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Chaos in randomly perturbed dynamical systems*

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1 Introduction

We consider systems of the form

$$\dot{x} = f(x) + \varepsilon(b(x)\eta(t) + c(x)), \quad x \in \mathbb{R}^n, \tag{1}$$

where $0 < \varepsilon \ll 1$ and $f, b, c : \mathbb{R}^n \to \mathbb{R}^n$ are C^N $(N \geq 2)$ with $f(0), b(0), c(0) = 0$ and $\mathrm{D}b(0) = 0$. Here $\eta(t)$ is a scalar stationary Gaussian process such that

$$\mathbb{E}[\eta(t)] = 0, \quad \mathbb{E}[\eta(t)\eta(t+\tau)] = r(\tau),$$

where $r : \mathbb{R} \to \mathbb{R}$ represents its autocorrelation function. Moreover, we require $r(\tau)$ to be continuous and absolutely integrable on $(-\infty, \infty)$, so that its spectrum is continuous. This implies via Maruyama's theorem [9] (see also [3]) that $\eta(t)$ is ergodic, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(\eta(t))\mathrm{d}t = \mathbb{E}[\phi(\eta(t))] \quad \text{a.s.}$$

for any measurable function $\phi : \mathbb{R} \to \mathbb{R}$. A little stronger requirement for $r(\tau)$ is also made at $\tau = 0$. See Section 2. In addition, we take $r(0) = 1$ without loss of generality. Thus, Eq. (1) represents a random perturbation of the deterministic system

$$\dot{x} = f(x). \tag{2}$$

We also assume that the origin $x = 0$ is a hyperbolic saddle with an isolated homoclinic orbit in the unperturbed system (2).

When $\eta(t)$ is a deterministic function, dynamical systems of the form (1) have been studied extensively. Especially, a global perturbation technique called *Melnikov's method* [10] was applied or extended to discuss chaotic dynamics of those systems. See, e.g., [4, 10, 12] for the periodic case, [15, 17] for the quasiperiodic case, and [7, 14] for the general, aperiodic case. In each case one computes an integral called the *Melnikov function* or *integral* to obtain conditions for the existence of chaos. Moreover, special bounded and

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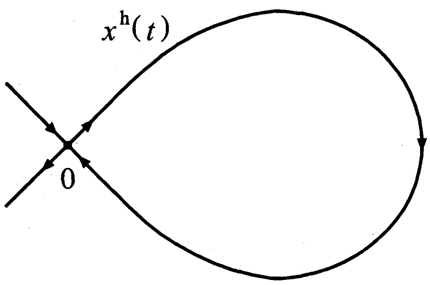

Figure 1: Assumption (A4).

unbounded random perturbations of two-dimensional systems were discussed by using similar approaches in [7] and [8], respectively. The latter case is included in our system (1) with

$$r(\tau) = \max\left(1 - \frac{|\tau|}{\Delta}, 0\right), \quad c(x) \equiv 0,$$

if $0 < \varepsilon/\sqrt{\Delta} \ll 1$ is replaced with ε, where Δ is a small positive constant.

In this article we review a recent result of [18] which shows that chaotic dynamics occurs almost surely in the general randomly perturbed systems of the form (1). This result is very contrast to the deterministic case, in which chaotic orbits exist only if the influence of $b(x)\eta(t)$ overcomes that of $c(x)$ in the perturbations. The approach used there is similar to that of [8] but a nice probabilistic property of the corresponding Melnikov functions is utilized. See [18] for the details and proofs. We also remark that randomly perturbed systems similar to (1) were also discussed in [5, 13] much earlier although such a fact was completely untouched and the treatments had a lack of mathematical rigor there.

2 Setup

As stated in Section 1, we first assume the following:

(A1) $f(0), b(0), c(0) = 0$ and $\mathrm{D}b(0) = 0$.

(A2) The autocorrelation function $r(\tau)$ for $\eta(t)$ is continuous and absolutely integrable on $(-\infty, \infty)$, and satisfies

$$1 - r(\tau) \le C|\tau|^\alpha \quad \text{as } \tau \to 0,$$

where $C, \alpha > 0$ are constants. Especially, $r(0) = 1$.

Assumption (A1) means that $x = 0$ is a constant solution to (1) for any $\varepsilon > 0$. By assumption (A2) $\eta(t)$ is continuous (and actually satisfies a Hölder condition) with probability one. See Section 9.2 of [2].

We make the following assumptions on the unperturbed system (2):

(A3) The origin $x = 0$ is a hyperbolic saddle equilibrium and the Jacobian matrix $\mathrm{D}f(0)$ has n_s and n_u eigenvalues with negative and positive real parts, respectively, such that $n_\mathrm{s} + n_u = n$.

(A4) The equilibrium $x = 0$ has a homoclinic orbit $x^{\mathrm{h}}(t)$, i.e., $\lim_{t\to\pm\infty} x^{\mathrm{h}}(t) = 0$. See Fig. 1.

Assumption (A3) and (A4) mean that the saddle $x = 0$ has n_{s}- and n_{u}-dimensional, stable and unstable manifolds, denoted by W_0^{s} and W_0^{u}, respectively, in (2), and W_0^{s} and W_0^{u} intersect along the homoclinic orbit $x = x^{\mathrm{h}}(t)$.

Consider the variational equation (VE) of (2) along $x^{\mathrm{h}}(t)$,

$$\dot{\xi} = \mathrm{D}f(x^{\mathrm{h}}(t))\xi, \quad \xi \in \mathbb{R}^n. \tag{3}$$

Obviously, $\xi = \dot{x}^{\mathrm{h}}(t)$ is a bounded solution of (3) with

$$\lim_{t\to\pm\infty} \dot{x}^{\mathrm{h}}(t) = 0.$$

We also assume the following on the VE (3).

(A5) Eq. (3) has no bounded solution that is independent of $\xi = \dot{x}^{\mathrm{h}}(t)$.

It follows from (A5) that

$$\dim(T_x W_0^{\mathrm{s}} \cap T_x W_0^{\mathrm{u}}) = 1$$

along $x = x^{\mathrm{h}}(t)$, $t \in \mathbb{R}$.

We turn to the randomly perturbed system (1) and give some preliminaries. We recommend the readers to refer to [1] for a general framework of our treatments if they are unfamiliar.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a canonical probability space, where the sample space is given by $\Omega = C(\mathbb{R}, \mathbb{R})$, $\mathscr{F}$ is the Borel σ-algebra of Ω, and $\mathbb{P}$ is a probability measure determined by the finite dimensional distribution of $\eta(t)$. According to the standard recipe [1], define a $\mathbb{P}$-preserving measurable flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ with $\theta_t : \Omega \to \Omega$ as

$$\theta_t\omega(\tau) = \omega(t + \tau)$$

for (1), where $\omega \in \Omega$ and $t, \tau \in \mathbb{R}$. We immediately see that

(i) $\theta_0 = \mathrm{id}$;

(ii) $\theta_t\theta_\tau = \theta_{t+\tau}$ for $t, \tau \in \mathbb{R}$;

(iii) $\theta_t\mathbb{P} = \mathbb{P}$ for $t \in \mathbb{R}$.

Let $D_1 \subset D_2 \subset \mathbb{R}^n$ be regions containing the homoclinic orbit $x^{\mathrm{h}}(t)$, i.e.,

$$D_j \supset \{x^{\mathrm{h}}(t) \mid t \in \mathbb{R}\} \cup \{0\}, \quad j = 1, 2,$$

and let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ bump function such that $0 \le \chi(x) \le 1$ for any $x \in \mathbb{R}^n$ and

$$\chi(x) = \begin{cases} 1 & \text{for } x \in D_1; \\ 0 & \text{for } x \in \mathbb{R}^n \setminus D_2. \end{cases}$$

Consider

$$\dot{x} = \tilde{f}(x) + \varepsilon(\tilde{b}(x)\eta(t) + \tilde{c}(x)), \tag{4}$$

where

$$\tilde{f}(x) = f(x)\chi(x), \quad \tilde{c}(x) = c(x)\chi(x), \quad \tilde{b}(x) = b(x)\chi(x).$$

Note that orbits of (4) are also those of (1) if they remain in the region D_1.

For given initial conditions, Eq. (4) has unique global solutions that are C^r about the initial values. See, e.g., [1] for the proof. We write the unique global solution with $x(0) = x_0 \in \mathbb{R}^n$ as $x = \varphi_\varepsilon(t,\omega)x_0$, $\omega \in \Omega$, and define a C^r global random dynamical system $\varphi_\varepsilon(t,\omega) : \mathbb{R}^n \to \mathbb{R}^n$ over θ, which satisfies a cocycle property:

(i) $\varphi_\varepsilon(0,\omega) = \mathrm{id}$;

(ii) $\varphi_\varepsilon(t+\tau,\omega) = \varphi_\varepsilon(t,\theta_\tau\omega)\varphi_\varepsilon(\tau,\omega)$ for $t,\tau \in \mathbb{R}$,

where $\omega \in \Omega$. In general, a random valuable $\bar{x}(\omega)$ satisfying

$$\varphi_\varepsilon(t,\omega)\bar{x}(\omega) = \bar{x}(\theta_t\omega) \quad \text{a.s.} \quad \text{for } t \in \mathbb{R}.$$

is called a *stationary solution* for (4). Since $f(0), b(0), c(0) = 0$ by assumption (A1), $\bar{x}(\omega) \equiv 0$ is a stationary solution.

Henceforth we write Ω_1 for some events whose probability is one, i.e., $\Omega_1 \in \mathscr{F}$ and $\mathbb{P}(\Omega_1) = 1$, by abuse of nomenclature.

3 Existence of transverse homoclinic orbits

Let E_0^{s} and E_0^{u} be, respectively, the stable and unstable subspaces of the linearized system at $x = 0$ for (2),

$$\dot{\xi} = \mathrm{D}f(0)\xi.$$

We have the following result for (4).

Theorem 1. *Let $\omega \in \Omega_1$. For any $T > 0$ fixed, there exists a bi-infinite sequence $\{q_j(\omega)\}_{j=-\infty}^{\infty}$, such that for $\varepsilon > 0$ sufficiently small, when $q \in [q_j(\omega) - T, q_j(\omega) + T]$, there exists n_{s}- and n_{u}-dimensional C^N manifolds, $W_{\varepsilon,q}^{\mathrm{s}}(\omega)$ and $W_{\varepsilon,q}^{\mathrm{u}}(\omega)$, which are $\mathscr{O}(\varepsilon)$-close to W_0^{s} and W_0^{u}, respectively, and satisfy the following properties:*

(ia) *$\varphi_\varepsilon(t,\theta_q\omega)x$ exponentially tends to 0 as $t \to \infty$ for $x \in W_{\varepsilon,q}^{\mathrm{s}}(\omega)$;*

(ib) *$\varphi_\varepsilon(t,\theta_q\omega)x$ exponentially tends to 0 as $t \to -\infty$ for $x \in W_{\varepsilon,q}^{\mathrm{u}}(\omega)$;*

(ii) *$W_{\varepsilon,q}^{\mathrm{s,u}}(\omega)$ are continuous in q;*

(iiia) *$\varphi_\varepsilon(t,\theta_q\omega)W_{\varepsilon,q}^{\mathrm{s}}(\omega) \subset W_{\varepsilon,q}^{\mathrm{s}}(\theta_t\omega)$ for $t+q \in [q_k(\omega) - T, q_k(\omega) + T]$ with $k \geq j$;*

(iiib) *$\varphi_\varepsilon(t,\theta_q\omega)W_{\varepsilon,q}^{\mathrm{u}}(\omega) \subset W_{\varepsilon,q}^{\mathrm{u}}(\theta_t\omega)$ for $t+q \in [q_k(\omega) - T, q_k(\omega) + T]$ with $k \leq j$;*

(iv) *For some constant $\delta > 0$ independent of $\varepsilon > 0$ and $\omega \in \Omega_1$, there exist C^N functions $h_{\varepsilon,q}^{\mathrm{s}} : E_0^{\mathrm{s}} \times \Omega_1 \to E_0^{\mathrm{u}}$ and $h_{\varepsilon,q}^{\mathrm{u}} : E_0^{\mathrm{u}} \times \Omega_1 \to E_0^{\mathrm{s}}$ such that*

$$W_{\varepsilon,q}^{\mathrm{s}}(\omega) \cap B_\delta = \{(s,u) \in (E_0^{\mathrm{s}} \times E_0^{\mathrm{u}}) \cap B_\delta \mid u = h_{\varepsilon,q}^{\mathrm{s}}(s,\omega)\}$$

and

$$W_{\varepsilon,q}^{\mathrm{u}}(\omega) \cap B_\delta = \{(s,u) \in (E_0^{\mathrm{s}} \times E_0^{\mathrm{u}}) \cap B_\delta \mid s = h_{\varepsilon,q}^{\mathrm{u}}(u,\omega)\},$$

where B_δ represents a closed ball centered at the origin with radius δ in $\mathbb{R}^n$, $h^{\mathrm{s,u}}_{\varepsilon,q}(0,\omega) = 0$ and $\mathrm{D}_s h^{\mathrm{s}}_{\varepsilon,q}(0,\omega), \mathrm{D}_u h^{\mathrm{u}}_{\varepsilon,q}(0,\omega) = \mathscr{O}(\varepsilon)$. Moreover, $h^{\mathrm{s}}_{\varepsilon,q}(s,\omega)$ and $h^{\mathrm{u}}_{\varepsilon,q}(u,\omega)$ are continuous in q, and C^N in ε as well as in s or u with bounded k-th order derivatives having bounds independent of $\omega \in \Omega_1$, $k = 1,\dots,N$.

To prove Theorem 1 a classical result on extreme values of Gaussian processes [11] is required in [18]. We refer to $W^{\mathrm{s}}_{\varepsilon,q}(\omega)$ and $W^{\mathrm{u}}_{\varepsilon,q}(\omega)$, respectively, as *stable and unstable manifolds at* $t = q$ for (1), when $\varepsilon > 0$ is sufficiently small.

If $W^{\mathrm{s}}_{\varepsilon,q}(\omega)$ and $W^{\mathrm{u}}_{\varepsilon,q}(\omega)$ intersect at $x \neq 0$, then Eq. (1) has a *homoclinic orbit* $x_\varepsilon(t,\omega)$ to the stationary solution $x = 0$, i.e.,

$$\lim_{t\to\pm\infty} x_\varepsilon(t,\omega) = 0.$$

We say that the homoclinic orbit $x_\varepsilon(t,\omega)$ is *transverse* if so is the intersection between $W^{\mathrm{s}}_{\varepsilon,q}(\omega)$ and $W^{\mathrm{u}}_{\varepsilon,q}(\omega)$. Now we state our main theorem.

Theorem 2. *For $\omega \in \Omega_1$ and $\varepsilon > 0$ sufficiently small, Eq. (1) has infinitely many transverse homoclinic orbits $x^j_\varepsilon(t,\omega)$, $j \in \mathbb{Z}$, such that $x^j_\varepsilon(t_j(\omega),\omega)$ lies in an $\mathscr{O}(\varepsilon)$-neighborhood of $x^{\mathrm{h}}(0)$ with $t_j(\omega) < t_{j+1}(\omega)$ for $j \in \mathbb{Z}$ and $\lim_{j\to\pm\infty} t_j(\omega) = \pm\infty$.*

A Melnikov-type approach and a classical result on level crossings of stochastic processes [2, 6] are used to prove Theorem 2 in [18]. We take the bi-infinite sequence $\{t_j(\omega)\}_{j=-\infty}^{\infty}$ such that $t_j(\theta_t\omega) = t_j(\omega) - t$, $j \in \mathbb{Z}$, for $t \in \mathbb{R}$.

4 Chaotic dynamics

Take the point $x^{\mathrm{h}}(0)$ such that it is $\mathscr{O}(1)$-distant from ∂B_δ, where $\delta > 0$ is sufficiently small as in Theorem 1. Let $T^\pm_\delta$ be time such that $T^-_\delta < 0 < T^+_\delta$, $x^{\mathrm{h}}(T^\pm_\delta) \in \partial B_\delta$ and $x^{\mathrm{h}}(t) \in B_\delta$ for $t \notin (T^-_\delta, T^+_\delta)$. We have

$$|T^\pm_\delta| = \mathscr{O}(|\log\delta|).$$

We choose a subsequence $\{\tau_j(\omega)\}_{j=-\infty}^{\infty}$ from the bi-infinite sequence $\{t_j(\omega)\}_{j=-\infty}^{\infty}$ given in Theorem 2, such that

$$\tau_{j+1}(\omega) - \tau_j(\omega) > T^+_\delta - T^-_\delta, \quad j \in \mathbb{Z}.$$

Note that $\tau_{j+1}(\theta_t\omega) - \tau_j(\theta_t\omega) = \tau_{j+1}(\omega) - \tau_j(\omega)$, $j \in \mathbb{Z}$, for $t \in \mathbb{R}$ since $t_j(\theta_t\omega) = t_j(\omega) - t$.

Let $a = \{a_j\}_{j=-\infty}^{\infty}$ denote a bi-infinite sequence with $a_j = 1$ or 2, $j \in \mathbb{Z}$. We denote the set of all such symbol sequences by Σ_2. Let $\sigma : \Sigma_2 \to \Sigma_2$ denote the shift map such that

$$\sigma(a)_j = a_{j+1}, \quad j \in \mathbb{Z}.$$

Define the *extended shift map* $\bar\sigma : \Sigma_2 \times \mathbb{Z} \to \Sigma_2 \times \mathbb{Z}$ as

$$\bar\sigma(a,j) = (\sigma(a), j+1).$$

Let $P_{\varepsilon,j}(\omega) = \varphi_\varepsilon(\tau_{j+1}(\omega) - \tau_j(\omega), \theta_{\tau_j(\omega)}\omega)$ and let

$$P_\varepsilon(\omega) : (x,j) \mapsto (P_{\varepsilon,j}(\omega)(x), j+1).$$

We now state our another main result.

Theorem 3. *For $\omega \in \Omega_1$ and $\varepsilon > 0$ sufficiently small, there exists a sequence of sets $\Lambda_j(\omega) \subset \mathbb{R}^n$, $j \in \mathbb{Z}$, with $P_j^\varepsilon(\omega)\Lambda_j(\omega) = \Lambda_{j+1}(\omega)$, such that the following diagram commutes*

$$\begin{array}{ccc} \Lambda(\omega) & \xrightarrow{P^\varepsilon} & \Lambda(\omega) \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ \Sigma_2 \times \mathbb{Z} & \xrightarrow{\bar{\sigma}} & \Sigma_2 \times \mathbb{Z} \end{array}$$

where $\Lambda_j(\omega)$, $j \in \mathbb{Z}$, are Cantor sets, $\Lambda(\omega) = \bigcup_{j=-\infty}^{\infty} \Lambda_j(\omega) \times \{j\}$, and $h(x;j) = (h_j(x), j)$ with $h_j(x)$ a homeomorphism mapping $\Lambda_j(\omega)$ onto Σ_2 such that the sequence $\{h_j^{-1}(x)\}_{j=-\infty}^{\infty}$ is equicontinuous.

The proof of Theorem 3 in [18] includes an extension of [16] for a description of chaos in the dynamics generated by sequences of maps. This theorem also implies that each orbit passing $\Lambda_j(\omega)$ at $t = \tau_j(\omega)$, $j \in \mathbb{Z}$, is unstable (of saddle type) and exhibits sensitive dependence on initial conditions.

5 Example

To illustrate the above theory we consider the randamoly perturbed Duffing oscillator,

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - x_1^3 + \varepsilon(x_1^2\eta(t) - \delta x_2), \tag{5}$$

where $\delta > 0$ is a constant and $\eta(t)$ is the stationary Ornstein-Uhlenbeck process with zero mean and

$$r(\tau) = \exp(-\gamma|\tau|)$$

with $\gamma > 0$ a constant. Similar systems were treated in [7, 8]. Assumptions (A1)-(A5) hold and the unperturbed homoclinic orbits are given by

$$x_\pm^{\mathrm{h}}(t) = (\pm\sqrt{2}\,\mathrm{sech}\,t, \mp\sqrt{2}\,\mathrm{sech}\,t\,\tanh t).$$

Applying Theorems 2 and 3, we show that there exist infinitely many transverse homoclinic orbits and chaotic dynamics occurs almost surely in (5) for any $\delta > 0$.

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