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THE SYMMETRIC INVARIANTS OF CENTRALIZERS AND FINITE W-ALGEBRAS

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1. Introduction

1.1. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero, let $\langle .,. \rangle$ be the Killing form of $\mathfrak{g}$ and let G be the adjoint group of $\mathfrak{g}$. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, we denote by $S(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$. Let $x \in \mathfrak{g}$ and denote by $\mathfrak{g}^x$ and G^x the centralizer of x in $\mathfrak{g}$ and G respectively. Then $\mathrm{Lie}(G^x) = \mathrm{Lie}(G_0^x) = \mathfrak{g}^x$ where G_0^x denotes the identity component of G^x. Moreover, $S(\mathfrak{g}^x)$ is a $\mathfrak{g}^x$-module and $S(\mathfrak{g}^x)^{\mathfrak{g}^x} = S(\mathfrak{g}^x)^{G_0^x}$. An interesting question, first raised by A. Premet, is the following:

Question 1. *Is the algebra $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ polynomial algebra in ℓ variables?*

In order to answer this question, thanks to the Jordan decomposition, one can assume that x is nilpotent. Besides, if $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial for some $x \in \mathfrak{g}$, then it is so for any element in the adjoint orbit $G(x)$ of x. If $x = 0$, it is well-known since Chevalley that $S(\mathfrak{g}^x)^{\mathfrak{g}^x} = S(\mathfrak{g})^{\mathfrak{g}}$ is polynomial in ℓ variables. At the opposite extreme, if x is a regular nilpotent element of $\mathfrak{g}$, then $\mathfrak{g}^x$ is abelian of dimension ℓ, [DV69], and $S(\mathfrak{g}^x)^{\mathfrak{g}^x} = S(\mathfrak{g}^x)$ is polynomial in ℓ variables too.

Let us say most simply that $x \in \mathfrak{g}$ *verifies the polynomiality condition* if $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra in ℓ variables.

A positive answer to Question 1 was suggested in [PPY07, Conjecture 0.1] for any simple $\mathfrak{g}$ and any $x \in \mathfrak{g}$. O. Yakimova has since discovered a counter-example in type $\mathbf{E}_8$, [Y07], disconfirming the conjecture. More precisely, the elements of the minimal nilpotent orbit in $\mathbf{E}_8$ do not verify the polynomiality condition. We provide here another counter-example in type $\mathbf{D}_7$ (cf. Proposition 1). In particular, one cannot expect a positive answer to [PPY07, Conjecture 0.1] for the simple Lie algebras of classical type. Question 1 still remains interesting and is positive for a large number of nilpotent elements $e \in \mathfrak{g}$ as it is explained below.

1.2. We briefly review in this paragraph what has been achieved so far about Question 1. Recall that the *index* of a finite-dimensional Lie algebra $\mathfrak{q}$, denoted by $\mathrm{ind}\,\mathfrak{q}$, is the minimal dimension of the stabilizers of linear forms on $\mathfrak{q}$ for the coadjoint representation, (cf. [Di74]):

$$\mathrm{ind}\,\mathfrak{q} := \min\{\dim \mathfrak{q}^{\xi} \ ; \ \xi \in \mathfrak{q}^*\} \ \text{ where } \ \mathfrak{q}^{\xi} := \{x \in \mathfrak{q} \ ; \ \xi([x,\mathfrak{q}]) = 0\}.$$

By [R63], if $\mathfrak{q}$ is algebraic, i.e., $\mathfrak{q}$ is the Lie algebra of some algebraic linear group Q, then the index of $\mathfrak{q}$ is the transcendental degree of the field of Q-invariant rational functions on $\mathfrak{q}^*$. The following result will be important for our purpose.

Theorem 1 ([CM10, Theorem 1.2]). *The index of $\mathfrak{g}^x$ equals ℓ for any $x \in \mathfrak{g}$.*

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Theorem 1 was first conjectured by Elashvili in the 90' motivated by a result of Bolsinov, [B91, Theorem 2.1]. It was proven by O. Yakimova when $\mathfrak{g}$ is a simple Lie algebra of classical type, [Y06], and checked by a computer programme by W. de Graaf when $\mathfrak{g}$ is a simple Lie algebra of exceptional type, [DeG08]. Before that, the result was established for some particular classes of nilpotent elements by D. Panyushev, [Pa03].

Theorem 1 is deeply related to Question 1. Indeed, thanks to Theorem 1, [PPY07, Theorem 0.3] applies and by [PPY07, Theorems 4.2 and 4.4], if $\mathfrak{g}$ is simple of type **A** or **C**, then all nilpotent elements of $\mathfrak{g}$ verify the polynomiality condition. The result for the type **A** was independently obtained by Brown and Brundan, [BB09]. In [PPY07], the authors also provide some examples of nilpotent elements satisfying the polynomiality condition in the simple Lie algebras of types **B** and **D**, and a few ones in the simple exceptional Lie algebras.

More recently, the analogue question to Question 1 for the positive characteristic was dealt with by L. Topley for the simple Lie algebras of types **A** and **C**, [T12].

1.3. The main goal of this piece of work is to continue the investigations of [PPY07]. Let us describe the main results. The following definition is central in our work:

Definition 1. *An element $x \in \mathfrak{g}$ is called a* good element of $\mathfrak{g}$ *if for some homogeneous elements $p_1, \dots, p_\ell$ of* $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$*, the nullvariety of $p_1, \dots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$.*

For example, by [PPY07, Theorem 5.4], all nilpotent elements of a simple Lie algebra of type **A** are good, and by [Y09, Corollary 8.2], the even nilpotent elements of $\mathfrak{g}$ are good if $\mathfrak{g}$ is of type **B** or **C** or if $\mathfrak{g}$ is of type **D** with odd rank. We rediscover here these results in a more general setting. We also show that the good elements verify the polynomiality condition. Moreover, x is good if and only if its nilpotent component in the Jordan decomposition is so.

Let e be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov Theorem, e is embedded into a $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Denote by $\mathcal{S}_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with e*. Identify the dual of $\mathfrak{g}$ with $\mathfrak{g}$, and the dual of $\mathfrak{g}^e$ with $\mathfrak{g}^f$, through the Killing form $\langle .,. \rangle$. For p in $\mathrm{S}(\mathfrak{g}) \simeq \mathbf{k}[\mathfrak{g}^*] \simeq \mathbf{k}[\mathfrak{g}]$, denote by ${}^e p$ the initial homogeneous component of its restriction to $\mathcal{S}_e$. According to [PPY07, Proposition 0.1], if p is in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Our main result is the following:

Theorem 2. *Suppose that for some homogeneous generators $q_1, \dots, q_\ell$ of* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$*, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. Then e is a good element of* $\mathfrak{g}$*. In particular,* $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ *is a polynomial algebra and* $\mathrm{S}(\mathfrak{g}^e)$ *is a free extension of* $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$*. Moreover, ${}^e q_1, \dots, {}^e q_\ell$ is a regular sequence in* $\mathrm{S}(\mathfrak{g}^e)$.

Theorem 2 applies to a great number of nilpotent orbits in the simple classical Lie algebras, and for some nilpotent orbits in the exceptional Lie algebras.

To state our results for the simple Lie algebras of types **B** and **D**, let us introduce some more notations. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V}) \subset \mathfrak{gl}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or 2ℓ. For x an endomorphism of $\mathbb{V}$ and for $i \in \{1, \dots, \dim \mathbb{V}\}$, denote by $Q_i(x)$ the coefficient of degree $\dim \mathbb{V} - i$ of the characteristic polynomial of x. Then for any x in $\mathfrak{g}$, $Q_i(x) = 0$ whenever i is odd. Define a generating family $q_1, \dots, q_\ell$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i = 1, \dots, \ell - 1$, set $q_i := Q_{2i}$. If $\dim \mathbb{V} = 2\ell + 1$, set $q_\ell = Q_{2\ell}$ and if $\dim \mathbb{V} = 2\ell$, let q_ℓ be a homogeneous element of degree ℓ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell} = q_\ell^2$. Denote by $\delta_1, \dots, \delta_\ell$ the degrees of ${}^e q_1, \dots, {}^e q_\ell$ respectively. By [PPY07, Theorem 2.1], if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0,$$

then the polynomials ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. In that event, by Theorem 2, e is good and we will say that e is *very good*. The very good nilpotent elements of $\mathfrak{g}$ can be characterized in term of their associated partitions of $\dim \mathbb{V}$. Theorem 2 also enables to obtain examples of good, but not very good, nilpotent elements of $\mathfrak{g}$; for them, there are a few more work to do.

Thus, we obtain a large number of good nilpotent elements, including all even nilpotent elements in type **B**, or in type **D** with odd rank. For the type **D** with even rank, we obtain the statement for some particular cases.

On the other hand, there are examples of elements that verify the polynomiality condition but that are not good; for example, the nilpotent elements of $\mathfrak{so}(\mathbb{k}^{10})$ associated with the partition $(3,3,2,2)$ or the nilpotent elements of $\mathfrak{so}(\mathbb{k}^{11})$ associated with the partition $(3,3,2,2,1)$. To deal with them, we use different techniques, more similar to those used in [PPY07].

As a result of all this, we observe for example that all nilpotent elements of $\mathfrak{so}(\mathbb{k}^n)$, with $n \leqslant 8$, are good and that all nilpotent elements of $\mathfrak{so}(\mathbb{k}^n)$, with $n \leqslant 13$, verify the polynomiality condition. In particular, by [PPY07, §3.9], this provides with $n = 7$ examples of good nilpotent elements for which the codimension of $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ in $(\mathfrak{g}^e)^*$ is 1. Here, $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ stands for the set of nonregular linear forms $x \in (\mathfrak{g}^e)^*$, i.e.,

$$(\mathfrak{g}^e)^*_{\mathrm{sing}} := \{x \in (\mathfrak{g}^e)^* \ ; \ \dim(\mathfrak{g}^e)^x > \operatorname{ind}\mathfrak{g}^e = \ell\}.$$

For such nilpotent elements, note that [PPY07, Theorem 0.3] does not apply.

Our results do not cover all nilpotent orbits in type **B** and **D**. As a matter of fact, we obtain a counter-example in type **D** to Premet's conjecture:

Proposition 1. *The nilpotent elements of* $\mathfrak{so}(\mathbb{k}^{14})$ *associated with the partition* $(3,3,2,2,2,2)$ *do not satisfy the polynomiality condition.*

1.4. The main ingredient to prove Theorem 2 is the finite W-algebra associated with the nilpotent orbit $G(e)$ which we emphasize the construction below. Our basic reference for the theory of finite W-algebras is [Pr02]. For i in $\mathbb{Z}$, let $\mathfrak{g}(i)$ be the i-eigenspace of $\operatorname{ad} h$ and set:

$$\mathfrak{p}_+ := \bigoplus_{i \geqslant 0} \mathfrak{g}(i).$$

Then $\mathfrak{p}_+$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}^e$. Let $\mathfrak{g}(-1)^0$ be a totally isotropic subspace of $\mathfrak{g}(-1)$ of maximal dimension with respect to the nondegenerate bilinear form

$$\mathfrak{g}(-1) \times \mathfrak{g}(-1) \longrightarrow \mathbb{k}, \qquad (x, y) \longmapsto \langle e\,, [x, y]\rangle$$

and set:

$$\mathfrak{m} := \mathfrak{g}(-1)^0 \oplus \bigoplus_{i \leqslant -2} \mathfrak{g}(i).$$

Then $\mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}$ with a derived subalgebra orthogonal to e. Denote by $\mathbb{k}_e$ the one dimensional $\mathrm{U}(\mathfrak{m})$-module defined by the character $x \mapsto \langle e\,, x\rangle$ of $\mathfrak{m}$, denote by $\tilde{Q}_e$ the induced module

$$\tilde{Q}_e := \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{m})} \mathbb{k}_e$$

and denote by $\tilde{H}_e$ the associative algebra

$$\tilde{H}_e := \operatorname{End}_{\mathfrak{g}}(\tilde{Q}_e)^{\mathrm{op}},$$

known as the *finite W-algebra* associated with e. If $e = 0$, then $\tilde{H}_e$ is isomorphic to the enveloping algebra $\mathrm{U}(\mathfrak{g})$ of $\mathfrak{g}$. If e is a regular nilpotent element, then $\tilde{H}_e$ identifies with the center of $\mathrm{U}(\mathfrak{g})$. More generally, by [Pr02, §6.1], the representation $\mathrm{U}(\mathfrak{g}) \to \operatorname{End}(\tilde{Q}_e)$ is injective on the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. The algebra $\tilde{H}_e$ is endowed with an increasing filtration, sometimes referred as the *Kazhdan filtration*, and one of the main theorems of [Pr02] states that the corresponding graded algebra is isomorphic to the graded algebra $\mathrm{S}(\mathfrak{g}^e)$. Here, $\mathrm{S}(\mathfrak{g}^e)$ is graded by the *Slodowy grading*.

Our idea is to reduce the problem modulo p for a sufficiently big prime integer p, and prove the analogue statement to Theorem 2 in characteristic p. More precisely, we construct a Lie algebra $\mathfrak{g}_K$ from $\mathfrak{g}$ over an algebraically closed field K of characteristic $p > 0$. The key advantage is essentially that the analogue H_e of the finite W-algebra $\tilde{H}_e$ in this setting is of finite dimension.

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