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Stationary wave to system of viscous conservation laws in half line

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1 Introduction

This paper is a survey of the paper [11] on large-time behavior of solutions to a system of viscous conservation laws over one-dimensional half space $\mathbb{R}_+ := (0, \infty)$,

$$U_t + f(U)_x = (G(U)U_x)_x, \quad x \in \mathbb{R}_+, \, t > 0. \tag{1.1}$$

Here $U = U(t,x)$ is an unknown m-vector function taking values in an open convex set $\mathscr{O}_U \subset \mathbb{R}^m$; $f(U)$ is a smooth m-vector function of $U \in \mathscr{O}_U$; $G(U)$ is a smooth $m \times m$ real matrix function.

Our main purpose is to show existence and asymptotic stability of a stationary solution to the system (1.1). To transfer the system to that in a symmetric form, by following the argument in [7], we firstly assume that the system (1.1) has a strictly convex entropy $\eta = \eta(U)$ satisfying

(i) $\eta(U)$ is a strictly convex scalar function, i.e., the Hessian matrix $D_U^2\eta(U)$ is positive definite for $U \in \mathscr{O}_U$.

(ii) There exists a smooth scalar function $q(U)$ (entropy flux) such that $D_U q(U) = D_U\eta(U) D_U f(U)$.

(iii) The matrix $G(U)(D_U^2\eta(U))^{-1}$ is real symmetric and non-negative for $U \in \mathscr{O}_U$.

Then we assume that

[A1] the system has the entropy function $\eta(U)$ satisfying (i) – (iii) above.

We introduce a new dependent variable $\hat{U}$ under assuming that there exists a diffeomorphism $U \mapsto \hat{U}$ from an open set $\mathscr{O}_U$ onto $\mathscr{O}_{\hat{U}}$. Notice that $\hat{U} = \hat{U}(U)$ is given by $\hat{U}(U) = {}^\top D_U\eta(U)$ and $D_U\hat{U} = D_U^2\eta$. Then the system (1.1) is deduced to the symmetric system for $\hat{U}$ as

$$\hat{A}^0(\hat{U})\hat{U}_t + \hat{A}(\hat{U})\hat{U}_x = (\hat{B}(\hat{U})\hat{U}_x)_x, \tag{1.2}$$

where

$$\begin{aligned}
&\hat{A}^0 := D_{\hat{U}}U = (D_U^2\eta)^{-1} && : \text{real symmetric and positive definite,}\\
&\hat{A} := D_U f D_{\hat{U}}U = D_U f (D_U^2\eta)^{-1} && : \text{real symmetric,}\\
&\hat{B} := G D_{\hat{U}}U = G(D_U^2\eta)^{-1} && : \text{real symmetric and non-negative definite.}
\end{aligned}$$

We next rewrite the system (1.2) to a normal form which is a coupled system of a symmetric hyperbolic system and a symmetric parabolic system. To do this, we suppose a following assumption:

[A2] The null space $\mathscr{N} := \ker \hat{B}(\hat{U})$ is independent of $\hat{U} \in \mathscr{O}_{\hat{U}}$.

Let $m_1 := \dim \mathscr{N}$ and $m_2 := m - m_1$. Here we assume that $\hat{B}(\hat{U})$ is the form of

$$\hat{B}(\hat{U}) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{B}_2(\hat{U}) \end{pmatrix}$$

without loss of generality (see [7]), where $\hat{B}_2(\hat{U})$ is an $m_2 \times m_2$ matrix and positive definite. Under the assumption **[A2]**, there exists a transformation $U \mapsto u$ which is a diffeomorphism from an open set $\mathscr{O}_U$ onto $\mathscr{O}_u$. Then we rewrite the system (1.2) to that for a new dependent variable u as

$$A^0(u)u_t + A(u)u_x = B(u)u_{xx} + g(u,u_x). \tag{1.3}$$

In (1.3), $A^0(u)$ is real symmetric and positive definite of the form

$$A^0(u) = \begin{pmatrix} A_1^0(u) & 0 \\ 0 & A_2^0(u) \end{pmatrix}$$

where $A_1^0(u)$ and $A_2^0(u)$ are real symmetric and positive definite; $A(u)$ is real symmetric of the form

$$A(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{pmatrix}$$

where $A_{11}(u)$ and $A_{22}(u)$ are symmetric and ${}^{\top}A_{12}(u) = A_{21}(u)$; $B(u)$ is real symmetric and non-negative definite of the form

$$B(u) = \begin{pmatrix} 0 & 0 \\ 0 & B_2(u) \end{pmatrix}$$

where $B_2(u)$ is real symmetric and positive definite; $g(u,u_x)$ is a nonlinear term of the form

$$g(u,u_x) = \begin{pmatrix} g_1(u,w_x) \\ g_2(u,u_x) \end{pmatrix}.$$

Using a notation $u = {}^{\top}(v,w)$ where $v = v(t,x) \in \mathbb{R}^{m_1}$ and $w = w(t,x) \in \mathbb{R}^{m_2}$, we deduce the system (1.3) to the decomposed form

$$A_1^0(u)v_t + A_{11}(u)v_x + A_{12}(u)w_x = g_1(u,w_x), \tag{1.4a}$$

$$A_2^0(u)w_t + A_{21}(u)v_x + A_{22}(u)w_x = B_2(u)w_{xx} + g_2(u,u_x). \tag{1.4b}$$

For the system (1.4), we put the following condition.

[A3] The matrix $A_{11}(u_+)$ is negative definite for some $u_+ \in \mathscr{O}_u$.

The assumption **[A3]** corresponds to the outflow problem for the model system of compressible viscous gases discussed in [5, 6, 13]. We prescribe the initial and the boundary conditions for (1.4) as

$$u(0,x) = u_0(x) = {}^{\top}(v_0, w_0)(x), \ \text{ i.e., } \ (v,w)(0,x) = (v_0,w_0)(x), \tag{1.5}$$

$$w(t,0) = w_b, \tag{1.6}$$

where $w_b \in \mathbb{R}^{m_2}$ is a constant. We assume that a spatial asymptotic state of the initial data is a constant:

$$\lim_{x\to\infty} u_0(x) = u_+ = {}^{\top}(v_+, w_+), \ \text{ i.e., } \ \lim_{x\to\infty}(v_0,w_0)(x) = (v_+,w_+). \tag{1.7}$$

We show the existence of a solution to the problem (1.4)–(1.6) globally in time under the smallness assumption on $|w_b - w_+|$. Thus the condition **[A3]** yields that the characteristics of the hyperbolic system (1.4a) around the boundary are negative. Therefore the boundary condition (1.6) is necessary and sufficient for the well-posedness.

For the heat-conductive model of compressible viscous gases in $\mathbb{R}^3$, Matsumura and Nishida in [8] show the asymptotic stability of a constant state (or a stationary solution corresponding to an external potential force) and establish a technical energy method. For the system (1.1) in the full space $\mathbb{R}^n$, Umeda, Kawashima and Shizuta in [16] consider a sufficient condition which guarantees a dissipative structure of the system (1.1) and show the asymptotic stability of the constant state.

For a barotropic model of compressible viscous gases in half space, Kawashima, Nishibata and Zhu in [6] consider an outflow problem, where a negative Dirichlet data for the velocity is imposed, and show the existence and the asymptotic stability of boundary layer solutions. The generalization of this problem to a multi-dimensional half space $\mathbb{R}^n_+ = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ is considered by Kagei and Kawashima in [2]. For the heat-conductive model, Kawashima, Nakamura, Nishibata and Zhu [5] prove the existence and the asymptotic stability of boundary layer solutions for the outflow problem. For the inflow problem, the barotropic model is considered in [10] and the heat-conductive model is considered in [1, 12, 14].

Notations. For vectors $u, v \in \mathbb{R}^m$, $|u|$ and $\langle u, v\rangle$ denote standard Euclidean norm and inner product, respectively. For a matrix A, ${}^{\top}A$ denotes a transport matrix of A. For $1 \le p \le \infty$, $L^p(\mathbb{R}_+)$ denotes a standard Lebesgue space over $\mathbb{R}_+$ equipped with a norm $\|\cdot\|_{L^p}$. For a non-negative integer s, $H^s(\mathbb{R}_+)$ denotes an s-th order Sobolev space over $\mathbb{R}_+$ in the L^2 sense with a norm $\|\cdot\|_{H^s}$. Notice that $H^0(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ and $\|\cdot\|_{H^0} = \|\cdot\|_{L^2}$. For a function $f = f(u)$, $D_u f(u)$ denotes a Fréchet derivative of f with respect to u. Especially, in the case of $u = {}^{\top}(u_1, \ldots, u_n) \in \mathbb{R}^n$ and $f(u) = {}^{\top}(f_1, \ldots, f_m)(u) \in \mathbb{R}^m$, the Fréchet derivative $D_u f = (\frac{\partial f_i}{\partial u_j})_{ij}$ is an $m \times n$ matrix. For a function $f = f(v,w)$, we sometimes abbreviate partial Fréchet derivatives $D_v f(v,w)$ and $D_w f(v,w)$ to $f_v(v,w)$ and $f_w(v,w)$, respectively.

2 Existence of stationary solution

The stationary wave $\tilde{U}(x)$ is defined as a smooth stationary solution to (1.1) which converges to a constant state $U_+ = U(u_+)$ as $x \to \infty$. Thus $\tilde{U}$ satisfies a system of equations

$$f(\tilde{U})_x = (G(\tilde{U})\tilde{U}_x)_x, \quad x \in \mathbb{R}_+. \tag{2.1}$$

Let $\tilde{u} = {}^\top(\tilde{v}, \tilde{w})$ be a stationary solution for (1.4). Since the transformation $U \mapsto u$ is a diffeomorphism, we have a relation between $\tilde{U}$ and $\tilde{u}$, i.e., $\tilde{u} = u(\tilde{U})$ and $\tilde{U} = U(\tilde{u})$. We assume that $\tilde{u}$ satisfies the same conditions (1.6) and (1.7), that is,

$$\tilde{w}(0) = w_b, \tag{2.2a}$$

$$\lim_{x\to\infty} \tilde{u}(x) = u_+, \text{ i.e., } \lim_{x\to\infty} (\tilde{v}, \tilde{w})(x) = (v_+, w_+). \tag{2.2b}$$

The existence of the stationary solution for the boundary value problem (2.1) and (2.2) is shown in Theorem 2.1 below. We note that the non-degenerate stationary solution exists if the number of negative characteristics is greater than the number of hyperbolic equations (1.4a). In order to handle the degenerate case, we have to assume that the matrix $D_U f(U_+)$ has a simple eigenvalue 0. Let $\mu(U)$ be an eigenvalue of the matrix $D_U f(U)$ satisfying $\mu(U_+) = 0$ and let $R(U)$ be a right eigenvector of $D_U f(U)$ corresponding to $\mu(U)$. We also use a notation $\#^-(A)$ which denotes the number of negative eigenvalues of a matrix A.

Theorem 2.1. *Assume that* **[A1]–[A3]** *hold and let* $\delta := |w_+ - w_b|$.

(i) (Non-degenerate case) *We assume that*

$$\#^-(D_U f(U_+)) > m_1$$

holds. Then there exists a local stable manifold $\mathcal{M}^s \subset \mathbb{R}^{m_2}$ *around the equilibrium* w_+ *such that if* $w_b \in \mathcal{M}^s$ *and* δ *is sufficiently small, then there exists a unique smooth solution* $\tilde{u}(x)$ *to* (2.1) *and* (2.2) *satisfying*

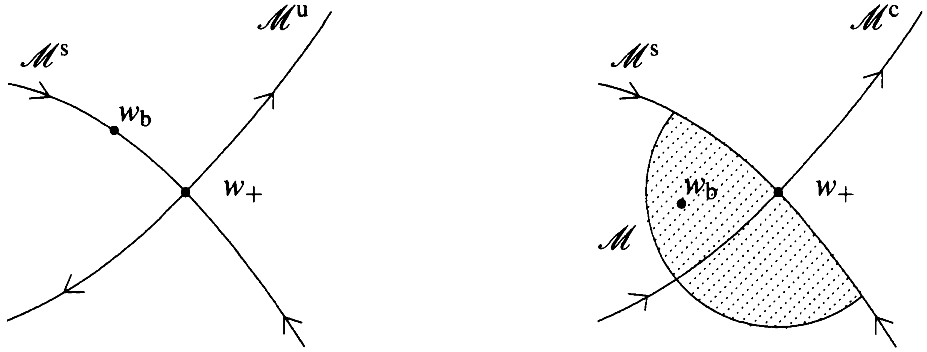
$$|\partial_x^k(\tilde{u}(x) - u_+)| \le C\delta e^{-cx} \text{ for } k = 0, 1, \ldots. \tag{2.3}$$

(ii) (Degenerate case) *We assume that* $D_U f(U_+)$ *has a simple eigenvalue 0, i.e.,* $\mu(U_+) = 0$. *Moreover we assume that the characteristic field corresponding to* $\mu(U_+) = 0$ *is genuinely nonlinear, that is,*

$$D_U \mu(U_+) R(U_+) \neq 0. \tag{2.4}$$

Then there exists a certain region $\mathcal{M} \subset \mathbb{R}^{m_2}$ *such that if* $w_b \in \mathcal{M}$ *and* δ *is sufficiently small, then there exists a unique smooth solution* $\tilde{u}(x)$ *satisfying*

$$|\partial_x^k(\tilde{u}(x) - u_+)| \le C\frac{\delta^{k+1}}{(1+\delta x)^{k+1}} + C\delta e^{-cx} \text{ for } k = 0, 1, \ldots.$$



(i) Non-degenerate case (ii) Degenerate case

Figure 1: The left figure shows the state space of $\tilde{w} \in \mathbb{R}^{m_2}$ for the non-degenerate case. The region $\mathscr{M}^{\mathrm{u}}$ means a local unstable manifold. The right figure is the degenerate case. $\mathscr{M}^{\mathrm{c}}$ is a local center manifold corresponding to eigenvalue 0.

Proof. Here we only give a brief outline of the proof for the non-degenerate case (i). For more details, see the paper [11]. Integrate (2.1) over (x,∞) with using a property $\tilde{U}_x(x) \to 0$ as $x \to \infty$ and symmetrize the resultant equation, we have

$$B(\tilde{u})\tilde{u}_x = A(u_+)(\tilde{u} - u_+) + O(|\tilde{u} - u_+|^2)$$

which is rewritten to

$$0 = A_{11}(u_+)(\tilde{v} - v_+) + A_{12}(u_+)(\tilde{w} - w_+) + O(|\tilde{u} - u_+|^2), \tag{2.5a}$$

$$B_2(\tilde{u})\tilde{w}_x = A_{21}(u_+)(\tilde{v} - v_+) + A_{22}(u_+)(\tilde{w} - w_+) + O(|\tilde{u} - u_+|^2). \tag{2.5b}$$

Due to the assumption **[A3]**, we solve (2.5a) with respect to $\tilde{v}$ by using the implicit function theorem. Thus $\tilde{v}$ is represented as a function of $\tilde{w}$

$$\tilde{v} - v_+ = \Gamma(\tilde{w} - w_+) + O(|\tilde{w} - w_+|^2), \quad \Gamma := -A_{11}(u_+)^{-1}A_{12}(u_+). \tag{2.6}$$

Substituting (2.6) in (2.5b), we get an $m_2 \times m_2$ system of differential equations for $\tilde{w}$

$$\tilde{w}_x = \tilde{A}(\tilde{w} - w_+) + O(|\tilde{w} - w_+|^2), \tag{2.7}$$

$$\tilde{A} := B_2(u_+)^{-1}(-A_{21}(u_+)A_{11}(u_+)^{-1}A_{12}(u_+) + A_{22}(u_+)).$$

Moreover, we see that

$$\#^-(\tilde{A}) = \#^-(A(u_+)) - \#^-(A_{11}(u_+)) = \#^-(f_U(U_+)) - m_1 > 0$$

which yields the existence of the local stable manifold. Therefore we complete the proof. □

3 Asymptotic stability of stationary solution

We next show the asymptotic stability of the stationary solution of which existence is shown in Theorem 2.1. To do this, we have to assume a condition which guarantees a dissipative structure of the system. This kind of dissipative structure is firstly studied by Kawashima in his doctorial thesis in 1984, and the following condition is imposed in [3, 15, 16].

[A4] Let $\lambda A^0(u_+)\phi = A(u_+)\phi$ and $B(u_+)\phi = 0$ for $\lambda \in \mathbb{R}$ and $\phi \in \mathbb{R}^m$. Then $\phi = 0$.

Kawashima proves the asymptotic stability of a constant state for the full space problem under the stability condition in his papers [3, 4, 15, 16]. The main purpose of the present paper is to show the asymptotic stability of the stationary solution in half space under the stability condition.

We first summarize a result on asymptotic stability of the non-degenerate stationary solution.

Theorem 3.1. *Assume that* **[A1]–[A4]** *hold and that the same assumptions as in Theorem* 2.1-(i) *hold. Then there exists a positive constant* ε_0 *such that if*

$$\|u_0 - \tilde{u}\|_{H^2} + \delta \leq \varepsilon_0,$$

the problem (1.4), (1.5) *and* (1.6) *has a unique solution* $u(t,x)$ *globally in time satisfying*

$$u - \tilde{u} \in C([0,\infty), H^2(\mathbb{R}_+)).$$

Moreover the solution u *converges to the stationary solution* $\tilde{u}$*:*

$$\lim_{t\to\infty} \|u(t) - \tilde{u}\|_{L^\infty} = 0. \tag{3.1}$$

In order to show asymptotic stability of the degenerate stationary solution, we have to assume convexity of the flux function $f(u)$ along a certain vector $\hat{r} \in \mathbb{R}^m$ defined by

$$\hat{r} := \big(D_U\mu(U_+)R(U_+)\big)D_U^2\eta(U_+)R(U_+). \tag{3.2}$$

Here we have assumed that $D_U\mu(U_+)R(U_+) \neq 0$ for existence of the degenerate stationary solution. Notice that the direction of $\hat{r}$ is uniquely determined since $\hat{r}$ is independent of the sign of $R(U_+)$. We also note that $\hat{r}$ is a right eigenvector of $\hat{A}(\hat{U}(U_+))$ corresponding to the eigenvalue 0. Using $\hat{r}$, we define a scalar function $\hat{f}(U)$ by

$$\hat{f}(U) := \langle \hat{r}, f(U)\rangle.$$

In Theorem 3.2, by assuming that $\hat{f}(U)$ is convex at $U = U_+$, we show asymptotic stability of the degenerate stationary solution.

Theorem 3.2. *Assume that* **[A1]–[A4]** *hold and that the same assumptions as in Theorem* 2.1-(ii) *hold. Moreover, we assume that a scalar function* $\hat{f}(U)$ *is convex at* $U = U_+$*, that is, the Hessian matrix* $D_U^2\hat{f}(U_+)$ *is non-negative definite. Then the same conclusion as in Theorem* 3.1 *holds true.*

The crucial point of proof of Theorems 3.1 and 3.2 is to obtain a uniform a priori estimate of a perturbation from the stationary solution

$$(\varphi, \psi)(t,x) := (v,w)(t,x) - (\tilde{v}, \tilde{w})(x).$$

We also use a notation $\xi(t,x) := u(t,x) - \tilde{u}(x) = (\varphi,\psi)(t,x)$. From (1.3), ξ satisfies

$$A^0(u)\xi_t + A(u)\xi_x = B(u)\xi_{xx} + h, \tag{3.3}$$
$$h := g - \tilde{g} - (A - \tilde{A})\tilde{u}_x + (B - \tilde{B})\tilde{u}_{xx},$$

where $\tilde{g} := g(\tilde{u},\tilde{u}_x)$, $\tilde{A} := A(\tilde{u})$, $\tilde{B} := B(\tilde{u})$. We also have the equation for (φ,ψ) from (1.4) as

$$A_1^0(u)\varphi_t + A_{11}(u)\varphi_x + A_{12}(u)\psi_x = h_1, \tag{3.4a}$$
$$A_2^0(u)\psi_t + A_{21}(u)\varphi_x + A_{22}(u)\psi_x = B_2(u)\psi_{xx} + h_2, \tag{3.4b}$$

where $h = {}^\top(h_1,h_2)$ and

$$h_1 := g_1 - \tilde{g}_1 - (A_{11} - \tilde{A}_{11})\tilde{v}_x - (A_{12} - \tilde{A}_{12})\tilde{w}_x,$$
$$h_2 := g_2 - \tilde{g}_2 - (A_{21} - \tilde{A}_{21})\tilde{v}_x - (A_{22} - \tilde{A}_{22})\tilde{w}_x + (B_2 - \tilde{B}_2)\tilde{w}_{xx},$$

where $\tilde{g}_1 := g_1(\tilde{u},\tilde{w}_x)$, $\tilde{g}_2 := g_2(\tilde{u},\tilde{u}_x)$, $\tilde{A}_{ij} := A_{ij}(u)$ $(i,j = 1,2)$, $\tilde{B}_2 := B_2(\tilde{u})$. The initial and the boundary conditions are prescribed as

$$(\varphi,\psi)(0,x) = (\varphi_0,\psi_0)(x) := (v_0,w_0)(x) - (\tilde{v},\tilde{w})(x), \tag{3.5}$$
$$\psi(t,0) = 0. \tag{3.6}$$

To summarize the a priori estimate for a solution (φ,ψ) in Sobolev space H^2, we define an energy norm $N(t)$

$$N(t) := \sup_{0\le\tau\le t} \|(\varphi,\psi)(\tau)\|_{H^2}.$$

Proposition 3.3. *Let $(\varphi,\psi) \in C([0,T];H^2(\mathbb{R}_+))$ be a solution to* (3.4)–(3.6) *for a certain $T > 0$. Then there exists a positive constant ε_1 such that if $N(t) + \delta \le \varepsilon_1$, the solution satisfies*

$$\|(\varphi,\psi)(t)\|_{H^2}^2 + \int_0^t \left(\|\varphi_x(\tau)\|_{H^1}^2 + \|\psi_x(\tau)\|_{H^2}^2\right) d\tau \le C\|(\varphi_0,\psi_0)\|_{H^2}^2. \tag{3.7}$$

The proof of Proposition 3.3 is divided into several steps. In this paper, we only show the brief derivation of the basic L^2 estimate for the non-degenerate case. To this end, we employ an energy form $\mathscr{E}$ defined by

$$\mathscr{E} := \eta(U) - \eta(\tilde{U}) - D_U\eta(\tilde{U})(U - \tilde{U}).$$

Note that, if $N(t)$ is sufficiently small, the energy form $\mathscr{E}$ is equivalent to $|(\varphi,\psi)|^2$ because the Hessian matrix $D_z^2\eta$ is positive. From a direct computation, we see that $\mathscr{E}$ satisfies

$$\mathscr{E}_t + \mathscr{F}_x + \langle B_2(u)\psi_x, \psi_x\rangle + \mathscr{G} = \mathscr{B}_x + \mathscr{R}, \tag{3.8}$$

where

$$\mathscr{F} := q(U) - q(\tilde{U}) - D_U\eta(\tilde{U})(f(U) - f(\tilde{U})),$$
$$\mathscr{B} := \big(D_U\eta(U) - D_U\eta(\tilde{U})\big)(G(U)U_x - G(\tilde{U})\tilde{U}_x),$$
$$\mathscr{G} := D_U\eta(\tilde{U})_x(f(U) - f(\tilde{U})) - \big(D_U\eta(U) - D_U\eta(\tilde{U})\big)f(\tilde{U})_x,$$

and $\mathscr{R}$ is a remainder term satisfying

$$\int_{\mathbb{R}_+} |\mathscr{R}|\,dx \leq C\int_{\mathbb{R}_+} (|\tilde{u}_x||\xi||\xi_x| + |\tilde{u}_x|^2|\xi|^2)\,dx \leq C\delta(|\varphi(t,0)|^2 + \|\xi_x(t)\|_{L^2}^2). \tag{3.9}$$

By using the boundary condition (3.6) and $A_{11}(u_+) < 0$ in **[A3]** as well as a smallness of $N(t)$, we have

$$\int_{\mathbb{R}_+} \mathscr{F}_x\,dx = -\mathscr{F}\big|_{x=0} \geq -\frac{1}{2}\langle A(\tilde{u})\xi, \xi\rangle\big|_{x=0} - C|\xi(t,0)|^3 \geq c|\varphi(t,0)|^2, \tag{3.10}$$

$$\left|\int_{\mathbb{R}_+} \mathscr{B}_x\,dx\right| = \left|\mathscr{B}|_{x=0}\right| \leq C(N(t)+\delta)|\varphi(t,0)|^2. \tag{3.11}$$

Due to the positivity of $B_2(u)$, we have

$$\int_{\mathbb{R}_+} \langle B_2(u)\psi_x, \psi_x\rangle\,dx \geq c\|\psi_x\|_{L^2}^2. \tag{3.12}$$

By using the expression of $\mathscr{G}$ as

$$\mathscr{G} = \frac{1}{2}\langle D_{\tilde{U}}^2\eta(\tilde{U})\tilde{U}_x, f_{UU}(\tilde{U})\Xi^2\rangle + O(|\tilde{U}_x||\Xi|^3 + |f(\tilde{U})_x||\Xi|^2), \tag{3.13}$$

where $\Xi := U - \tilde{U}$, we estimate the integral of $\mathscr{G}$ as

$$\int_{\mathbb{R}_+} |\mathscr{G}|\,dx \leq C\int_{\mathbb{R}_+} |\tilde{u}_x||\xi|^2\,dx \leq C\delta(|\varphi(t,0)|^2 + \|\xi_x(t)\|_{L^2}^2). \tag{3.14}$$

Here we have utilized the Poincaré type inequality

$$\int_{\mathbb{R}_+} e^{-cx}|\xi(x)|^2\,dx \leq C(|\xi(0)|^2 + \|\xi_x\|_{L^2}^2)$$

in deriving (3.9) and (3.14). Therefore, combining the estimates obtained above, we obtain the basic L^2 estimate for the non-degenerate case as

$$\|(\varphi,\psi)(t)\|_{L^2}^2 + \int_0^t (\|\psi_x(\tau)\|_{L^2}^2 + |\varphi(\tau,0)|^2)\,d\tau \\ \leq C\|(\varphi_0,\psi_0)\|_{L^2}^2 + C\delta\int_0^t \|\varphi_x(\tau)\|_{L^2}^2\,d\tau \tag{3.15}$$

provided that $N(t)+\delta$ is sufficiently small. For the degenerate case, owing to the algebraic convergence of the degenerate stationary solution, we can not obtain the same estimate (3.14) of $\mathscr{G}$ as the non-degenerate case. To overcome this difficulty, we show that the leading part of $\mathscr{G}$ is non-negative under the assumption in Theorem 3.2 that $\hat{f}(U)$ is convex.

To complete the proof of Proposition 3.3, we derive the estimates for the higher order derivatives. To do this, we combine the energy method in half space discussed in [9] and the dissipative estimate of the hyperbolic part under the stability condition. For more detailed computations, see [11].

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