SOME RECENT RESULTS ON THE CAUCHY PROBLEM FOR GRAVITY WATER WAVES

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1. INTRODUCTION

Consider the incompressible Euler equation with free surface, referred to below as the water waves equation. This equation describes the motion of surface waves over an incompressible liquid flow, moving under the restoring force of gravitation. This is in fact a system of two nonlinear equations: the incompressible Euler equation in the interior of the domain and an equation for the free surface parametrization. This system differs strongly from the usual incompressible Euler equation. Indeed, in general, one assumes that the flow is irrotational. Instead, the main difficulty is that there is a free boundary.

At time t, we assume that the fluid domain $\Omega(t)$ is of the form

$$\Omega(t) = \left\{ (x,y) \in \mathbf{R}^d \times \mathbf{R} \, ; \, y < \eta(t,x) \right\},$$

where the free surface elevation η is an unknown and $d \geq 1$ is the dimension of the free surface. The equation dictating the dynamics in the case of infinitesimally small waves reads:

$$(1.1) \qquad\qquad \partial_t^2 \eta + g|D_x|\eta = 0,$$

where $g > 0$ is the gravity acceleration and $|D_x|$ is the Fourier multiplier defined by $|D_x|e^{ix\cdot\xi} = |\xi|e^{ix\cdot\xi}$ (so that $|D_x|^2 = -\Delta$). Notice that this is *not* a partial differential equation since $|D_x|$ is a nonlocal operator.

In 1815, using the Fourier transform, Cauchy ([19]) reduced the study of the linear equation (1.1) to the analysis of plane waves of the form $\eta(t,x) = \varepsilon \exp(i(k \cdot x - \omega t))$. The key point is the fact that the wave number $|k|$ and the angular frequency ω are related by the dispersion relation

$$\omega^2 = g|k|.$$

Once this is established, the solutions can be completely described by using microlocal analysis. For instance, for $d = 1$, the stationary phase formula implies that, for sufficiently decaying initial data, when the time t goes to $+\infty$, the asymptotic behavior of η is given

by the real part of the following expression

(1.2)
$$\frac{\varepsilon}{\sqrt{t}}\alpha\left(\frac{x}{t}\right)\exp\left(i\frac{t^2}{4|x|}\right)$$

for some bounded fonction α.

Likewise, one can study the Cauchy problem and determine the asymptotic behavior of the solutions of the nonlinear equations. In recent years, considerable progress has been made on this problem and we describe in this paper some of these results.

2. EQUATIONS

Recall that we assume that the fluid domain $\Omega(t)$ is of the form

$$\Omega(t) = \left\{ (x,y) \in \mathbf{R}^d \times \mathbf{R} \, ; \, y < \eta(t,x) \right\},$$

where the free surface elevation η is an unknown and $d \geq 1$. The velocity field $v \colon \Omega \to \mathbf{R}^{d+1}$ is irrotational, by assumption, and hence $v = \nabla_{x,y}\phi$ for some velocity potential $\phi \colon \Omega \to \mathbf{R}$ satisfying

(2.1)
$$\Delta_{x,y}\phi = 0, \quad \partial_t\phi + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0,$$

where $g > 0$ is the gravity acceleration and $P = P(t,x,y)$ is the pressure. The problem is then given by 2 boundary conditions:

(2.2)
$$\begin{cases} \partial_t\eta = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi & \text{on } \partial\Omega, \\ P = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nabla = \nabla_x$ and ∂_n is the normal derivative, so that $\sqrt{1+|\nabla\eta|^2}\,\partial_n\phi = \partial_y\phi - \nabla\eta\cdot\nabla\phi$.

We work below with the Craig-Sulem-Zakharov formulation of the water waves equations: this is a reduction to a system on the boundary obtained by introducing the Dirichlet–Neumann operator. By definition, the Dirichlet–Neumann operator maps a function defined on the boundary of an open set to the normal derivative of its harmonic extension. This operator plays a central role in the analysis of the water waves equations since the works by Walter Craig and Catherine Sulem.

Following Zakharov ([52]) and Craig and Sulem ([23]), denote by ψ the trace of the potential on the free surface:

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

and introduce the Dirichlet-Neumann operator $G(\eta)$, which is given by

$$G(\eta)\psi = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta}.$$

Then η, ψ are functions of $t \in \mathbf{R}$ and $x \in \mathbf{R}^d$ only, satisfying (see [23])

$$(2.3) \qquad \begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g\eta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{\left(\nabla\eta \cdot \nabla\psi + G(\eta)\psi\right)^2}{1 + |\nabla\eta|^2} = 0. \end{cases}$$

It is not obvious that it is equivalent to solve (2.1)–(2.2) or (2.3). This equivalence is proved in [3]. More precisely, it is proved in [3] that, if a solution (η, ψ) of System (2.3) belongs to $C^0([0,T]; H^\sigma(\mathbf{R}^d) \times H^\sigma(\mathbf{R}^d))$ for some $T > 0$ and $\sigma > 1 + d/2$, then one can define a velocity potential ϕ and a pressure P satisfying (2.1)–(2.2). Notice that $\sigma = 1 + d/2$ corresponds to the scaling index. Indeed, if ϕ and η are solutions of the gravity water waves equations, then ϕ_λ and η_λ defined by

$$(2.4) \qquad \phi_\lambda(t,x,y) = \lambda^{-3/2}\phi(\sqrt{\lambda}t, \lambda x, \lambda y), \quad \eta_\lambda(t,x) = \lambda^{-1}\eta(\sqrt{\lambda}t, \lambda x),$$

solve the same system of equations. For η, the (homogeneous) Sobolev spaces invariant by this scaling is $\dot{H}^{1+d/2}(\mathbf{R}^d)$.

3. DIRICHLET-NEUMANN OPERATOR

Many results have been obtained about water waves equations, using energy estimates or bifurcation results. These methods led to spectacular progress (local in time well-posedness of the Cauchy problem or existence of extremal waves...). However, much less is known for the water waves equations than for many other dispersive equations. This is because of two key properties of the water waves equations: they are quasi-linear equations (instead of semi-linear as Benjamin-Ono or KdV for instance) and secondly they are not partial differential equations but instead a pseudo-differential system, involving the Dirichlet-Neumann operator which is nonlocal and also depends nonlinearly on the unknown η.

In particular, an important step is to study the Dirichlet-Neumann operator. The literature on this subject is now well known and we only quote some papers where this operator is used to study various qualitative properties of the equations: see Craig, Schanz and Sulem [24] (derivation of the modulation equations), Lannes [38] (analysis of the Cauchy problem), Iguchi [32, 33] and Alvarez-Samaniego and Lannes [12] (analysis of asymptotic limits), Iooss and Plotnikov [34] (small divisors problem). We refer to the book by Lannes [39] for much more references. We report in this section on a paradifferential analysis of the Dirichlet-Neumann operator, which is used to study the Cauchy problem for the water waves equations as explained later.

In this section the time variable is seen as a parameter and we skip it. Assume that $\eta \in C^\infty(\mathbf{R}^d)$ is bounded together with all its derivatives. Then, for any $\psi \in H^{\frac{1}{2}}(\mathbf{R}^d)$ there is a unique variational solution $\phi \in H^1(\Omega)$ to the problem

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega = \{y < \eta(x)\}, \quad \phi|_{y=\eta} = \psi.$$

Notice that $\nabla_{x,y}\phi$ belongs only to $L^2(\Omega)$, so it is not obvious that one can consider the trace $\partial_n\phi|_{\partial\Omega}$. However, since $\Delta_{x,y}\phi = 0$, one can express the normal derivative in terms of the tangential derivatives and prove that $\sqrt{1 + |\nabla\eta|^2}\partial_n\phi|_{\partial\Omega}$ is well-defined.

Namely, one obtains that if $\eta \in C^\infty(\mathbf{R}^d)$ and $\psi \in H^{\frac{1}{2}}(\mathbf{R}^d)$ then

$$G(\eta)\psi(x) = \sqrt{1 + |\nabla\eta|^2}\partial_n\phi|_{y=\eta(x)} = \partial_y\phi(x, \eta(x)) - \nabla\eta(x) \cdot \nabla\phi(x, \eta(x))$$

is well-defined and belongs to $H^{-\frac{1}{2}}(\mathbf{R}^d)$. In addition, it follows from classical elliptic regularity results that, for any $s \geq \frac{1}{2}$, $G(\eta)$ is bounded from $H^s(\mathbf{R}^d)$ into $H^{s-1}(\mathbf{R}^d)$. It is known that this property still holds in the case where η has limited regularity. Namely, for $s \geq 0$ large enough, we have

$$\|G(\eta)\psi\|_{H^s} \leq C\big(\|\eta\|_{H^{s+1}}\big)\|\psi\|_{H^{s+1}}.$$

Various variants of this estimate have been obtained by Craig and Nicholls ([22]), Wu ([49]), Günther and Prokert [29] and Lannes [38]. In [2], it is proved that this estimate holds for any $s > d/2$.

On the other hand, it is known since Calderón that, for $\eta \in C^\infty$ bounded together with all its derivatives, $G(\eta)$ is a pseudo-differential operator whose principal symbol is the product of the length of the covector ξ by the coefficient $\sqrt{1 + |\nabla\eta|^2}$. Namely, given a symbol $a = a(x, \xi)$ we define the pseudo-differential operator $\mathrm{Op}(a)$ by

$$\mathrm{Op}(a)u(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi}a(x, \xi)\widehat{u}(\xi)\, d\xi$$

$$= \frac{1}{(2\pi)^d} \iint e^{i(x-y)\cdot\xi}a(x, \xi)u(y)\, dy\, d\xi,$$

and then

$$G(\eta)\psi = \mathrm{Op}(\lambda^{(1)})\psi + R_0(\eta)\psi,$$

where

(3.1) $$\lambda^{(1)} = \sqrt{(1 + |\nabla\eta|^2)|\xi|^2 - (\nabla\eta \cdot \xi)^2},$$

and where the remainder $R_0(\eta)\psi$ satisfies the following property: there exists K such that, for all $s \geq 1/2$,

$$\|R_0(\eta)\psi\|_{H^s} \leq C\left(\|\eta\|_{H^{s+K}}\right)\|\psi\|_{H^s}.$$

This allows to approximate $G(\eta)$ by $\mathrm{Op}(\lambda^{(1)})$ (which is of order 1, that is bounded from H^μ into $H^{\mu-1}$ for any $\mu \in \mathbf{R}$), modulo the remainder $R_0(\eta)$ which is of order 0. Actually, we have an approximation at any order (see [13, 44]).

Theorem 3.1. *Assume that $\eta \in C^\infty$ is bounded together with all its derivatives. There exists a sequence of symbols $(\lambda^{(-k)})_{k \in \mathbb{N}}$, where $\lambda^{(-k)}$ is homogeneous of order $-k$ in ξ, such that for all $\ell \in \mathbb{N}$,*

$$G(\eta)\psi = \text{Op}\big(\lambda^{(1)} + \lambda^{(0)} + \cdots + \lambda^{(-\ell)}\big)\psi + R_{-\ell}(\eta)\psi,$$

where $R_{-\ell}(\eta)$ satisfies the following property: there exists K_ℓ such that for all $s \geq \frac{1}{2}$ we have

(3.2) $$\|R_{-\ell}(\eta)\psi\|_{H^{s+\ell}} \leq C\left(\|\eta\|_{H^{s+K_\ell}}\right)\|\psi\|_{H^s}.$$

Remark 3.2. If $d = 1$ then $\lambda^{(1)} = |\xi|$ and $\lambda^{(k)} = 0$ for any $k \leq 0$ and hence one has the following key simplification

$$\text{Op}\big(\lambda^{(1)} + \lambda^{(0)} + \cdots + \lambda^{(-\ell)}\big) = |D_x|.$$

In general, for $d \geq 2$, we have $\lambda^{(k)} \neq 0$ for $k \leq 0$ and $\eta \neq 0$. Indeed,

(3.3) $$\lambda^{(0)} = \frac{1 + |\nabla\eta|^2}{2\lambda^{(1)}}\left\{\text{div}\left(\alpha^{(1)}\nabla\eta\right) + i\partial_\xi\lambda^{(1)} \cdot \nabla\alpha^{(1)}\right\},$$

with

$$\alpha^{(1)} = \frac{1}{1 + |\nabla\eta|^2}\left(\lambda^{(1)} + i\nabla\eta \cdot \xi\right).$$

3.1. Paradifferential analysis.

For the analysis of the water waves equations, one has to assume that η and ψ have essentially the same regularity. Then, the constant K_ℓ which appears in (3.2) corresponds to a loss of derivatives with respect to the coefficients (indeed, the proof of Theorem 3.1 gives that K_ℓ is a large enough number). A paradifferential approach is introduced in [1] to obtain the dependence of $G(\eta)$ and the remainders in η, thereby obtaining estimates without losses of derivatives.

To introduce this paradifferential approach, there are two preliminaries observations that should be added.

(1) We begin by recalling from Lannes [38] (see also [33, 34]) a formula for the derivative of $G(\eta)\psi$ with respect to η:

(3.4) $$\lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\left(G(\eta + \varepsilon\dot\eta)\psi - G(\eta)\psi\right) = -G(\eta)(B\dot\eta) - \text{div}(V\dot\eta),$$

where

$$B = (\partial_y\phi)(x, \eta(x)) = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1 + |\nabla\eta|^2}, \quad V = (\nabla\phi)(x, \eta(x)) = \nabla\psi - B\nabla\eta.$$

(See the original paper [38] for the assumptions under which the formula holds.)

(2) Our second observation is that the symbols $\lambda^{(0)}, \lambda^{(-1)}, \ldots$ in Theorem 3.1 are defined by induction so that $\lambda^{(k)}$ involves only derivatives of η of order $\leq |k| + 2$. Then the

definition of the full symbol λ of $G(\eta)$ can be extended for $\eta \notin C^\infty$ in the following obvious manner: we consider in the asymptotic expansion

$$\lambda \sim \lambda^{(1)} + \lambda^{(0)} + \lambda^{(-1)} + \cdots$$

only the terms which are meaningful. This means that, for $\eta \in C^{k+2} \setminus C^{k+3}$ with $k \in \mathbf{N}$, we set

$$(3.5) \qquad \lambda = \lambda^{(1)} + \lambda^{(0)} + \cdots + \lambda^{(-k)}.$$

As defined, the symbol λ belongs to the spaces introduced by Bony of pluri-homogeneous symbols, and we here review some notations about Bony's paradifferential calculus (following [15, 40]).

Notation 3.3. For $\rho \in \mathbf{N}$, we denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the Sobolev spaces of L^∞ functions whose derivatives of order ρ are in L^∞. For $\rho \in]0, +\infty[\setminus\mathbf{N}$, we denote by $C^\rho(\mathbf{R}^d)$ or $W^{\rho,\infty}(\mathbf{R}^d)$ the space functions in $W^{[\rho],\infty}(\mathbf{R}^d)$ whose derivatives of order $[\rho]$ are uniformly Hölder continuous with exponent $\rho - [\rho]$.

Definition 3.4 (homogeneous symbol). *Given real numbers $\rho \geq 0$ and $m \in \mathbf{R}$, Γ_ρ^m denotes the space of functions $a(x, \xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ which are homogeneous of degree m and C^∞ with respect to $\xi \neq 0$, and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x, \xi)$ belongs to $W^{\rho,\infty}(\mathbf{R}^d)$ and*

$$(3.6) \qquad \sup_{|\xi|=1} \left\| \partial_\xi^\alpha a(\cdot, \xi) \right\|_{W^{\rho,\infty}} < +\infty.$$

Definition 3.5 (Pluri-homogeneous symbol). *Given $\rho > 0$ and $m \in \mathbf{R}$, Σ_ρ^m denotes the space of symbols $a(x, \xi)$ such that*

$$a = \sum_{0 \leq j < \rho} a^{(m-j)} \qquad (j \in \mathbf{N}),$$

where $a^{(m-j)} \in \Gamma_{\rho-j}^{m-j}$ (we say that $a^{(m)}$ is the principal symbol of a).

Notice that, if $a \in \Gamma_\rho^m$ then

$$\exists C_\alpha, \quad \forall |\xi| \geq 1/2, \quad \left\| \partial_\xi^\alpha a(\cdot, \xi) \right\|_{W^{\rho,\infty}} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}$$

so, for any function ζ such that $\zeta = 0$ on a neighborhood of the origin and $\zeta = 1$ for $|\xi| \geq 1$, the symbol $a(x, \xi)\zeta(\xi)$ belongs to the usual symbol classes of symbols of order m and regularity $W^{\rho,\infty}$ in x.

Now fix a cut-off function ζ as above and introduce a C^∞ function χ homogeneous of degree 0 and satisfying, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta, \eta) = 1 \quad \text{if} \quad |\theta| \leq \varepsilon_1 |\eta|, \qquad \chi(\theta, \eta) = 0 \quad \text{if} \quad |\theta| \geq \varepsilon_2 |\eta|.$$

Given a symbol a, we define the paradifferential operator T_a by

$$(3.7) \qquad \widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \zeta(\eta) \widehat{u}(\eta) \, d\eta,$$

where $\widehat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi) \, dx$ is the Fourier transform of a with respect to the first variable.

The following remark, although elementary, is important for our purposes.

Remark 3.6. If $q(D_x)$ is a Fourier multiplier with symbol $q(\xi)$, then we do not have $q(D_x) = T_q$, because of the cut-off function ζ. However, we have $Q(D_x) = T_q$ provided that $q(\xi) = 0$ for $|\xi| \leq 1$. In general, $q(D_x) - T_q$ is a smoothing operator, mapping $H^{-\infty}$ to H^{∞}.

We are now ready to state the following result.

Theorem 3.7 (from [1]). *Let $d \geq 1$ and $s > 2 + d/2$. If $\eta \in H^s$ and $\psi \in H^s$, then*

$$G(\eta)\psi = T_\lambda (\psi - T_B \eta) - T_V \cdot \nabla \eta - T_{\mathrm{div}\, V} \eta + R(\eta)\psi,$$

where the remainder $R(\eta)\psi$ is twice more regular than the unknowns:

$$(3.8) \qquad \|R(\eta)\psi\|_{H^{2s-K}} \leq C \left(\|\eta\|_{H^s} \right) \|\psi\|_{H^s},$$

where K depends only on d and where the coefficients B and V are as in (3.4).

Remark 3.8. *i*) The unknown $\psi - T_B \eta$ is referred to as the good unknown of Alinhac (following the ideas introduced by Alinhac in [8, 9]). For the study of the linearized water waves equations, this amounts to introduce $\delta \psi - B \delta \eta$. The fact that this leads to a key cancelation was first observed by Lannes in [38].

ii) If $d = 1$, then $T_\lambda = |D_x|$ modulo a smoothing operator of order $-\infty$, so

$$G(\eta)\psi = |D_x| \left(\psi - T_B \eta \right) - \partial_x (T_V \eta) + \tilde{R}(\eta)\psi$$

where $\tilde{R}(\eta)\psi$ satisfies (3.8).

One can complement the previous results in several directions. For instance, the next result, proved with Riccardo Montalto, contains an analysis of the quadratic terms.

Theorem 3.9. *Let $d \geq 1$ and fix $s_0 > 3 + d/2$. For any $s > s_0$ there exists a nondecreasing function $C \colon \mathbf{R}_+ \to \mathbf{R}_+$ such that the following property holds. For any $\eta \in H^s$ and $\psi \in H^s$,*

$$(3.9) \qquad \begin{aligned} G(\eta)\psi &= \left(|D_x| + T_{\lambda - |\xi|} \right) \left(\psi - T_B \eta \right) - T_V \cdot \nabla \eta - T_{\mathrm{div}\, V} \eta \\ &\quad + F_{\leq 2}(\eta)\psi + F_{\geq 3}(\eta)\psi, \end{aligned}$$

where the coefficients B and V are as in (3.4),

$$F_{\leq 2}(\eta)\psi = -|D_x|R(|D_x|\psi, \eta) - \operatorname{div} R(\nabla\psi, \eta)$$

and where $F_{\geq 3}(\eta)\psi$ *satisfies*

(3.10) $\qquad \|F_{\geq 3}(\eta)\psi\|_{H^{s+s_0-K}} \leq C\left(\|\eta\|_{H^{s_0}}\right)\left\{\|\eta\|_{H^{s_0}}^2 \|\psi\|_{H^s} + \|\eta\|_{H^{s_0}} \|\psi\|_{H^{s_0}} \|\eta\|_{H^s}\right\}$,

where $K = 5/2 + d/2$.

The estimate (3.10) is tame: it is linear in the higher order norms. We refer to [6] or de Poyferré [25] for other tame estimates where the lower order norms depend only on Hölder norms.

3.2. Paralinearization of the Dirichlet-Neumann operator in rough domains.

As already mentioned, in the case of smooth domains, it is known that, modulo a smoothing operator, $G(\eta)$ is a pseudo-differential operator whose principal symbol is given by

(3.11) $\qquad \lambda^{(1)}(x, \xi) := \sqrt{(1 + |\nabla\eta(x)|^2)|\xi|^2 - (\nabla\eta(x) \cdot \xi)^2}$.

Notice that $\lambda^{(1)}$ is well-defined for any C^1 function η. The results of this section aim at comparing $G(\eta)$ to the paradifferential operator $T_{\lambda^{(1)}}$ when η has limited regularity.

Proposition 3.10 (from [2]). *Let* $d \geq 1$ *and* $s > \frac{1}{2} + \frac{d}{2}$. *For any* $\frac{1}{2} \leq \sigma \leq s$ *and any*

$$0 < \varepsilon \leq \frac{1}{2}, \qquad \varepsilon < s - \frac{1}{2} - \frac{d}{2},$$

there exists a non-decreasing function $C: \mathbf{R}_+ \to \mathbf{R}_+$ *such that*

$$\|G(\eta)f - T_{\lambda^{(1)}}f\|_{H^{\sigma-1+\varepsilon}(\mathbf{R}^d)} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}\right)\|f\|_{H^{\sigma}(\mathbf{R}^d)}.$$

Consequently, for any $s > 1 + d/2$, by applying the previous estimate with $\varepsilon = 1/2$,

(3.12) $\qquad \|G(\eta)f - T_{\lambda^{(1)}}f\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}\right)\|f\|_{H^s(\mathbf{R}^d)}$.

When η is a smooth function, one expects that $G(\eta) - T_{\lambda^{(1)}}$ is of order 0 which means that $G(\eta)f - T_{\lambda^{(1)}}f$ has the same regularity as f. Indeed, this is the conclusion of Theorem 3.1 applied with $\ell = 0$. On the other hand, (3.12) gives only that this difference is of order $1/2$ (this means that it maps H^s to $H^{s-1/2}$). This is because we allow η to be only $1/2$-derivative more regular than f. This is tailored to the analysis of gravity water waves since, for scaling reasons, it is natural to assume that η is $1/2$-derivative more regular than the trace of the velocity on the free surface (see (2.4)).

One can improve (3.12) by proving that $G(\eta) - T_{\lambda^{(1)}}$ is of order $1/2$ assuming only that $s > 3/4 + d/2$ together with sharp Hölder regularity assumptions on both η and f (these Hölder assumptions are the ones that hold by Sobolev injections for $s > 1 + d/2$). Since,

for the evolution equations, Hölder norms are controlled only in some L^p spaces in time (by Strichartz estimates), we need to precise the dependence of the constants.

Proposition 3.11 (from [4]). *Let $d \geq 1$ and consider real numbers s, r such that*

$$s > \frac{3}{4} + \frac{d}{2}, \qquad r > 1.$$

Consider $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \cap W^{r+\frac{1}{2},\infty}(\mathbf{R}^d)$ and $f \in H^s(\mathbf{R}^d) \cap W^{r,\infty}(\mathbf{R}^d)$, then $G(\eta)f$ belongs to $H^{s-\frac{1}{2}}(\mathbf{R}^d)$ and

$$(3.13) \qquad \|G(\eta)f - T_{\lambda^{(1)}} f\|_{H^{s-\frac{1}{2}}} \leq C\big(\|\eta\|_{H^{s+\frac{1}{2}}} + \|f\|_{H^s}\big) \Big\{1 + \|\eta\|_{W^{r+\frac{1}{2},\infty}} + \|f\|_{W^{r,\infty}}\Big\}$$

for some non-decreasing function $C \colon \mathbf{R}^+ \to \mathbf{R}^+$ depending only on s and r.

Remark. This estimate is tame in the following sense. When $s < 1 + d/2$ and $r > 1$, for oscillating functions, we have

$$\left\|u\!\left(\frac{x}{\varepsilon}\right)\right\|_{W^{r+\frac{1}{2},\infty}} \sim \left(\frac{1}{\varepsilon}\right)^{r+\frac{1}{2}} \gg \left(\frac{1}{\varepsilon}\right)^{s+\frac{1}{2}-\frac{d}{2}} \sim \left\|u\!\left(\frac{x}{\varepsilon}\right)\right\|_{H^{s+\frac{1}{2}}}.$$

Consequently, the estimate (3.13) is *linear* with respect to the highest norm.

We now consider the difference of two Dirichlet-Neumann operators. The first estimate follows directly from variational arguments.

Proposition 3.12. *There exists a non decreasing function $\mathcal{F} \colon \mathbf{R}^+ \to \mathbf{R}^+$ such that, for all $\eta_j \in W^{1,\infty}(\mathbf{R}^d)$, $j = 1, 2$ and all $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$,*

$$\big\|(G(\eta_1) - G(\eta_2))f\big\|_{H^{-\frac{1}{2}}} \leq \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}\big) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}.$$

The following proposition allows to consider the much more difficult case where $\eta_1 - \eta_2$ is controlled only in $W^{\alpha,\infty}$ for some $0 < \alpha < 1$.

Proposition 3.13 (from [4]). *Assume $d \geq 1$ and consider real numbers s, r such that*

$$s > \frac{3}{4} + \frac{d}{2}, \qquad s + \frac{1}{4} - \frac{d}{2} > r > 1.$$

Then there exists a non decreasing function $\mathcal{F} \colon \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\|G(\eta_1)f - G(\eta_2)f\|_{H^{s-\frac{3}{2}}}$$
$$\leq \mathcal{F}\big(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}}}\big) \Big\{ \|\eta_1 - \eta_2\|_{W^{r-\frac{1}{2},\infty}} \|f\|_{H^s}$$
$$+ \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \big(\|f\|_{H^s} + \|f\|_{W^{r,\infty}} \big) \Big\}.$$

There are quite difficulties which appear for $s < 1 + d/2$, in particular for $d = 1$. For instance, one has to estimate the $H^{s-\frac{3}{2}}$-norm of various products of the form uv with $u \in H^{s-1}$ and $v \in H^{s-\frac{3}{2}}$. For $s < 1 + d/2$, the product is no longer bounded from $H^{s-1} \times H^{s-\frac{3}{2}}$ to $H^{s-\frac{3}{2}}$ and, clearly, one has to further assume some control in Hölder (or

Zygmund) norms. Namely, we assume that $u \in H^{s-1} \cap L^\infty$ and $v \in H^{s-\frac{3}{2}} \cap C_*^{-\frac{1}{2}}$. Then, paralinearizing the product $uv = T_u v + T_v u + R(u,v)$ and using the usual estimate for paraproducts, one obtains

$$\|T_u v\|_{H^{s-\frac{3}{2}}} \lesssim \|u\|_{L^\infty} \|v\|_{H^{s-\frac{3}{2}}}, \quad \|T_v u\|_{H^{s-\frac{3}{2}}} \lesssim \|v\|_{C_*^{-\frac{1}{2}}} \|u\|_{H^{s-1}}$$

so the only difficulty is to estimate the $H^{s-\frac{3}{2}}$-norm of the remainder $R(u,v)$. However, the usual estimate for remainders requires that $s - 3/2 > 0$, which does not hold in general under the assumption $s > 3/4 + d/2$. To circumvent this problem, one has to factor out some derivative. This means that one has to replace $R(u,v)$ by $\partial_x R(\tilde{u}, \tilde{v})$ for some functions, say, $\tilde{u} \in L^\infty$ and $\tilde{v} \in H^{s-\frac{1}{2}}$. Now, since $s - 1/2 > 0$, one can estimate the $H^{s-\frac{1}{2}}$-norm of $R(\tilde{u}, \tilde{v})$ by means of the usual estimate for remainders.

4. LOCAL WELL-POSEDNESS

Many results have been obtained in the study of the Cauchy problem for the water waves equations, starting from the pioneering work of Nalimov [41] who proved in 1974 that the Cauchy problem is well-posed locally in time, in the framework of Sobolev spaces, under an additional smallness assumption on the data. We also refer the reader to Yoshihara [47, 48] and Craig [20]. Without smallness assumption, the well-posedness of the Cauchy problem was proved only in the 1990s by Wu for the case without surface tension (see [49]) and by Beyer-Günther in [14] in the case with surface tension. Several extensions of their results have been obtained by many authors. We refer in particular to Iguchi's paper [33] and Lannes' paper [38] where the Cauchy problem is studied using the Dirichlet-Neumann operator.

As already mentioned, for scaling reasons, it is natural to assume that η is $1/2$ more regular than the trace $\underline{v}(t,x) = v(t,x,\eta(t,x))$ of the velocity of the fluid at the free surface. In other words, it is natural to assume that, at time 0, $(\eta_0, \underline{v}_0)$ belongs to $H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)^{d+1}$, and it is natural to ask the following question: how small can s be taken to ensure local well-posedness? In [2], local in time well-posedness is proved for $s > 1 + d/2$. Therefore, in terms of Sobolev embeddings, the initial surfaces considered in [2] turn out to be only of $C^{3/2}$-class and consequently have unbounded curvature, while the initial velocities are only Lipschitz. On the other hand, it is well-known that water waves are dispersive waves and one expects to improve this well-posedness result by taking benefit of dispersive effects. This improvement is performed in [4]. To describe the main result of [4], we need to introduce the Taylor coefficient. This is the function a defined by $a = -\partial_y P|_{y=\eta}$ and we say that the Taylor sign condition is satisfied if

$$a = -\partial_y P|_\Sigma \geq c > 0.$$

This means that the pressure increases going from the air into the fluid domain. This is always satisfied when there is no bottom (as proved by Wu) or for small smooth perturbations of flat bottoms (as proved by Lannes).

Theorem 4.1 (from [4]). *Let*

$$s > 1 + \frac{d}{2} - \mu \quad \text{with} \quad \begin{cases} \mu = \frac{1}{24} & \text{if } d = 1, \\ \mu = \frac{1}{12} & \text{if } d \geq 2, \end{cases}$$

and set $\mathcal{H}^s = H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times (H^s)^{d+1}$ *where* $H^\sigma = H^\sigma(\mathbf{R}^d)$.

Then for any initial data such that $(\eta_0, \psi_0, \underline{v}_0) \in \mathcal{H}^s$ *and satisfying the Taylor sign condition, there exist* $T > 0$ *and a solution of the water waves system (unique in a suitable space) such that* $(\eta, \psi, \underline{v}) \in C^0([0, T]; \mathcal{H}^s)$.

Remark 4.2. • In view of Sobolev embeddings, our assumptions require that $\underline{v}_0$ belongs to the Hölder space $W^{1-\mu,\infty}$. However, we construct solutions such that $\underline{v}$ is still in $L^2((0, T); W^{1,\infty})$.

• One important conclusion is that, in dimension $d = 1$, one can consider initial free surface whose curvature does not belong to L^2.

To prove this result, we follow several steps. The first one is (roughly speaking) to reduce the water waves equations to a quasilinear wave type equation. This is the property that the equations can be reduced to a very simple form

(4.1) $$\partial_t u + T_V \cdot \nabla u + i T_\gamma u = f$$

where $V = v_x \,|_{y=\eta}$ is the horizontal component of the velocity field at the interface, T_V is a paraproduct and T_γ is a paradifferential operator of order $1/2$ with symbol

$$\gamma = \sqrt{a \lambda^{(1)}}$$

where, as above,

$$\lambda^{(1)} = \sqrt{\left(1 + |\nabla \eta|^2\right)|\xi|^2 - \left(\xi \cdot \nabla \eta\right)^2},$$

where $a = -\partial_y P|_{y=\eta}$ is the Taylor coefficient and λ is the principal symbol of the Dirichlet-Neumann operator. When $d = 1$, $\lambda^{(1)}$ simplifies to $|\xi|$ so $T_\gamma u = T_{\sqrt{a}} |D_x|^{\frac{1}{2}} u$.

The second step in the proof consists in proving that the solutions of the water waves system enjoy dispersive estimates (Strichartz-type inequalities). This is done by constructing parametrices on small time intervals tailored to the size of the frequencies considered (in the spirit of the works by Lebeau, Bahouri-Chemin, Tataru, and Burq-Gérard-Tzvetkov). We prove the following result in [4].

Theorem 4.3. *Let $I = [0, T]$, $d \geq 1$. Let μ be such that $\mu < \frac{1}{24}$ if $d = 1$ and $\mu < \frac{1}{12}$ if $d \geq 2$.*

Let $s \in \mathbf{R}$ and $f \in L^{\infty}(I; H^s(\mathbf{R}^d))$. Let $u \in C^0(I; H^s(\mathbf{R}^d))$ be a solution of (4.1). Then one can find $k = k(d)$ such that

$$\|u\|_{L^p(I; C_*^{s - \frac{d}{2} + \mu}(\mathbf{R}^d))} \leq \mathcal{F}\big(\|V\|_{E_0} + \mathcal{N}_k(\gamma)\big)\Big\{\|f\|_{L^p(I; H^s(\mathbf{R}^d))} + \|u\|_{C^0(I; H^s(\mathbf{R}^d))}\Big\}$$

where C_*^r is the Zygmund space of order $r \in \mathbf{R}$ (one has $C_*^r = W^{r, \infty}$ for $r \in (0, +\infty) \setminus \mathbf{N}$), $p = 4$ if $d = 1$ and $p = 2$ if $d \geq 2$, $E_0 = L^p(I; W^{1, \infty}(\mathbf{R}^d))^d$ and $\mathcal{N}_k(\gamma) = \sum_{|\beta| \leq k} \|D_{\xi}^{\beta} \gamma\|_{L^{\infty}(I \times \mathbf{R}^d \times \mathcal{C})}$ with $\mathcal{C} = \{\frac{1}{10} \leq |\xi| \leq 10\}$.

To conclude this section, as an illustration of the relevance of this low regularity Cauchy theory, we mention that we solve in [5] a question raised by Boussinesq in 1910 [16] on the water waves problem in a canal. In [16], Boussinesq suggested to reduce the water waves system in a canal to the same system on $\mathbf{R}^3$ with periodic conditions with respect to one variable, by a simple reflection/periodization procedure (with respect to the normal variable to the boundary of the canal). However, this idea remained inapplicable for the simple reason that the even extension of a smooth function on the half line is in general merely Lipschitz continuous (due to the singularity at the origin). As a consequence, even if one starts with a smooth initial domain, the reflected/periodized domain will only be Lipschitz continuous. In [5] we are able to take benefit of an elementary observation which shows that actually, as soon as we are looking for *reasonably smooth* solutions, the angle between the free surface and the vertical boundary of the canal is a right angle. Consequently, the reflected/periodized domain enjoy additional smoothness (namely up to C^3), which is enough to apply our *rough data* Cauchy theory and to show that the strategy suggested in [16] does indeed apply. This appears to be the first result on Cauchy theory for the water-wave system in a domain with boundary. Moreover, it is proved in [5] that one can work in the uniformly local Sobolev spaces introduced by Kato ([37]).

5. GLOBAL WELL-POSEDNESS

Blow-up results or global existence results for the water waves equation have been widely studied in recent years.

Concerning singularities, Castro, Córdoba, Fefferman, Gancedo and Gómez-Serrano obtained several striking results (see for instance [18]) about the problem of finding one water-wave solution such that, at time 0, the fluid interface is a graph, at a later time

$t_1 > 0$ the fluid interface is not a graph, and, at a later time $t_2 > t_1$, the fluid self-intersects. Consequently global well-posedness does not hold for arbitrarily large initial data.

For 3D waves (with a 2D free surface), Sijue Wu ([51]) and Pierre Germain-Nader Masmoudi-Jalal Shatah ([27]) proved global well-posedness results for smooth, small, and decaying at infinity Cauchy data. The analysis of 2D waves is more delicate since the solutions of the linearized equations decay at a lower speed (namely their $L^\infty(dx)$-norm is $O(1/\sqrt{t})$ whereas the solutions of the 3D problem have an L^∞-norm in $O(1/t)$). It is then more delicate to control the nonlinear perturbation in large time.

With Jean-Marc Delort, we prove in [7] a global existence result for 2D waves (so that the interface is 1D and the fluid domain 2D). We obtain moreover an asymptotic description in physical coordinates of the solution, which shows that modified scattering holds (this means that the behavior of the solutions to the nonlinear equations is different from the one of the solutions to the linear equations; compare (1.2) and (5.1)). This result has been obtained independently by Alexandru Ionescu and Fabio Pusateri (see [31]) and also by Mihaela Ifrim and Daniel Tataru ([30]).

Theorem 5.1. *For $\varepsilon \ll 1$, the Cauchy problem with initial data $(\eta, \psi)|_{t=1} = \varepsilon(\eta_0, \psi_0)$ where η_0, ψ_0 belong to $C_0^\infty(\mathbf{R})$ has a unique global in time solution. Moreover, $u = |D_x|^{\frac{1}{2}} \psi + i\eta$ satisfies*

$$(5.1) \qquad u(t,x) = \frac{\varepsilon}{\sqrt{t}} \alpha\left(\frac{x}{t}\right) \exp\left[i\frac{t}{4|x/t|} + i\frac{\varepsilon^2}{64}\frac{|\alpha(x/t)|^2}{|x/t|^5} \log t\right] + t^{-\frac{1}{2}-\kappa} \rho(t,x)$$

where $\kappa > 0$, $\alpha \in C_b^0(\mathbf{R})$, $\rho \in L^\infty$.

The proof is based on dispersive estimates, following the approaches initiated by Klainerman or Shatah. More precisely, the proof is based on a bootstrap argument involving L^2 and L^∞ estimates. The L^2 bounds rely on a normal forms paradifferential method allowing one to obtain energy estimates on the Eulerian formulation of the water waves equations. The L^∞ estimates are obtained by interpreting the equation in a semi-classical way, and combining Klainerman vector fields with the description of the solution in terms of semi-classical lagrangian distributions.

6. RESONANCES

Zakharov and his collaborators study since 1968 the interactions of water waves. Introduce a new unknown a by

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}\left(a_k + \overline{a_{-k}}\right), \qquad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}\left(a_k - \overline{a_{-k}}\right), \qquad \omega_k = \sqrt{g|k|},$$

where the u_k are Fourier coefficients. Then a solves an hamiltonian equation

$$(6.1) \qquad \partial_t a = i\partial_{\bar{a}}H(a,\bar{a}).$$

Since there are no 3 waves resonances, which means that

$$|k|^{\frac{1}{2}} = |k_1|^{\frac{1}{2}} + |k_2|^{\frac{1}{2}} \text{ and } k = k_1 + k_2 \quad \Rightarrow \quad k = k_1 = k_2 = 0,$$

one can (formally) eliminate the quadratic terms in $\partial_{\bar{a}}H(a,\bar{a})$ by means of a canonical transformation. Consequently, starting from (6.1), Zakharov obtains ([52]) an equation of the form

$$(6.2) \qquad i\dot{b}_k = \omega_k b_k + \frac{1}{2}\int T^{k_2 k_3}_{k k_1} b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3}\, dk_1 dk_2 dk_3 + \cdots$$

One can try to simplify further this equation. To do so, consider the equation for the 4 waves resonances, that is

$$(6.3) \qquad \sqrt{g|k|} + \sqrt{g|k_1|} = \sqrt{g|k_2|} + \sqrt{g|k_3|}, \quad k + k_1 = k_2 + k_3.$$

There are trivial solutions:

$$k = k_2, \quad k_1 = k_3.$$

One can check that the nontrivial solutions are of the form

$$
\begin{aligned}
k &= a(1+\zeta)^2, & k_1 &= a(1+\zeta)^2\zeta^2, \\
k_2 &= -a\zeta^2, & k_3 &= a(1+\zeta+\zeta^2)^2,
\end{aligned}
$$

with $0 < \zeta < 1$ and $a > 0$.

Dyachenko and Zakharov obtain a striking result in [26].

Theorem 6.1 (Dyachenko and Zakharov, 1994). *For any non trivial solutions* (k, k_1, k_2, k_3) *of* (6.3), *one has*

$$T^{kk_1}_{k_2 k_3} = 0.$$

The question to obtain some rigorous mathematical results from this beautiful observation is still open. There are however some rigorous mathematical justifications concerning the normal forms methods. In this direction, the first goal is to eliminate the quadratic component of the nonlinearity (as Zakharov did formally already in his paper [52]). In practice, one looks for a local diffeomorphism at 0 in H^s, for s large enough, so that the equation obtained by conjugation by this diffeomorphim be of the form of an equation with a cubic nonlinearity (while the water waves equation contains quadratic terms). Such nonlinear changes of unknowns, reducing the water waves equation to a cubic equation, have been known for quite a time (see Craig [21] or Iooss and Plotnikov [35]). However, these transformations were losing derivatives, as a consequence of the quasi-linear character of the problem. The article [6] introduces a paradifferential change of unknown,

without loss of derivatives, which eliminates the part of the quadratic terms that bring non zero contributions in a Sobolev energy inequality.

Following a classical strategy first used by Sijue Wu for the water waves equations, these transformations are used to study the Cauchy problem on large time intervals. Another motivation to study these nonlinear changes of unknowns is that one can deduce from (6.2) that the equation describing the dynamics of the envelop of a plane wave is the nonlinear Schrödinger equation. We refer to [45, 46] for a rigorous derivation of the nonlinear Schrödinger equation from the gravity water waves equation.

To conclude, let us discuss a related question concerning the nonlinear interactions of several plane waves. There are indeed several recent results in this direction and they are related to the so-called trivial resonances. Roughly speaking, the question can be stated as follows: consider a solution of the linearized equation of the form

$$U_\varepsilon = \varepsilon \sum_{1 \le j \le n} \exp(i(k_j \cdot x - \omega(k_j)t))$$

where $\omega(k) = \sqrt{g|k|}$ is the dispersion relation of the linearized water waves equations. Then one asks if there exist global in time solutions of the nonlinear equations of the form $U_\varepsilon + O(\varepsilon^2)$. This problem was initiated by Reeder and Shinbrot [43] who considered the special case with surface tension, $n = 2$ and two wave-vectors k_1 and k_2 which are mirror images (as $k_1 = (1, \tau)$ and $k_2 = (1, -\tau)$; this happens when one studies the problem of the nonlinear interaction of a 2D periodic plane wave with its reflection off a flat vertical wall). Their result was later improved by Craig and Nicholls [22] and Groves and Haragus [28], obtaining the existence of solutions from bifurcation analysis.

The first results about the nonlinear interaction of two periodic plane gravity water waves (thus without surface tension) came only recently in a beautiful series of papers of Iooss, Plotnikov and Toland (see [42, 36, 34]). This was a well known problem which remained open because of small divisors problems. The study of the general problem of the interaction of any finite number of different harmonics (instead of only two as in the previous mentioned papers) is one important open question. This question is related to KAM theory for infinite dimensional dynamical systems, which has attracted a lot of interest in recent years. However, the existing literature on KAM and Nash-Moser theory for PDEs mainly deals with semi-linear problems, with well-known results obtained by Craig, Bourgain, Kuksin, Kappeler-Pöschel among many others. The extension of KAM and Nash-Moser theory to quasi-linear PDEs is a very recent subject, which counts very few results (see [10, 11] and the references therein).

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