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CONFUSION AND UNITY IN HANDLING OF HEAT MOTION AND FLUID MOTION IN THE 19TH CENTURY

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ABSTRACT.

We discuss the confusion and unity in handling of heat motion and fluid motion in the 19th century, explaining the theoretical background. These two theories or themes are studied by Fourier at the first time and lasted in the academic activities in accompany with the arrival of continuum. These two theories or themes are studied by Fourier at the first and and the last academic activity in the arrival of continuum. The other study on the communication of heat between two bodies problem has been researched by Prévost 1792 [29], which precedes Fourier's manuscript 1807 [11].

The hydrodynamists like Navier, Poisson, Cauchy are propose the wave equations in the elasticity, and the last two hydrodynamists proposes the total equations in unity on the continuum. From the viewpoint of mathematics on the continuum, we pick up Poisson's direct method for definite integral in regarding to the problems between real and imaginary, that is the life-work theme Euler and Laplace also struggled to solve it.

We point out this problem based on the then continuum concept, which is the bridge point over classical mechanics into classical quantum mechanics like Boltzmann, and moreover into new quantum mechanics like Schrödinger. Through this wide range as possible, we like to attention to mathematical aspect of Fourier and his surrondings, in the viewpoint of unity of heat and fluid on the continuum.

Our motivation in this paper is to consider the confusion and unity on the continuum from both the positive and negative mathematical viewpoints.

1. INTRODICTION

[1,2,3] How does the wave occur ? Newton 1686 [22] shows his principle on the wave motion in the water pressure.

> The pressure doesn't propagate by the fluid of the secondary linear strait, except for the particle of adjacent fluid. If the adjacent particles a, b, c, d, e propagate in the straight line, press from a to e ; the particle e progresses separately into the oblique points f and g, and without sustained pressure, and moreover, to the particles h and k ; m as it is fixed in another direction, it presses for the particle into propping up ; the unsustained pressure goes separately into the particles l and m, and as this way, it follows successively and limitlessly. thus it will occur so many time, inaccurately, to the particle in the indirect adjacency. Q.E.D. [22, pp.354-5] (trans. from Latin, mine.)

Date: 2015/01/07.

[1]Translation from Latin/French/German into English mine, except for Boltzmann.

[2]To establish a time line of these contributor, we list for easy reference the year of their birth and death: Newton (1643-1727), Euler(1707-83), d'Alembert(1717-83), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Poisson(1781-1840), Cauchy(1789-1857), Dirichlet(1805-59), Riemann(1826-66), Boltzmann(1844-1906), Hilbert(1862-1943), Schrödinger(1887-1961).

[3]We use ($\Downarrow$) means our remark not original, when we want to avoid the confusions between our opinion and sic. ($\Leftarrow$) means our translation in citing the origin.

What is the fluid ? According to today's diffinition, it is called the fluid is a *limitlessly free continuum.* Where does *continuum* come from in the historical view ?

2. THE ORIGIN OF EIGENVALUE PROBLEM

Euler 1748 [6] says the height of the vibrating cord is calculated by the linear, first-ordered expression as follows :

$$y = \alpha \sin \frac{\pi x}{a} + \beta \sin \frac{2\pi x}{a} + \gamma \sin \frac{3\pi x}{a} + \cdots \tag{1}$$

Lagrange 1759 [13] descrives as the introductional expression of the trigonometric series by P_ν and Q_ν as follows :

$$P_\nu \equiv Y_1 \sin \frac{\nu\varpi}{2m} + Y_2 \sin \frac{2\nu\varpi}{2m} + Y_3 \sin \frac{3\nu\varpi}{2m} + \cdots + Y_{m-1} \sin \frac{(m-1)\nu\varpi}{2m} \tag{2}$$

where, Q_ν has the same linear, first-ordered combination with coefficients $V_1, V_2, \cdots$ instead of $Y_1, Y_2, \cdots$. The indicies of P and Q show simply the *valeur particulières* (eigenvalues) of ν which (the *valeur particulières*) belong to them (P_ν and Q_ν, respectively). [13, pp.79-80] (trans. mine.) Remark. Lagrange's ϖ is equal to π. In (1), we can see in case we assume $a = 2m$ and $\alpha, \beta, \gamma, \cdots$ are equal to $Y_1, Y_2, Y_3, \cdots$, then $x = \nu$ in Lagrange's P_ν in (2) or Q_ν reffering to *valeur particulière* (eigenvalue), namely (1) = (2).

3. THE HEAT AND FLUID THEORIES IN THE 19TH CENTURY

3.1. The theory of heat communication in the Prévost's essay.

Prévost [29] discuss the communication of heat between two corps in earlier than Fourier, who corresponds with Prévost, according to Grattan-Guiness [11, p.23].

His principles are as follows : all the corps radiate the heat without relation to the temperature. The heat equilibrium is induced with the equal quantity of heat by the heat communication. These principles become shared with Fourier successively. (cf. Table 1.)

3.2. The outline of the situations surrounding Fourier.

About the situations around Fourier, we can summarize as follows :

1. Fourier's manuscript 1807, which had been unknown for us until 1972, I. Grattan-Guinness [11] discovered it. Fourier's paper 1812 based on the manuscript was prized by the academy of France. We consider that Fourier, in his life work of the heat theory, begins with the communication theory, and he devoted in establishing this theme as the priority.
2. Owing to the arrival of continuum theory, many mathematical physical works are introduced, such as that Fourier and Poisson struggle to deduce the trigonometric series in the heat theory and heat diffusion equations. In the curent of formularizing process of the fluid dynamics, Navier, Poisson, Cauchy and Stokes struggle to deduce the wave equations and the Navier-Stokes equations. Of cource, there are many proceding researches before these topics, however, for lack of space, we must pick up at least, the essentials such as following contents :
3. Fourier [9] combines heat theory with the Euler's equations of incompressible fluid dynamics and proposes the equation of heat motion in fluid in 1820, however, this paper was published in 1833 after 13 years, it was after 3 years since Fourier passed away. Fourier seems to have been doutful to publish it in life.
4. After Fourier's commnunication theory, the gas theorists like Maxwell, Kirchhoff, Boltzmann [2] study the transport equations with the concept of collision and transport of the molecules in mass. In both principles, we see almost same relation between the Fourier's communication and transport of heat molecules and the Boltzmann's collision and transport of gas molecules.
5. Since 1811, Poisson issued many papers on the definite integral, containing transcendental,

TABLE 1. The four books on physico-mathematical theories of heat

bibliography	Prévost 1792 [29] 1751-1839	Fourier 1808,11,22 [8] 1768-1830	Poisson 1823, 35 [25] 1781-1840	Poincaré 1895 [26] 1854-1912
title	*Rec ches physico-mécaniques sur la cheleur*	*Théorie analytique de la chaleur*	*Théorie mathématique de la chaleur*	*Théorie analytique de la propagation de la chaleur*
total page	232	[8] 541 except for contents	[25] 552	314
point/ merit	communication theory · only quantity of heat depends on the heat communication, not with temperature	· heat theory · trigonometric series	· check on Fourier's method mathematically with another method · proving convergence of series	introduction of · Dirichlet principle, · harmonic function · the methods by Fourier, Laplace, Cauchy, Riemann · analogous equations with heat such as the equation of cord vibration, telegraph
contributions	to Fourier's communication theory	· trigonometric series after followers until now · molecular theory to Navier	to Sturm and Liouville	to Hilbert and Fredholm · integral equation · harmonic function
remark	Poisson [25] introduces as an essay		Poincaré 1895 doesn't mention this one at all	Fredholm refers on Poincaré's harmonic function

and remarked on the necessity of careful handling to the diversion from real to imaginary, especially, to Fourier explicitly. To Euler and Laplace, Poisson owes many knowledge, and builds up his principle of integral, consulting Lagrange, Lacroix, Legendre, etc. On the other hand, Poisson feels incompatibility with Laplace's 'passage', on which Laplace had issued a paper in 1809, entitled : On the 'reciprocal' passage of results between real and imaginary. in 1782-3.

6. To these passages, Poisson proposed the direct, double integral in 1811, 13, 15, 20 and 23. The one analytic method of Poisson 1811 is using the round braket, contrary to the Euler's integral 1781. The multiple integral itself was discussed and practical by Laplace in 1782, about 20 years before, when Poisson applied it to his analysis in 1806.

7. As a contemporary, Fourier is made a victim by Poisson. To Fourier's main work : *The analytical theory of heat* in 1822, and to the relating papers, Poisson points the diversion applying the what-Poisson-called-it 'algebraic' theorem of De Gua or the method of cascades by Roll, to transcendental equation. Moreover, about their contrarieties, Darboux, the editor of *Œuvres de Fourier*, evaluates on the correctness of Poisson's reasonings in 1888. Drichlet also mentions about Fourier's method as a sort of *singularity of passage* from the finite to the infinite.

3.3. The preliminary discources on Fourier from the Nota to I.Grattan-Guinness.

3.3.1. *The Fourier's Oeuvres edited by G. Darboux.*
The preliminary discource by Fourier, edited by G. Barboux, says in 1820 :

> G. Darboux says in his first edition in 1888 : The works relating to the heat theory by Fourier appear in the late 18C. It has been submitted to the Academy of

Science, in Dec. 21, 1807. his first publication is unknown for us : we don't know except for an extract of 4 pages of BSP in 1808 ; It was read by the Committee, however, may be withdrawn by Fourier during 1810. The Committee of Academy, held in 1811, decided the following judgment : " Make clear the mathematical theory on the propagation of heat, and compare this theory with the exact result of experiments." (trans. mine.) [4]

3.3.2. *The Fourier 1822 by A. Freeman and The Fourier 1807 edited by I. Grattan-Guinness.*

In 1878, A. Freeman published the first English translated Fourier's second version, of which the preliminary is completely the same as G. Darboux 1888, ten years later than A. Freeman. In 1972, I. Grattan-Guinness discovered the manuscript 1807. He pays attentions to the Avertissment in the second edition by G. Darboux as above we mention. We are thankful to Grattan-Guinness for the showing one of the paragraph of ¶.136 (Des températures finales et de la courbe qui les présente.), and its belonging figure[5] of the Fourier's Manuscript 1807, *Théorie de la propagation de la chaleur*, edited and commented by Grattan-Guinness [11, p.371-2].

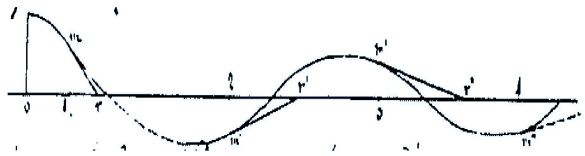

fig.1 An exponential decay of diffusion in Fourier's Manuscript 1807

4. THE THEORETICAL CONTRARIETIES TO FOURIER

4.1. Lagrange and Fourier on the trigonometric series.

Riemann studies the history of research on Fourier series up to then (*Geschichte der Frage über die Darstellbarkeit einer willkührlich gegebenen Function durch eine trigonometrische Reihe*, [30, pp.4-17].) We cite one paragraph of his interesting description from the view of mathematical history as follows :

> (⇐) When Fourier submitted his first work to the Academy française[6] (21, Dec., 1807) on the heat, representing a completely arbitrary (graphically), given functions with the trigonometric series, at first, gray-haired Lagrange[7] irritates so much, however, refuses flatly. The paper is called now being belonged to the Arcive of the Parisian Academy française. (id. According to Mr. Professor Dirichlet's oral presentation.) Therefore, after Poisson inspects carefully through the paper,[8] promptly argues that in the paper of Lagrange, there is a paragraph on the vibration of string, where Fourier may have discovered the descriptive method.[9] To refuse this defect of the statement telling clearly on the rivalry relation between Fourier and Poisson, we would like to back to the Lagrange's

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4(⇓) About the extract, same as above footnote. Lagrange was a member of the Committee of judgement and poses against Fourier's paper 1807. cf [30]. G.Darboux lists as follows : Lagrange, Laplace, Malus, Haüe and Legendre. [4, p.vii].

5This figure is the Fourier's original. [11, p.370]. In this figure, on the x axis, there are the numbers 1, 2, 3, 4

6(⇓) i.e. French Academy.

7(⇓) Lagrange was then seventy-one years old.

8id.

9id.

papers, so we can reach the event in the Academy nothing have been clear yet. [30, p.10] (trans. mine.)

Riemann cites exactly the French original as follows :

($\Leftarrow$) In fact, a paragraph cited by Poisson is the expression :

$$y = 2\int Y \sin X\pi dX \ \sin x\pi + 2\int Y \sin 2X\pi dX \ \sin 2x\pi + \cdots + 2\int Y \sin nX\pi dX \ \sin nx\pi, \tag{3}$$

So, If $x = X$, then $y = Y$, and Y is the ordinate confronting to the abscissa X. This formula doesn't coincide with the Fourier's series[10] ; there is sufficiently the capability of some mistake; however, it is only a simple outlook, because Lagrange uses $\int dx$ as the integral symbol. Today, it is to be used by $\sum \Delta X$. When we inspect through his papers, it is beyond believable that he expresses a completely arbitrary function by series expansion with infinite sins. [30, pp.10-11] (trans. mine.)

Lagrange had stated (3) in his paper of the motion of sound in 1762-65. [14, p.553]

4.2. The trials to seek the mathematical rigours on heat theories.

Poisson [23] traces Fourier's work of heat theory, from the another point of view. Poisson emphasizes, in the head paragraph of his paper, that although he totally takes the different approaches to formulate the heat differential equations or to solove the various problems or to deduce the solutions from them, the results by Poisson are coincident with Fourier's. Poisson [23] considers the proving on the convergence of series of periodic quantities by Lagrange and Fourier as the manner lacking the exactitude and vigorousness, and wants to make up to it. Poisson proposes the different and complex type of heat equation with Fourier's. For example, we assume that interior ray extends to sensible distance, which forces of heat may affect the phenomena, the terms of series between before and after should be differente.

We remark that Fourier's integral problems are handled in the scope on the infinite solid in Fourier 1822 [8]. We must pay attention to that these considerations have been capable on the continuum theory.

Poincaré 1895 [26] proves the existence of the function satisfying the Dirichlet condition :

Thèoréme. - Si une fonction $f(x)$ satisfait à la condition de Dirichlet dans l'intervalle $(-\pi, \pi)$, elle pourra être reprèsentée dans ce même intervalle par une sêrie de Fourier, c'est-à-dire que l'on aura :

$$\pi f(x) = \frac{1}{2}\int_{-\pi}^{\pi} f(x)\, dx + \sum \cos mx \int_{-\pi}^{\pi} f(x) \cos mx \, dx + \sum \sin mx \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

[26, p.57, §38] (cf. Table 1.)

5. Confusions and unify on continuum theory

The hysico-mathematicians are must construct at first the physical structure, then allpies the mathematical concept on it. The former is necessary to fit with the actual phenomena. Arago 1829 [1] seeks to separate these items to Navier 1829 [21] in the current of dispute with Poisson and Arago. This is comes from the word what-Navier-called *l'une sur l'autre*,he fails to explain exactly it, and since then, his theories and the equations are neglected up to the top of 20th century. We consider that the confusions and unify are as follows :

- Poisson and Fourier discuss on the handling of De Gua's theory into the transcendental equations. Without clear explanation, Fourier passed away in 1830.

10($\Downarrow$) This means two interpretations : one means the series by Fourier, the other today's conventionally used nomenclature : 'the Fourier series'. Judging from Riemann's young days, in 1867, this may mean the former. In generally, the trigonometric series is used then.

- On the attraction and replusion of molecule, Navier depends on Fourier's principle of heat molecule. The then hysico-mathematicians had little evaluated Navier until the top of 20th century. For formulation of heat motion in the fluid, Fourier cites not Navier's fluid equations, but Euler's fluid equations.
- The hydrodynamists like Navier, Poisson, Cauchy are propose the wave equations in the elasticity, and the last two hydrodynamists proposes the total equations in unity on the continuum.
- On the formulation of heat motion in the fluid, Fourier had submitted this paper, however, until his death, he has not published it, in which he seems to aim the unity of hydro- and thermodynamics, however, he has given up it.

5.1. **A comment on continuum by Duhamel.**

Duhamel 1829 [5] points out the theory of continuum from the viewpoint of scientific history.

> ($\Leftarrow$) The forces against the separation of composition of corps whether rigid or solid, are zero or doesn't exist in the state we discuss. It doesn't start to occur it only when we dare to separate it, or alternate the distance between the molecules. Namely, if we represent this forces by integral, when force turns into zero, even after the alternation of the molecular distance, say, if the body is separeting, it means that the body doesn't resist at all against the separation, this is unthinkable so. (J4-2)
>
> We will talk about the fact that as the same way in 1821, when Mr. Navier proposes the molecular activities and regards the body as the countinuum, Mr. Poisson also has gotten the same idea.
>
> This method inspecting the molecular actions are originally used in the study of capillary phenomena by Mr. Laplace. Mr. Navier has gotten afterward the nice idea to deduce the theory of elastic solid ; however, all the researchers and physico-mathematicians have supposed the molecules adjoining corps, and Poisson is the first of coincidence with calculations with the physical structures. (J5-1)
>
> In addition to, although the hypotheses of continuum theory have been actually so inexact, however, have played big roles in the science, In the roles, have played, the theories by M. Laplace have welcomed by the researchers. This observation on the molecular activities, in the bulk of special problems, above all, in the continuum theory, it has the very countless merits to have to sweep out the all special hypotheses. (J5-2) [5, p.99] (trans. mine.)

5.2. **Attraction and repulsion.**

Here, we show one of Fourier's contexts which Navier depends on and esteems as the authority of hysico-mathematicians.

> ($\Leftarrow$) ¶54. The equilibre which keeps in the interior of a solid mass between the *repulsive force* due to heat and the molecular *attraction* is stable ; namely, which restablish by iiself, when it troubles by an accidental cause. If the molecules are places in the distance which is convenient to the equilibre, and if an exterior force make this distance without the temperature changes by the heat, the effect of *attraction* begins surpass it and makes the molecules at the initial position, after a multitude of oscilation which becomes more and more insensible. A resemble effect operates when a mechanic cause shortens the initial distance of the

TABLE 2. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" were fixed. (HD : hydrodynamics, N : non-linear, g.d : grad.div, $C : \frac{\Delta}{gr.dv}$ in elastic or fluid. Δ : tensor coefficient of the main axis in Laplacian.)

no	name/ prob	the kinetic equations	Δ	g.d	C
1 N	Euler (1752 -55)[7] fluid	$\begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{cases} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$			
2	Lagrange (1782)[15] fluid	$\begin{cases} \Delta\left[\left(\frac{d^2x}{dt^2} + X\right)\frac{\partial x}{\partial a} + \left(\frac{d^2y}{dt^2} + Y\right)\frac{\partial y}{\partial a} + \left(\frac{d^2z}{dt^2} + Z\right)\frac{\partial z}{\partial a}\right] - \frac{\partial\lambda}{\partial a} = 0, \\ \Delta\left[\left(\frac{d^2x}{dt^2} + X\right)\frac{\partial x}{\partial b} + \left(\frac{d^2y}{dt^2} + Y\right)\frac{\partial y}{\partial b} + \left(\frac{d^2z}{dt^2} + Z\right)\frac{\partial z}{\partial b}\right] - \frac{\partial\lambda}{\partial b} = 0, \\ \Delta\left[\left(\frac{d^2x}{dt^2} + X\right)\frac{\partial x}{\partial c} + \left(\frac{d^2y}{dt^2} + Y\right)\frac{\partial y}{\partial c} + \left(\frac{d^2z}{dt^2} + Z\right)\frac{\partial z}{\partial c}\right] - \frac{\partial\lambda}{\partial c} = 0 \end{cases}$ where, $\mathbf{a} = (a, b, c)$: position on $t = 0$, $\mathbf{X} = (x,\ y,\ z)$: position on $t = t$, $\mathbf{X} = (X,\ Y,\ Z)$: outer force, Δ : density, λ : pressure. Today, these are described as follows : $\rho\sum_{j=1}^{3}\frac{\partial x_j}{\partial a_i}\left(\frac{\partial^2 x_j}{\partial t^2} - K_j\right) = -\frac{\partial p}{\partial a_i},\ (i = 1, 2, 3),$			
3	Navier (1827) [19] elastic sol.	$(6\text{-}1)_{Ne} \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases}$ where Π is density of the solid, g is acceleration of gravity.	ε	2ε	$\frac{1}{2}$
4	Navier (1827) [20] fluid	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) \\ \quad - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) \\ \quad - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) \\ \quad - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w \end{cases}$	ε	2ε	$\frac{1}{2}$
5	Poisson ('31)[24] elastic in gen.	$\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases}$	$\frac{a^2}{3}$	$\frac{2a^2}{3}$	$\frac{1}{2}$

molecules ; this is the origin of sonic or flexible vibration of corps and of all the effect of elasticity. [8, pp.31-2] (trans. mine.)

6. Fourier's heat equation of motion in fluid

Fourier esteems Euler's fluid dynamic equations, saying in the preface of "The analysis of the heat motion in the fluid." We cite Fourier's English translated paper as follows :

> To solve this, we must consider, a given space interior of mass, for example, by the volume of a rectangular prism composed of six sides, of which the position is given. We investigate all the successive alterations which the quality of heat contained in the space of prism obeys. This quantity alternates instantly and constantly, and becomes very different by the two things. One is the property, the molecules of fluid have, to communicate their heat with sufficiently near molecules, when the temperatures are not equal.
>
> The question is reduced into to calculate separately : the heat receiving from the space of prism due to the communication and the heat receiving from the

TABLE 3. (Continued from Table 2.) The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" were fixed.

no	name/ prob	the kinetic equations	Δ	g.d	C
6	Poisson ('31) [24] fluid in general eq.	$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) \\ \quad + \frac{\alpha}{3}(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) \\ \quad + \frac{\alpha}{3}(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) \\ \quad + \frac{\alpha}{3}(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \end{cases}$ $\Rightarrow \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \end{cases}$ $\begin{cases} \text{where ,} \\ \varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \\ \beta \equiv \alpha(K+k) \end{cases}$	β	$\frac{\beta}{3}$	3
7	Stokes ('49)[31] fluid	$(12)_S$ $\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases}$	μ	$\frac{\mu}{3}$	3
8 N	Prandtl (1905) [27], HD	$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla(V+p) = k\nabla^2 v, \quad \text{div } \mathbf{v} = 0$	k		
9 N	Prandtl (1934) [28], HD	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ $= X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$ for incompressible, it is simplified as follows : $\text{div } \mathbf{w} = 0, \quad \frac{D\mathbf{w}}{dt} = g - \frac{1}{\rho}\text{grad } p + \nu\Delta\mathbf{w}$	ν	$\frac{\nu}{3}$	3

space due to the motion of molecules.

We know the analytic expression of communicated heat, and the first point of the question is plainly cleared. The rest is the calculation of transported heat : it depend on only the velocity of molecules and the direction which they take in their motion. [9, pp.507-514.]. (trans. mine.)

Fourier combines heat theory with the Euler's equation of incompressible fluid dynamics and proposes the equation of heat motion in fluid in 1820, however, this paper was published in 1833 after 13 years, it was after 3 years since Fourier passed away. Fourier seems to have been doutful to publish it in life. Here, ε is the variable density and θ is the variable temperature of the molecule respectively. K : proper conductance of mass, C : the constant of specific heat, h : the constant determining dilatation, e : density at $\theta = 0$.

$$\begin{cases} \frac{1}{\varepsilon}\frac{dp}{dx} + \frac{d\alpha}{dt} + \alpha\frac{d\alpha}{dx} + \beta\frac{d\alpha}{dy} + \gamma\frac{d\alpha}{dz} - X = 0, \\ \frac{1}{\varepsilon}\frac{dp}{dy} + \frac{d\beta}{dt} + \alpha\frac{d\beta}{dx} + \beta\frac{d\beta}{dy} + \gamma\frac{d\beta}{dz} - Y = 0, \\ \frac{1}{\varepsilon}\frac{dp}{dz} + \frac{d\gamma}{dt} + \alpha\frac{d\gamma}{dx} + \beta\frac{d\gamma}{dy} + \gamma\frac{d\gamma}{dz} - Z = 0. \\ \frac{d\varepsilon}{dt} + \frac{d}{dx}(\varepsilon\alpha) + \frac{d}{dy}(\varepsilon\beta) + \frac{d}{dz}(\varepsilon\gamma) = 0, \quad \varepsilon = e(1 + h\theta). \\ \frac{d\theta}{dt} = \frac{K}{C}\left(\frac{d^2\theta}{dx^2} + \frac{d^2\theta}{dy^2} + \frac{d^2\theta}{dz^2}\right) - \left[\frac{d}{dx}(\alpha\theta) + \frac{d}{dy}(\beta\theta) + \frac{d}{dz}(\gamma\theta)\right]. \end{cases}$$

where, α, β, γ, p, ε, θ are the function of x, y, z, t. X, Y, Z are the outer forces. We think, Fourier seems to feel an inferiority complex to the fluid dynamics by Euler and he divers the Euler equation as the transport equation from Euler 1755 [7]. (cf. Table 2.)

7. POISSON'S PARADIGM OF UNIVERSAL TRUTH ON THE DEFINITE INTEGRAL

Poisson mentions the universality of the method to solve the differential equations. Poisson attacks the definite integral by Euler and Laplace, and Fourier's analytical theory of heat, and manages to construct universal truth in the paradigms.

One of the paradigms is made by Euler and Laplace. Laplace succeeds to Euler and states the passage from real to imaginary or reciprocal passage between two, which we mention in below.

The other contradictory problem is Fourier's application of De Gua. The diversion is Fourier's essential tool for the analytical theory of heat.

Dirichlet calls these passages a sort of *singularity of passage* from the finite to the infinite. cf. Chapter 1. We think that Poisson's strategy is to destruct both paradigms and make his own paradigm to establish the univarsal truth between mathematics and physics.

8. *La valeur particulière* AND THE EIGENVALUE

We confirm the identity of *valeur particulière* with the eigenvalue. We would pay attention to the historical fact that it has been develloped for the linear differential equation on the heat deffusion, or the trigonometric serirs in the analysis including string or sonic oscillation and the process redefined by Hilbert in 1904.

- We think the eigenvalue is translated from *la valeur particulière* into German word *der Eigenwert* by the Hilbert 1904 and is expatiated by Courant-Hilbert 1924 [3]. The word *eigenfunction* is combined corresponding to the word : *eigenvalue.*
- In the bibliographies of the earlier centuries, for example, Lagrange 1759, Fourier 1822, Poisson 1823, 1835, Cauchy 1823, Sturm 1836, Liouville 1836, Poincaré 1895, et al. use la *valeur particulière.* Sturm and Liouville owe to Poisson's preceding works of now so-called Sturm-Liouville type differential equation of the second order.
- In the first English translation of Fourier's main work [8], Freeman 1878 [10] uses 'the particular value' to all the over 43 original words in this book.
- Wilkinson 1952 uses *eigenvalue* without using the other English word : *proper value* or particular value in recognition of its nomenclature of eigenvalue.
- Today's French word : *la valeur propre*, used by Chatelin 1988, et al., may be reimported from German *Eigenwert* after Wilkinson's English word *eigenvalue.*
- The then French usage of *la fonction particulière / le espace particulière* corresponding to the eigenfunction / eigenspace / eigenvector aren't distinct in these days, however, the correspondency between the eigenvalue and the function is visible, for example, such as the expression in Poisson [25] or Sturm [33] or the expression in Liouville [16], in spite of the fact that its usage aren't so distinct as after Hilbert.
- On the other hand, the word *valeur caractèristique* aren't used as the eigenvalue.
- At last, we can recognize Euler 1748 on the cord vibration as one of the origin of eigenvalue problem. It is because the two equations (1) and (2) are the same.

9. THE CARRIED-OVER TO THE NEXT CENTURY UNIFYING THE LEGACIES IN THE 19TH C.

In 1878, ten years earlier than G. Darboux, A. Freeman [10] published the first English translated Fourier's second version 1822. To this work, Lord Kelvin (William Thomson) contributes to import the Fourier's theory into the England academic society. [11] The microscopical description of hydromechanics equations are followed by the description of equations of gas theory by Maxwell, Kirchhoff and Boltzmann. Above all, in 1872, Boltzmann formulated the Boltzmann equations. After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872. They contributed to formulate the fluid equations and to fix the Navier-Stokes equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called Navier-Stokes equations" in 1905 , in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of

[11] A.Freeman puts the name of W. Thomson in his acknoledgement. cf. [10, errata].

two coefficients as 3 : 1, which arose from Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845. From Fourier's equation of heat, Boltzmann's gas transport equation is deduced. (cf. Table 2, 3).

10. Conclusions

1. We consider our problem as the totality among the definite integral, the trigonometric series, etc., for Poisson's objection to Fourier is relating the universal and fundamental problem of analytics, as we show Poisson's analytical/mathematical thought or sight in the Chapter 7, etc. In fact, Poisson's work-span covers them.
2. Prévost's preceding works are cited and transferred to the Fourier's communication theory of heat, and Prévost guides Fourie to the primary academic theme.
3. Boltzmann's concept of collision and transport with entropy and probability are treated as the classical quantum mechanics. In this sense, Fourier's communication theory and the equation of motion in the fluid stand on the communication point between the classical mechanics and new quantum mechanics by Schrödinger.
4. Owing to the arrival of continuum, we are able to discuss the solution of the problem on the continuous space of mathematics. As Duhamel says, at first, Poisson performs it with the concept of mathematically infinite continuity. This allows us to discuss, without depending on the microscopic-description, by the vectorially description, like Saint-Venant, Stokes.
5. Although the confusion of knowledges on continuum, the unity in the mathematics are gained, however, the applicabilities of the unite or general equations are then not yet defined, which comes from the misunderstandings interphysico-mathematics, such as the identity of fluid and elasticity, or, fluid and heat.
6. Sturm-Liouville type differential equations of heat diffusion problems [16, 32] are redefined by Hilbert [12] using the second order differential operator $\mathcal{L}$ and as the *EigenWert* problem translating from the traditionally used nomenclature *la valeur particulière.*
7. About the describability of the trigonometric series of an arbitrary function, nobody succeeds in it including Fourier, himself. Up to the middle of or after the 20th century, these collaborations are continued, finally in 1966, by Carleson proved in L^2, and in 1968, by Hunt in L^p.

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