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Note on total and partial functions in second-order arithmetic

Makoto Fujiwara*
Mathematical Institute
Tohoku University
6-3, Aramaki Aoba, Aoba-ku
Sendai, Miyagi, Japan
sb0m29@math.tohoku.ac.jp

Takashi Sato
Mathematical Institute
Tohoku University

Abstract

We analyze the total and partial extendability for partial functions in second-order arithmetic. In particular, we show that the partial extendability for consistent set of finite partial functions and that for consistent sequence of partial functions are equivalent to WKL (weak König's lemma) over RCA_0, despite the fact that their total extendability derives ACA (arithmetical comprehension).

1 Introduction

The mainstream of reverse mathematics has been developed in second-order arithmetic, whose language is set-based and functions are represented as their graphs [4]. For this reason, some results on partial functions in reverse mathematics do not correspond to those in computability theory while many results in computability theory can be transformed into the corresponding results in reverse mathematics straightforwardly. For example, it is known in computability theory that every computable partial $\{0,1\}$-valued function has a total extension with low degree, and also that there exists a computable partial $\{0,1\}$-valued function which has no computable total extension (cf. [1, Lemma 8.17]). One can, however, easily see that the assertion that every partial $\{0,1\}$-valued function has a total extension is provable in RCA_0. On the other hand, as we will see in Section 2, the assertion that every partial function has a total extension is equivalent to arithmetical comprehension over RCA_0. That is to say, the treatment of partial functions in second-order arithmetic sometimes causes a different situation from computability theory. Based on this insight, we investigate some basic properties on partial functions in the context of reverse mathematics. We recall that in our setting, a function f is a set of pairs such that if $(n,m),(n,m')\in f$ then $m=m'$. A function f is said to be *total* if for all n there exists m such that $(n,m)\in f$ and to be *partial* otherwise. We use a notation $f:X\to Y$ only for total functions and $f:\subset X\to Y$ denotes that f is (a graph of) a (possibly partial) function from X to Y. For $f:\subset X\to Y$, $\mathrm{dom}(f)$ denotes the domain of f. We refer the reader to Simpson's book [4] for other basic definitions and comprehensive treatment of ordinary mathematics in second-order arithmetic as well as the coding of pairs, finite sets, finite sequences etc.

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2 Totalization of Partial Function

We first show that totalization of a partial function requires arithmetical comprehension scheme.

Theorem 1. *The following are equivalent over* RCA_0.

1. ACA.
2. *Every partial function* $f :\subset \mathbb{N} \to \mathbb{N}$ *has its domain* $\mathrm{dom}(f)$, *i.e., a set* $X \subseteq \mathbb{N}$ *such that* $\forall n(n \in X \leftrightarrow \exists m(f(n) = m))$.
3. *Every partial function* $f :\subset \mathbb{N} \to \mathbb{N}$ *has its total extension* $\tilde{f} : \mathbb{N} \to \mathbb{N}$, *i.e., a total function* $\tilde{f} : \mathbb{N} \to \mathbb{N}$ *such that* $\forall n \forall m(f(n) = m \to \tilde{f}(n) = m)$.

Proof. We reason in RCA_0.

$(1 \to 2)$ is trivial since $\mathrm{dom}(f)$ is Σ^0_1 definable in f.

$(2 \to 3)$ is shown by taking the total extension $\tilde{f}$, by Σ^0_0 comprehension, as

$$\tilde{f} := \{(n,m) : (n,m) \in f \vee (n \notin \mathrm{dom}(f) \wedge m = 0)\}.$$

$(3 \to 1)$ is shown via [4, Lemma III.1.3] as follows. Let $g : \mathbb{N} \to \mathbb{N}$ be a one-to-one function. Take a partial inverse function f of g as $f := \{(n,m) : g(m) = n\}$ by Σ^0_0 comprehension. By our assumption 3, there exists a total extension $\tilde{f}$ of f. Then it is easy to see that

$$\exists n(g(n) = m) \leftrightarrow g(\tilde{f}(m)) = m)$$

holds for all m. Thus we have the image of g by Σ^0_0 comprehension. □

For further understanding, let us consider computable partial functions as their program codes in second-order arithmetic. Note that one can construct the program code for a given (graph of) computable partial function in RCA_0 , while it seems to be impossible to construct the corresponding graph for a given program code of computable partial function in RCA_0. This indicates that for a computable partial function, its graph has more information than its program code. The previous theorem states that even if partial functions are given as their graphs, the totalization requires arithmetical comprehension. On the other hand, one can easily see that RCA_0 proves that every graph of partial $\{0,1\}$-valued function has a total extension in contrast to [1, Lemma 8.17] in computability theory. If partial $\{0,1\}$-valued functions are represented as their program codes, the corresponding fact to [1, Lemma 8.17] holds in second-order arithmetic:

Proposition 2. *The following assertion is provable in* WKL_0 *but is not provable in* RCA[1]*: for any program code e of computable partial* $\{0,1\}$*-valued function (i.e.,* $\{e\} :\subset \mathbb{N} \to 2$*), there exists its total extension* $f_e : \mathbb{N} \to 2$, *i.e.,* $T(e,i,z) \to f_e(i) = U(z)$ *where* $T(e,i,z)$ *expresses that the Turing machine with Gödel number e applied to the input i terminates with a computation whose Gödel number is z and* $U(z)$ *is its output.*

Proof. To show our assertion in WKL_0, by Σ^0_0-comprehension, take a set B of all $t \in 2^{<\mathbb{N}}$ such that $\forall z,k < t(T(e,k,z) \to U(k) = t(k))$. Then one can easily see that T is an infinite binary tree and that a path through B obtained by WKL is a total extension of $\{e\}$.

On the other hand, it is not true in the minimum ω-model REC $:= \{A \in P(\omega) : A$ is computable$\}$ of RCA (cf. [1, Lemma 8.17]). □

If we consider the corresponding assertion for general computable partial (not necessarily $\{0,1\}$-valued) functions, it is verified in the same manner by using König's lemma (which is equivalent to ACA over RCA_0 [4, Lemma III.7.2]) instead of weak König's lemma.

[1] RCA denotes the system RCA_0+full (second-order) induction scheme.

3 Total and Partial Extension of Consistent Partial Functions

Next we consider compactness-like property on partial functions. For $\gamma_1 :\subset X \to Y$ and $\gamma_2 :\subset X \to Y$, $\gamma_1 \preceq \gamma_2$ denotes that γ_2 is an extension of γ_1, i.e., $dom(\gamma_1) \subseteq dom(\gamma_2)$ and $\gamma_1(x) = \gamma_2(x)$ for $x \in dom(\gamma_1)$. In general, a set $\mathscr{F}$ of partial functions is said to be *consistent* (cf. [2, Section 1.10]) if for all finite subset $\mathscr{G}$ of $\mathscr{F}$, there exists a (partial) function f such that $\gamma \preceq f$ for each $\gamma \in \mathscr{G}$. As mentioned in [2, Section 1.10], one may think of a total function $f : X \to Y$ as being built up from tokens of information, each of which is a partial function $\sigma :\subset X \to Y$ with finite domain. Motivated by this idea, we first define the notion of consistency over sets of finite partial functions in RCA_0 to investigate the extendability of consistent set of finite partial functions.

Definition 3. A set $\mathscr{F}$ of (codes of) finite partial functions is *consistent* if for all n, $\gamma_1 \in \mathscr{F}$ and $\gamma_2 \in \mathscr{F}$ such that $n \in dom(\gamma_1) \cap dom(\gamma_2)$, $\gamma_1(n) = \gamma_2(n)$ holds. [2]

On the other hand, it holds that any consistent set $\mathscr{F}$ of (not necessarily finite) partial functions has a total extension f, namely, f is a total function from X to Y such that $\gamma \preceq f$ for any $\gamma \in \mathscr{F}$ (cf. [2, Section 1.10]). Then our another goal is to investigate the strength of this assertion. Here the obstacle is that a set $\mathscr{F}$ of (not necessarily finite) partial functions is not naturally represented in the language of second-order arithmetic.[3] To deal with the analogue of this assertion in second-order arithmetic, we introduce the notion of consistency over sequences of partial functions.

Definition 4. A sequence $\langle f_i \rangle_{i \in \mathbb{N}}$ of partial functions is *consistent* if for all $i, j, n \in \mathbb{N}$ such that $n \in dom(f_i) \cap dom(f_j)$, $f_i(n) = f_j(n)$ holds.

We are now ready to mention our main results.

Theorem 5. *The following are equivalent over* RCA_0.

1. WKL.
2. *Every consistent sequence* $\langle f_i \rangle_{i \in \mathbb{N}}$ *of partial functions has a partial extension* f*, i.e., a graph* f *of partial function such that* $f_i \subset f$ *for all* $i \in \mathbb{N}$.
3. *Every consistent set* $\mathscr{F}$ *of (codes of) finite partial functions has a partial extension* f*, i.e., a graph* f *of partial function such that* $\sigma \subset f$ *for all* $\sigma \in \mathscr{F}$.

Proof. We reason in RCA_0.

($1 \to 2$): Let $\varphi(n,m) :\equiv \exists i(f_i(n) = m)$ and $\psi(n,m) :\equiv \exists i(f_i(n) \neq m)$ respectively. Then φ and ψ are Σ^0_1 and there is no (n,m) satisfying both of φ and ψ. By Σ^0_1 separation (derived from WKL [4, Lemma IV.4.4]), we have a set X such that

$$(\varphi(n,m) \to (n,m) \in X) \wedge ((n,m) \in X \to \neg\psi(n,m))$$

for all $n, m \in \mathbb{N}$. Let

$$f := \{(n,m) : (n,m) \in X \wedge \forall m' < m((n,m') \notin X)\}$$

[2]One can easily see that this condition is equivalent to the definition of consistency in [2, Section 1.10] over RCA_0.

[3]It is possible and expected to formalize this assertion in finite-type extension RCA_0^ω (cf. [3]) of RCA_0 and investigate its strength.

by Σ^0_0 comprehension. Then f is a desired partial function.

(2 → 3): Enumerate $\mathscr{F}$ as $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ and define a consistent sequence $\langle f_i\rangle_{i\in\mathbb{N}}$ such that each f_i is a partial function coded by σ_i.

(3 → 1): We assume 3 to show WKL via [4, Lemma IV.4.4]. Let $g,h:\mathbb{N}\to\mathbb{N}$ be one-to-one functions such that $g(i)\neq h(j)$ for all $i,j\in\mathbb{N}$. By Σ^0_0 comprehension, take a sequence $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ such that each σ_i is the code of (graph of) finite partial function

$$\{(2g(i),0),(2h(i),0),(2i+1,1)\}.$$

Since σ_i contains $(2i+1,1)$, we have $i\leq\sigma_i$ for each i (under the standard coding of finite sets e.g. in [4]). Therefore, there exists the set $\mathscr{F}$ of σ_i's in RCA_0. Since $\mathscr{F}$ is trivially consistent, by our assumption 3, we have its partial extension f. Taking $X:=\{n:(2n,0)\in f\}$, it follows that $n\in X$ if n is in the range of g and $n\notin X$ if n is in the range of h. This completes the proof. □

Corollary 6. *The following are equivalent over* RCA_0.

1. ACA.
2. *Every consistent sequence* $\langle f_i\rangle_{i\in\mathbb{N}}$ *of partial functions has a total extension* f.
3. *Every consistent set* $\mathscr{F}$ *of finite partial functions has a total extension* f.
4. *Every consistent set* $\mathscr{F}$ *of finite partial functions with single domain has a total extension* f.

Proof. (1 → 2) follows from Theorem 5 and Theorem 1. (2 → 3) and (3 → 4) is easy. We show (4 → 1) over RCA_0 via Theorem 1. Let $f:\subset\mathbb{N}\to\mathbb{N}$ be a graph of partial function. Define $\mathscr{F}$ as the set of codes of $\{(n,m)\}$'s such that $(n,m)\in f$. Then $\mathscr{F}$ is a consistent set of finite partial functions with single domain and its total extension is clearly a total extension of f. □

At the end, we investigate the above assertions especially for $\{0,1\}$-valued functions. In contrast to the previous case, all of the corresponding assertions are equivalent to weak König's lemma over RCA_0:

Proposition 7. *The following are equivalent over* RCA_0.

1. WKL.
2. *Every consistent sequence* $\langle f_i\rangle_{i\in\mathbb{N}}$ *of partial* $\{0,1\}$*-valued functions has a total extension* f.
3. *Every consistent sequence* $\langle f_i\rangle_{i\in\mathbb{N}}$ *of partial* $\{0,1\}$*-valued functions has a partial extension* f.
4. *Every consistent set* $\mathscr{F}$ *of finite partial* $\{0,1\}$*-valued functions has a total extension* f.
5. *Every consistent set* $\mathscr{F}$ *of finite partial* $\{0,1\}$*-valued functions has a partial extension* f.

Proof. We reason in RCA_0.

(1 → 2): The idea of proof is the same as for Proposition 2. Let B be the set of all $t\in 2^{<\mathbb{N}}$ such that $\forall i,k<\mathrm{lh}(t)\forall v<2((k,v)\in f_i\to t(i)=v)$. Then B is an infinite binary tree. By WKL, there exists an infinite path p through B. It is easy to see that p is a desired total function.

(2 → 3), (2 → 4), (3 → 5), (4 → 5) are easy.

(5 → 1): The proof of (3 → 1) in Theorem 5 works as well. In fact, the consistent set $\mathscr{F}$ which we construct there consists of partial $\{0,1\}$-valued functions. □

The above proof suggests that Proposition 7 is a reverse mathematical analogue of [1, Lemma 8.17] in computability theory.

Remark 8. By careful inspection, it is found that all equivalences presented in this paper can be established even over RCA_0^* without the scheme of Σ_1^0 induction (See [4, Definition X.4.1] for the precise definition of RCA_0^*).

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