1)^2AB + (m-1)\left\{(m+1)(A+B)-1\right\}\right]B^{[m]}}{B(B+m-1)\left((m+1)A+m\right)\left((m+1)A-1\right)A^{[m]}} \\ &> 0, \end{aligned}$$

as desired.

Appendix B
From (9), we have

$$\begin{aligned} q_m(t,y) &= \int_0^{1-t} \left\{ \int_0^y p_m(t+r,z)dz \right\} e^{-yr} \left(\frac{1-t-r}{1-t} \right)^m dr \\ &= \frac{e^{-(1-t)y}}{\{(1-t)y\}^m} \int_t^1 \left\{ \int_0^y p_m(s,z)dz \right\} e^{(1-s)y}[(1-s)y]^m ds. \quad (A1) \end{aligned}$$

However, we have, from (8),

$$\begin{aligned} \int_0^y p_m(s,z)dz &= \int_0^y \frac{m!e^{-(1-s)z}}{z\{(1-s)z\}^m} K_m((1-s)z)dz \\ &= \int_0^{(1-s)y} \frac{m!e^{-x}}{x^{m+1}} K_m(x)dx \\ &= L_m((1-s)y). \quad (A2) \end{aligned}$$

Substituting (A2) into (A1) immediately yields (10).

Appendix C
It is easy to see from the definitions of $K_m(c)$ and $M_m(c)$ that

$$\frac{d}{dc}\left\{ \frac{M_m(c)}{K_m(c)} \right\} = \frac{c^m e^c R_m(c)}{m!(K_m(c))^2},$$

where $R_m(c) = L_m(c)K_m(c) - M_m(c)$. Hence, to show that $M_m(c)/K_m(c)$ is increasing in c, it suffices to show that $R_m(c)$ is increasing, because $R_m(0) = 0$. This can be easily seen from

$$\frac{dR_m(c)}{dc} = \frac{m!e^{-c}}{c^{m+1}}(K_m(c))^2 > 0.$$

Appendix D

First observe that, from (8) and (A2),

$$p_m\left(s,\frac{c}{1-s}\right)=\frac{m!e^{-c}(1-s)}{c^{m+1}}K_m(c) \qquad (A3)$$

and

$$\int_0^{c/(1-t)} p_m(t,y)dy = L_m(c). \qquad (A4)$$

Substituting (A3) and (A4) into (14) and combining it with (12) and (13) yields

$$\begin{aligned} v_m^* &= \frac{m!e^{-c}K_m(c)}{c^{m+1}}\int_0^1\int_0^t (1-s)^{m+1}f_S(s)f_T(t)dsdt \\ &+\frac{L_m(c)}{c}\int_0^1\int_0^s(1-t)^{m+1}f_T(t)f_S(s)dtds. \qquad (A5)\end{aligned}$$

Interchanging the order of integration, the first bivariate integral becomes

$$c\int_0^1\left\{\int_s^1 c(1-t)^{c-1}dt\right\}(1-s)^{m-c-1}se^{-\frac{cs}{1-s}}ds = c\int_0^1 s(1-s)^{m-1}e^{-\frac{cs}{1-s}}ds \quad (A6)$$

The second bivariate integral is equal to

$$\int_0^1\left\{\int_0^s c(1-t)^{m+c}dt\right\}f_S(s)ds=\left(\frac{c}{m+c+1}\right)\left[1-c\int_0^1 s(1-s)^{m-1}e^{-\frac{cs}{1-s}}ds\right] \quad (A7)$$

However, letting $x=c(1-s)^{-1}$, we have

$$\begin{aligned}\int_0^1 s(1-s)^{m-1}e^{-\frac{cs}{1-s}}ds &= \int_c^\infty\left(\frac{x-c}{x}\right)\left(\frac{c}{x}\right)^{m-1}e^{-(x-c)}\frac{cdx}{x^2}\\ &= e^c\left[c^m\int_c^\infty\frac{e^{-x}}{x^{m+1}}dx-c^{m+1}\int_c^\infty\frac{e^{-x}}{x^{m+2}}dx\right]\\ &= e^c\left[\frac{c^m}{m!}I_m(c)-\frac{c^{m+1}}{(m+1)!}I_{m+1}(c)\right]. \qquad (A8)\end{aligned}$$

Substituting (A6)-(A8) into (A5) gives (6) as desired.

Remark 3. See Tamaki (2016) for more detail of the problems considered here.

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