数理解析研究所講究録
第 1952 巻 2015 年 46-54

ON A STABILITY OF HEAT KERNEL ESTIMATES UNDER FEYNMAN-KAC PERTURBATIONS FOR DIFFUSION PROCESSES

Daehong Kim and Kazuhiro Kuwae

1. Preliminaries

This note is an intermediate report of [14] for diffusion cases. Let (E,d) be a locally compact complete separable metric space. We assume that E is connected and unbounded, and any closed ball of (E,d) is compact. Let $\mathfrak{m}$ be a Radon measure with full support. We write $B_r(x)=\{y\in E\mid d(x,y)<r\}$ and $V(x,r):=\mathfrak{m}(B_r(x))$. We consider a strongly local irreducible regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(E;\mathfrak{m})$ and let $\mathbf{X}=(\Omega,X_t,\mathbf{P}_x)_{x\in E}$ be an $\mathfrak{m}$-symmetric diffusion process associated to $(\mathcal{E},\mathcal{F})$ (see [6] for details). We always assume that $\mathbf{X}$ admits a jointly continuous heat kernel $p_t(x,y)$.

For $\alpha>0$, we define the α-order resolvent kernel

$$R_\alpha(x,y)=\int_0^\infty e^{-\alpha t}p_t(x,y)\mathrm{d}t,\quad x,y\in E.$$

When the process $\mathbf{X}$ is transient, we can define 0-order resolvent kernel $R(x,y):=R_0(x,y)<\infty$ for $x,y\in E$ with $x\neq y$. $R(x,y)$ is called the Green kernel of $\mathbf{X}$. For a non-negative Borel measure ν, we write $R_\alpha\nu(x):=\int_E R_\alpha(x,y)\nu(\mathrm{d}y)$, $R\nu(x):=R_0\nu(x)$ and $R_\alpha f(x)=R_\alpha\nu(x)$ when $\nu(\mathrm{d}x)=f(x)\mathrm{d}x$ for any $f\in\mathcal{B}_+(E)$ or $f\in\mathcal{B}_b(E)$. Here $\mathcal{B}_+(E)$ (resp. $\mathcal{B}_b(E)$) denotes the space of non-negative (resp. bounded) Borel functions on E. The space of bounded continuous functions on E will be denoted as $C_b(E)$. Let E_∂ be the one point compactification of E. An increasing sequence $\{F_k\}$ of closed sets is said to be a *strict $\mathcal{E}$-nest* if $\mathbf{P}_x(\lim_{k\to\infty}\sigma_{F_k^c}=\infty)=1$ $\mathfrak{m}$-a.e. $x\in E$. Here $\sigma_{F_k^c}:=\inf\{t>0\mid X_t\in E\setminus F_k\}$ is the first hitting time of X_t to $F_k^c:=E\setminus F_k$. A function f defined on E (resp. E_∂) is said to be *$\mathcal{E}$-quasi continuous* (resp. *strictly $\mathcal{E}$-quasi continuous*) if there exists an $\mathcal{E}$-nest (resp. a strict $\mathcal{E}$-nest) $\{F_k\}$ of closed sets such that $f|_{F_k}$ (resp. $f|_{F_k\cup\{\partial\}}$) is continuous for each $k\in\mathbb{N}$. Since $\mathbf{X}$ is conservative, any $\mathcal{E}$-nest $\{F_k\}$ of closed sets is automatically a strict $\mathcal{E}$-nest. The regularity of the given Dirichlt form $(\mathcal{E},\mathcal{F})$ for $\mathbf{X}$ tells us that there exists an $\mathcal{E}$-nest of compact sets ([18, Chapter V, Proposition 2.12]). Denote by $QC(E_\partial)$ the family of all strictly $\mathcal{E}$-quasi continuous functions on E_∂. Let $(\mathcal{E},\mathcal{F}_e)$ be the extended Dirichlet space of $(\mathcal{E},\mathcal{F})$ and any element $f\in\mathcal{F}_e$ admits a strictly $\mathcal{E}$-quasi continuous version $\tilde{f}$ with $\tilde{f}(\partial)=0$ (the proof is similar to the proof of [6, Theorem 2.1.3] if $\mathbf{X}$ is transient, for general case, it can be proved by way of time change method). Throughout this paper, we always take a strictly $\mathcal{E}$-quasi continuous version of the element of $\mathcal{F}_e$, that is, we omit *tilde* from $\tilde{f}$ for $f\in\mathcal{F}_e$.

Let $S_1(\mathbf{X})$ be the family of positive smooth measures in the strict sense ([6]). A measure $\nu \in S_1(\mathbf{X})$ is said to be *of Dynkin class* (resp. *Green-bounded*) *with respect to* $\mathbf{X}$ if $\sup_{x\in E} R_\beta\nu(x) < \infty$ for some $\beta > 0$ (resp. $\sup_{x\in E} R\nu(x) < \infty$). A measure $\nu \in S_1(\mathbf{X})$ is said to be *of Kato class* (resp. *of extended Kato class*) *with respect to* $\mathbf{X}$ if $\lim_{\beta\to\infty}\sup_{x\in E} R_\beta\nu(x) = 0$ (resp. $\lim_{\beta\to\infty}\sup_{x\in E} R_\beta\nu(x) < 1$). Denote by $S_D^1(\mathbf{X})$ (resp. $S_{D_0}^1(\mathbf{X})$) the family of measures of Dynkin class (resp. of Green-bounded) and by $S_K^1(\mathbf{X})$ (resp. $S_{EK}^1(\mathbf{X})$) the family of measures of Kato class (resp. of extended Kato class). Clearly, $S_K^1(\mathbf{X}) \subset S_{EK}^1(\mathbf{X}) \subset S_D^1(\mathbf{X})$ and $S_{D_0}^1(\mathbf{X}) \subset S_D^1(\mathbf{X})$. Note that any measure $\nu \in S_D^1(\mathbf{X})$ is a positive Radon measure in view of the Stollmann-Voigt's inequality: $\int_E u^2 d\nu \le \|R_\beta\nu\|_\infty \mathcal{E}_\beta(u,u)$, $u \in \mathcal{F}, \beta \ge 0$ ([19, Theorem 3.1]). Conversely, any positive Radon measure ν satisfying $\sup_{x\in E} R_\alpha\nu(x) < \infty$ for some $\alpha > 0$ always belongs to $S_1(\mathbf{X})$ in view of [17, Proposition 3.1].

We say that a positive continuous additive functional (PCAF in abbreviation) in the strict sense A^ν of $\mathbf{X}$ and a positive measure $\nu \in S_1(\mathbf{X})$ are in the Revuz correspondence if they satisfy for any bounded $f \in \mathcal{B}_+(E)$,

$$\int_E f(x)\nu(\mathrm{d}x) = \uparrow \lim_{t\downarrow 0}\frac{1}{t}\int_E \mathbf{E}_x\left[\int_0^t f(X_s)\mathrm{d}A_s^\nu\right]\mathfrak{m}(\mathrm{d}x).$$

It is known that the family of equivalence classes of the set of PCAFs in the strict sense and the family of positive measures belonging to $S_1(\mathbf{X})$ are in one to one correspondence under the Revuz correspondence ([6, Theorem 5.1.4]).

A function f on E is said to be *locally in* $\mathcal{F}$ *in the broad sense* (denoted as $f \in \dot{\mathcal{F}}_{\rm loc}$) if there is an increasing sequence of finely open Borel sets $\{E_n\}$ with $\bigcup_{n=1}^\infty E_n = E$ q.e. and for every $n \ge 1$, there is $f_n \in \mathcal{F}$ such that $f = f_n$ $\mathfrak{m}$-a.e. on E_n. A function f on E is said to be *locally in* $\mathcal{F}$ *in the ordinary sense* (denoted as $f \in \mathcal{F}_{\rm loc}$) if for any relatively compact open set G, there exists an element $f_G \in \mathcal{F}$ such that $f = f_G$ $\mathfrak{m}$-a.e. on G. Clearly $\mathcal{F}_{\rm loc} \subset \dot{\mathcal{F}}_{\rm loc}$. It is shown in [16, Theorem 4.1], $\mathcal{F}_e \subset \dot{\mathcal{F}}_{\rm loc}$.

Take a bounded $u \in \dot{\mathcal{F}}_{\rm loc} \cap QC(E_\partial)$. In [15, Theorem 6.2(1)], the authors proved under the condition $\mu_{\langle u\rangle} \in S_D^1(\mathbf{X})$ that the additive functional $u(X_t) - u(X_0)$ admits the following decomposition:

$$u(X_t) - u(X_0) = M_t^u + N_t^u \quad t \in [0, +\infty[\quad \mathbf{P}_x\text{-a.s.} \tag{1.1}$$

for q.e. $x \in E$, where M^u is a square integrable martingale additive functional, $\mu_{\langle u\rangle}$ is the Revuz measure associated with the quadratic variational processes (or the sharp bracket PCAF) $\langle M^u\rangle$ of M^u, and N^u is a continuous additive functional (CAF in abbreviation) locally of zero energy. We moreover note that (1.1) holds for all $x \in E$ as the strict decomposition provided u is (nearly) Borel and finely continuous on E ([15, Theorem 6.2(2)]). Note that N^u is not a process of finite variation in general. Note that $\mathcal{E}(f,f) = \frac{1}{2}\mu_{\langle f\rangle}(E)$ provided $f \in \mathcal{F}_e$.

2. Green-tight measures of Kato class

We introduce some notions of Green-tight Kato class measures in the strict sense.

Definition 2.1 (Green-tight Kato class measures). Let $\nu \in S_1(\mathbf{X})$.

(1) ν is said to be *of Green-tight Kato class with respect to* $\mathbf{X}$ if $\nu \in S^1_K(\mathbf{X})$ and for any $\varepsilon > 0$ there exists a compact subset $K = K(\varepsilon)$ of E such that

$$\sup_{x \in E} R(\mathbf{1}_{K^c}\nu)(x) < \varepsilon.$$

(2) ν is said to be *of semi-Green-tight extended Kato class with respect to* $\mathbf{X}$ if $\nu \in S^1_{EK}(\mathbf{X})$ and there exists a compact subset K of E such that

$$\sup_{x \in E} R(\mathbf{1}_{K^c}\nu)(x) < 1.$$

(3) ν is said to be *of Green-tight Kato class in the sense of Chen with respect to* $\mathbf{X}$ if for any $\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{x \in E} R(\mathbf{1}_{K^c \cup B}\nu)(x) < \varepsilon.$$

(4) ν is said to be *of semi-Green-tight extended Kato class in the sense of Chen with respect to* $\mathbf{X}$ if there exists a Borel subset K of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{x \in E} R(\mathbf{1}_{K^c \cup B}\nu)(x) < 1.$$

We denote by $S^1_{K_\infty}(\mathbf{X})$ (resp. $S^1_{CK_\infty}(\mathbf{X})$, $S^1_{K_1}(\mathbf{X})$, $S^1_{CK_1}(\mathbf{X})$) the family of Green-tight Kato class measures (resp. the family of Green-tight Kato class measures in the sense of Chen, the family of semi-Green-tight extended Kato class measures, the family of semi-Green-tight Kato class measures in the sense of Chen) with respect to $\mathbf{X}$

Remark 2.2. It is known that $S^1_{CK_\infty}(\mathbf{X}) \subset S^1_{CK_1}(\mathbf{X}) \subset S^1_{D_0}(\mathbf{X}) \cap S^1_{EK}(\mathbf{X})$. Moreover, $S^1_{CK_\infty}(\mathbf{X}) \subset S^1_{K_\infty}(\mathbf{X}) \subset S^1_K(\mathbf{X})$ and $S^1_{CK_1}(\mathbf{X}) \subset S^1_{K_1}(\mathbf{X}) \subset S^1_{EK}(\mathbf{X})$ hold in general (see [3],[11]). However, we have $S^1_{K_\infty}(\mathbf{X}) = S^1_{CK_\infty}(\mathbf{X})$ and $S^1_{CK_1}(\mathbf{X}) \cap S^1_K(\mathbf{X}) = S^1_{K_1}(\mathbf{X}) \cap S^1_K(\mathbf{X})$ provided $\mathbf{X}$ has resolvent Feller property (see [11, Lemma 4.1]).

In order to introduce the new (semi-)Green-tight measures of (extended) Kato class, we explain the notion of weighted capacity of the Dirichlet form associated with the time changed process:

Let $\nu \in S_1(\mathbf{X})$ and denote by A^ν_t the PCAF in the strict sense associated to ν in Revuz correspondence. Denote by S^ν_o the support of A^ν defined by $\mathsf{S}^\nu_o := \{x \in E \mid \mathbf{P}_x(R = 0) = 1\}$, where $R(\omega) := \inf\{t > 0 \mid A^\nu_t(\omega) > 0\}$. S^ν_o is nothing but the fine support of ν, i.e., the topological support of ν with respect to the fine topology of $\mathbf{X}$. Let $(\check{\mathbf{X}}, \nu)$ be the time changed process of $\mathbf{X}$ by A^ν_t and $(\check{\mathcal{E}}, \check{\mathcal{F}})$ the associated Dirichlet form on $L^2(\mathsf{S}^\nu; \nu)$, where S^ν is the support of ν. It is known that $(\check{\mathcal{E}}, \check{\mathcal{F}})$ is a regular Dirichlet form having $\mathcal{C}|_{\mathsf{S}^\nu}$ as its core and $\mathsf{S}^\nu \setminus \mathsf{S}^\nu_o$ is $\check{\mathcal{E}}$-polar, i.e., 1-capacity 0 set with respect to $(\check{\mathcal{E}}, \check{\mathcal{F}})$. The life time of $(\check{\mathbf{X}}, \nu)$ is given by A^ν_ζ. Let $C^\nu : 2^E \to [0, +\infty]$ be the weighted 1-capacity with respect to $(\check{\mathcal{E}}, \check{\mathcal{F}})$.

Now we introduce a new class of (semi-)Green-tight measures of (extended) Kato class.

Definition 2.3 (Natural (semi-)Green-tight measures of (extended) Kato class). Let $\alpha \geq 0$ and $\nu \in S_1(\mathbf{X})$.

(1) ν is said to be an *α-order natural Green-tight measure of Kato class with respect to* $\mathbf{X}$ if $\nu \in S^1_D(\mathbf{X})$ ($\nu \in S^1_{D_0}(\mathbf{X})$ for $\alpha = 0$) and for any $\varepsilon > 0$ there exists a closed subset $K = K(\varepsilon)$ of E and a constant $\delta > 0$ such that for all Borel set $B \subset K$ with $C^\nu(B) < \delta$,

$$\sup_{x\in E} \mathbf{E}_x \left[\int_0^{\tau_{B\cup K^c}} e^{-\alpha t} \mathrm{d}A^\nu_t \right] < \varepsilon.$$

(2) ν is said to be a 0*-order natural semi-Green-tight measure of extended Kato class with respect to* $\mathbf{X}$ if $\nu \in S^1_{D_0}(\mathbf{X})$ and there exists a closed subset K of E and a constant $\delta > 0$ such that for all Borel set $B \subset K$ with $C^\nu(B) < \delta$,

$$\sup_{x\in E} \mathbf{E}_x \left[A^\nu_{\tau_{B\cup K^c}} \right] < 1.$$

In view of the resolvent equation, for positive α, the α-order natural Green-tightness is independent of the choice of $\alpha > 0$. Let denote by $S^1_{NK^+_\infty}(\mathbf{X})$ the family of positive order natural Green-tight measures of Kato class with respect to $\mathbf{X}$. The class $S^1_{NK_\infty}(\mathbf{X})$ (resp. $S^1_{NK_1}(\mathbf{X})$) is then denoted as the family of 0-order natural Green-tight measures of Kato class (resp. the family of 0-order natural semi-Green-tight measures of extended Kato class) with respect to $\mathbf{X}$.

Remark 2.4.

(1) It is proved in [11, Lemma 4.4] that $S^1_{CK_1}(\mathbf{X}) \subset S^1_{NK_1}(\mathbf{X}) \subset S^1_{EK}(\mathbf{X}) \cap S^1_{D_0}(\mathbf{X})$ and $S^1_{CK_\infty}(\mathbf{X}) \subset S^1_{NK_\infty}(\mathbf{X}) \subset S^1_K(\mathbf{X}) \cap S^1_{D_0}(\mathbf{X})$.

(2) The advantage of the new semi-Green-tight measures of extended Kato class is that $S^1_{NK_1}(\mathbf{X})$ is stable under some Girsanov transformation (see Corollaries 5.1 and 5.2 in [11]).

Definition 2.5. Let $R^z(x,y)$ be the Green kernel of Doob's $R(\cdot,z)$-transformed process $\mathbf{X}^z$ of $\mathbf{X}$ defined by

$$R^z(x,y) := \frac{R(x,y)R(y,z)}{R(x,z)}, \quad x,y \in E \setminus \{z\} \text{ with } x \neq y$$

and $R^z\nu(x) := \int_E R^z(x,y)\nu(\mathrm{d}y)$.

(1) A measure $\nu \in S_1(\mathbf{X})$ is said to be *conditionally Green-bounded in the sense of Chen with respect to* $\mathbf{X}$ if

$$\sup_{(x,z)\in E\times E, x\neq z} R^z\nu(x) < \infty.$$

(2) A measure $\nu \in S_1(\mathbf{X})$ is said to be *of conditionally Green-tight Kato class in the sense of Chen with respect to* $\mathbf{X}$ if for any $\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{(x,z)\in E\times E, x\neq z} R^z(\mathbf{1}_{K^c\cup B}\nu)(x) < \varepsilon.$$

(3) A measure $\nu \in S_1(\mathbf{X})$ is said to be *of conditionally semi-Green-tight extended Kato class in the sense of Chen with respect to* $\mathbf{X}$ if there exists a Borel subset K of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{(x,z)\in E\times E, x\neq z} R^z(\mathbf{1}_{K^c\cup B}\nu)(x) < 1.$$

Let denote by $S^1_{CS_\infty}(\mathbf{X})$ (resp. $S^1_{CS_1}(\mathbf{X})$, $S^1_{DS_0}(\mathbf{X})$) the family of conditionally Green-tight Kato class measures (resp. the family of conditionally semi-Green-tight extended Kato class measures, the family of conditionally Green-bounded measures) in the sense of Chen. It is known in general that $S^1_{CS_\infty}(\mathbf{X}) \subset S^1_{CS_1}(\mathbf{X}) \subset S^1_{DS_0}(\mathbf{X})$, $S^1_{CS_\infty}(\mathbf{X}) \subset S^1_{CK_\infty}(\mathbf{X})$, $S^1_{CS_1}(\mathbf{X}) \subset S^1_{CK_1}(\mathbf{X})$ and similarly $S^1_{DS_0}(\mathbf{X}) \subset S^1_{D_0}(\mathbf{X})$ (cf. [3, 4]).

3. Results

Throughout this note, we assume that Ψ is a fixed continuous increasing bijection on $]0, +\infty[$ satisfying that for all $0 < r < R$,

$$C_\Psi^{-1}\left(\frac{R}{r}\right)^{\beta} \leq \frac{\Psi(R)}{\Psi(r)} \leq C_\Psi\left(\frac{R}{r}\right)^{\beta'} \tag{3.1}$$

for some $1 < \beta \leq \beta'$ and $C_\Psi \geq 1$. We consider the following condition: there exists large $L > 0$ such that

$$\operatorname*{ess\text{-}sup}_{s\geq L} \frac{s\Psi'(s)}{\Psi(s)} < \infty. \tag{3.2}$$

Example 3.1. Take $\beta_1, \beta_2 \in]1, +\infty[$ and set

$$\Psi(s) := \begin{cases} s^{\beta_1}, & s \in [0,1], \\ s^{\beta_2}, & s \in [1, +\infty[. \end{cases}$$

Then (3.1) and (3.2) for some large $L > 0$ are satisfied.

We further assume the *volume doubling condition* (*VD*) (see [8, Definition 1.1]): there exists a constant $C_D > 0$ such that

$$V(x, 2r) \leq C_D V(x, r) \quad \text{for all } x \in E, \quad r \in]0, +\infty[.$$

We set

$$\Phi(s,t) := \sup_{r>0}\left\{\frac{s}{r} - \frac{t}{\Psi(r)}\right\}.$$

Definition 3.2. For $c \in]0, 1]$, we say that $(E, d, \mathfrak{m}, \mathbf{X})$ satisfies $(UE)^{c,*}_\Psi$ if the heat kernel $p_t(x, y)$ of $\mathbf{X}$ exists and satisfies the following upper estimate

$$p_t(x,y) \leq \frac{Ce^{kt}}{V(x, \Psi^{-1}(t))}\exp\left(-\frac{1}{2}\Phi\left(cd(x,y), t\right)\right) \tag{3.3}$$

for all $t > 0$ and $\mathfrak{m}$-a.e. $x, y \in E$, where $C > 0$ and $k \geq 0$ are constants independent of x, y, t.

For $c \in [1,+\infty[$, we say that $(E,d,\mathfrak{m},\mathbf{X})$ satisfies $(LE)_{\Psi}^{c;*}$ if the heat kernel $p_t(x,y)$ of $\mathbf{X}$ exists and satisfies the following lower estimate

$$p_t(x,y) \geq \frac{Ce^{-kt}}{V(x,\Psi^{-1}(t))}\exp\left(-c\Phi\left(cd(x,y),t\right)\right) \tag{3.4}$$

for all $t>0$ and $\mathfrak{m}$-a.e. $x,y \in E$, where $C>0$ and $k \geq 0$ are constants independent of x,y,t.

We say that $(E,d,\mathfrak{m},\mathbf{X})$ satisfies $(UE)_{\Psi}^{*}$ (resp. $(LE)_{\Psi}^{*}$) if it satisfies $(UE)_{\Psi}^{c;*}$ (resp. $(LE)_{\Psi}^{c;*}$) for some $c \in]0,1]$ (resp. $c \in [1,+\infty[$). In particular, we say that $(E,d,\mathfrak{m},\mathbf{X})$ satisfies $(UE)_{\Psi}$ (resp. $(LE)_{\Psi}$) if it satisfies $(UE)_{\Psi}^{*}$ (resp. $(LE)_{\Psi}^{*}$) with $k=0$

Remark 3.3.

(1) Clearly, $\Phi(s,t) = t\Phi(s/t,1)$. If $\Psi(r) = Cr^{\beta}$ with some $C>0$ and $\beta > 1$, then $\Phi(s,1) = cs^{\beta/(\beta-1)}$. Consequently, under (3.1), we always have $\Phi(s,1) \geq cs^{\beta/(\beta-1)}$ for some $c>0$.

(2) It is known (cf. [2],[7]) that $(UE)_{\Psi}$+$(LE)_{\Psi}$ implies that the heat kernel $p_t(x,y)$ admits a locally Hölder continuous in x,y version, so that (3.3) is a posteriori true for all $x,y \in E$.

(3) By [8, Theorem 1.17], $(UE)_{\Psi}$+$(LE)_{\Psi}$ is stable under bounded perturbations, that is, if two strongly local regular Dirichlet forms $(\mathcal{E}^{(1)},\mathcal{F}^{(1)})$ and $(\mathcal{E}^{(2)},\mathcal{F}^{(2)})$ having common domain $\mathcal{F}^{(1)} = \mathcal{F}^{(2)}$, satisfy $C^{-1}\mathcal{E}^{(1)} \leq \mathcal{E}^{(2)} \leq C\mathcal{E}^{(1)}$ on $\mathcal{F}^{(1)} \times \mathcal{F}^{(1)}$ and $(\mathcal{E}^{(1)},\mathcal{F}^{(1)})$ admits $(UE)_{\Psi}$+$(LE)_{\Psi}$ for some positive constants, then $(\mathcal{E}^{(2)},\mathcal{F}^{(2)})$ also does for some constants.

(4) By [8, Theorem 1.17], $(UE)_{\Psi}$+$(LE)_{\Psi}$ is equivalent to $(G)_{\Psi}$ (see [8] for $(G)_{\Psi}$). In this case, under the transience of $\mathbf{X}$, we know the existence of global Green kernel $R(x,y)$ and it satisfies that there exist $\kappa \in]0,1[$ and $C \in]0,+\infty[$ such that for $x,y \in E$ with $x \neq y$

$$C^{-1}\int_{\kappa d(x,y)}^{\infty}\frac{\Psi(s)\mathrm{d}s}{sV(x,s)} \leq R(x,y) \leq C\int_{\kappa d(x,y)}^{\infty}\frac{\Psi(s)\mathrm{d}s}{sV(x,s)}.$$

For a bounded finely continuous function $u \in \dot{\mathcal{F}}_{\mathrm{loc}} \cap QC(E_\partial)$ satisfying $\mu_{\langle u\rangle} \in S_D^1(\mathbf{X})$, we consider the transforms by the additive functionals $A_t := N_t^u + A_t^{\mu}$ of the form

$$e_A(t) := \exp(A_t), \quad t \geq 0, \tag{3.5}$$

where N_t^u is the continuous additive functional of zero quadratic variation appeared in a Fukushima decomposition of $u(X_t)-u(X_0)$, (see (1.1)), A_t^{μ} is the continuous additive functional of $\mathbf{X}$ with a signed smooth measure $\mu := \mu_1 - \mu_2$ in the strict sense as its Revuz measure. Note that N^u is not necessarily of bounded variation. The transform (3.5) defines a semigroup, namely, the generalized Feynman-Kac semigroup

$$P_t^A f(x) := \mathbf{E}_x[e_A(t)f(X_t)], \quad f \in \mathcal{B}_b(E),\ t \geq 0. \tag{3.6}$$

The purpose of this note is to give the analytic condition on u and μ under which $p_t^A(x,y)$ on $]0,+\infty[\times E \times E$ also satisfies $(UE)_{\Psi}$+$(LE)_{\Psi}$.

Define the quadratic form $(\mathcal{Q},\mathcal{F})$ by

$$(3.7)\quad \mathcal{Q}(f,g):=\mathcal{E}(f,g)+\frac{1}{2}\int_E f(x)\mu_{\langle u,g\rangle}(\mathrm{d}x)+\frac{1}{2}\int_E g(x)\mu_{\langle u,f\rangle}(\mathrm{d}x)-\int_E fg\mathrm{d}\mu.$$

Then it is well-defined for $f,g\in\mathcal{F}$ provided $\mu_{\langle u\rangle}\in S_D^1(\mathbf{X})$, $\mu_1+\mu_2\in S_D^1(\mathbf{X})$. Moreover, if $\mathbf{X}$ is transient, $\mathcal{Q}$ is extended to $\mathcal{F}_e\times\mathcal{F}_e$ with the same expression (3.7) provided $\mu_{\langle u\rangle}\in S_{D_0}^1(\mathbf{X})$, $\mu_1+\mu_2\in S_{D_0}^1(\mathbf{X})$. The L^2-generator $\mathcal{L}^{\mathcal{Q}}$ associated to $(\mathcal{Q},D(\mathcal{Q}))$ can be expressed in the formal form $\mathcal{L}^{\mathcal{Q}}:=\mathcal{L}^{\mathcal{E}}+\mathcal{L}^{\mathcal{E}}u+\mu$, where $\mathcal{L}^{\mathcal{E}}$ is the infinitesimal generator for the semigroup of $\mathbf{X}$.

Under $\mu_{\langle u\rangle}+\mu_1+\mu_2\in S_D^1(\mathbf{X})$ (resp. $\mu_{\langle u\rangle}+\mu_1+\mu_2\in S_{D_0}^1(\mathbf{X})$), we set for $\alpha>0$ (resp. $\alpha=0$)

$$(3.8)\qquad \lambda^{\mathcal{Q}_\alpha}(\bar{\mu}_1):=\inf\left\{\mathcal{Q}_\alpha(f,f)\;\middle|\; f\in\mathcal{C},\ \int_E f^2\mathrm{d}\bar{\mu}_1=1\right\},$$

where $\bar{\mu}_1:=\frac{1}{2}\mu_{\langle u\rangle}+\mu_1$. Let $\lambda^{\mathcal{Q}_0}(\bar{\mu}_1):=\lambda^{\mathcal{Q}}(\bar{\mu}_1)$.

Our main theorem is the following:

Theorem 3.4. *Assume that $\mathbf{X}$ is transient. Suppose that the heat kernel $p_t(x,y)$ satisfies $(UE)_\Psi$+$(LE)_\Psi$ for all $x,y\in E$. Let $u\in\mathcal{F}_{\mathrm{loc}}\cap QC(E_\partial)$ be a bounded finely continuous (nearly) Borel function on E. Assume $\mu_1\in S_{NK_1}^1(\mathbf{X})$, $\mu_{\langle u\rangle}\in S_{NK_\infty}^1(\mathbf{X})$ and $\mu_2\in S_{D_0}^1(\mathbf{X})$. Then we have the following:*

1. *$\lambda^{\mathcal{Q}}(\bar{\mu}_1)>0$ implies that the integral kernel $p_t^A(x,y)$ satisfies $(UE)_\Psi$+$(LE)_\Psi$ for all $x,y\in E$.*
2. *Suppose that $\mu_1\in S_{CS_1}^1(\mathbf{X})$, $\mu_{\langle u\rangle}\in S_{CS_\infty}^1(\mathbf{X})$ and $\mu_2\in S_{DS_0}^1(\mathbf{X})$ hold. If $p_t^A(x,y)$ satisfies $(UE)_\Psi$ for all $x,y\in E$ and* (3.2) *holds for some $L>0$, then $\lambda^{\mathcal{Q}}(\bar{\mu}_1)>0$.*

Next corollary gives a stability of the short time estimates for integral kernel $p_t^A(x,y)$ without assuming the transience of $\mathbf{X}$. Denote by $\mathbf{X}^{(\alpha)}$ the α-subprocess killed at rate αm.

Corollary 3.5. *Suppose that $p_t(x,y)$ satisfies $(UE)_\Psi$+$(LE)_\Psi$ for all $x,y\in E$. Let $u\in\mathcal{F}_{\mathrm{loc}}\cap QC(E_\partial)$ be a bounded finely continuous (nearly) Borel function on E. Assume $\mu_1\in S_{NK_1}^1(\mathbf{X}^{(\alpha)})$, $\mu_{\langle u\rangle}\in S_{NK_\infty^\pm}^1(\mathbf{X})$ and $\mu_2\in S_D^1(\mathbf{X})$.*

Then we have the following:

1. *$\lambda^{\mathcal{Q}_\alpha}(\bar{\mu}_1)>0$ implies that the integral kernel $p_t^A(x,y)$ satisfies $(UE)_\Psi^*$+$(LE)_\Psi^*$ for all $x,y\in E$.*
2. *Suppose that $\mu_1\in S_{CS_1}^1(\mathbf{X}^{(\alpha)})$, $\mu_{\langle u\rangle}\in S_{CS_\infty}^1(\mathbf{X}^{(\alpha)})$ and $\mu_2\in S_{DS_0}^1(\mathbf{X}^{(\alpha)})$ hold. If $p_t^A(x,y)$ satisfies $(UE)_\Psi^*$ for all $x,y\in E$ and $\alpha>k$, then $\lambda^{\mathcal{Q}_\alpha}(\bar{\mu}_1)>0$. Here k is the constant appeared in $(UE)_\Psi^*$.*

Our result, in particular Theorem 3.4(1),(2), extend the result on the stability of Li-Yau estimates for the heat kernel of Riemannian manifold proved by Takeda [20]. One of the main progress in our result is to add the perturbation by continuous additive functional of locally of zero energy. The global integral kernel estimate under such perturbations was firstly done by Glover-Rao-Song [9, Theorem 2.9] (see also Glover-Rao-Šikić-Song [10, Proposition 1.4]) in the framework of Brownian motion. But the stability of global full integral kernel estimates have not been treated in this direction. Indeed, the global estimates shown in [9, Theorem 2.9] is weaker than the heat kernel of Brownian motion, because of the lack of Green-tightness of Kato class measures.

As a corollary of Corollary 3.5(1), we have the following theorem:

Theorem 3.6. *Let $u \in \mathcal{F}_{\mathrm{loc}} \cap QC(E_\partial)$ be a bounded finely continuous (nearly) Borel function on E. Assume $\mu_1 \in S^1_{EK}(\mathbf{X})$, $\mu_{\langle u\rangle} \in S^1_K(\mathbf{X})$ and $\mu_2 \in S^1_D(\mathbf{X})$. Then, $p_t^A(x,y)$ satisfies $(UE)^*_\Psi$+$(LE)^*_\Psi$ (resp. $(UE)^*_\Psi$+$(LE)^*_\Psi$) for all $x,y \in E$ provided $p_t(x,y)$ satisfies $(UE)_\Psi$ +$(LE)_\Psi$ (resp. $(UE)^*_\Psi$+$(LE)^*_\Psi$) for all $x,y \in E$.*

Theorem 3.6 also extends the previous known results on the integral kernel estimates under the perturbation by measures of Kato classes and provides a stability of the short time estimates for integral kernel. The conditions for measures in Theorem 3.6 are very mild comparing the known results.

REFERENCES

[1] M. Aizenman and B. Simon, *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure. Appl. Math. **35** (1982), no. 2, 209–273.

[2] M. Barlow, A. Grigoryan and T. Kumagai, *On the equivalence of parabolic Harnack inequalities and heat kernel estimates*, J. Math. Soc. Japan **64** (2012), no. 4, 1091–1146.

[3] Z.-Q. Chen, *Gaugeability and conditional gaugeability*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4639–4679.

[4] Z.-Q. Chen and R. Song, *General gauge and conditional gauge theorems*, Ann. Prob. **30** (2002), no. 3, 1313–1339.

[5] G. De Leva, D. Kim and K. Kuwae, *L^p-independence of spectral bounds of Feynman-Kac semigroups by continuous additive functionals*, J. Funct. Anal. **259** (2010), no. 3, 690–730.

[6] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Second revised and extended edition. de Gruyter Studies in Mathematics, **19**. Walter de Gruyter & Co., Berlin, 2011.

[7] A. Grigoryan, *Heat kernels and Green functions on metric measure spaces*, Canad. J. Math. **66** (2014), no. 3, 641–699.

[8] A. Grigoryan, J. Hu and K.-S. Lau, *Generalized capacity, Harnack inequality and heat kernels on metric spaces*, preprint, 2014.

[9] J. Glover, M. Rao and R. Song, *Generalized Schödinger semigroups*, in "Seminar on Stochastic Processes", 1992 (E. Cinlar, K. L. Chung, and M. J. Sharpe Eds.) Birkhäuser, Boston 1993.

[10] J. Glover, M. Rao, H. Šikić and R. Song, *Quadratic forms corresponding to the generalized Schrödinger semigroups*, J. Funct. Anal. **125** (1994), no. 2, 358–378.

[11] D. Kim and K. Kuwae, *Analytic characterizations of gaugeability for generalized Feynman-Kac functionals*, preprint (2015), to appear in Transactions of AMS.

[12] D. Kim and K. Kuwae, *General analytic characterization of gaugeability for Feynman-Kac functionals*, preprint (2013).

[13] D. Kim and K. Kuwae, *On a stability of heat kernel estimates under generalized non-local Feynman-Kac perturbations for stable-like processes*, Festschrift Masatoshi Fukushima, In Honor of Masatoshi Fukushima's Sanju, Interdisciplinary Mathematical Sciences **17**, 287–305. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

[14] D. Kim and K. Kuwae, *On a stability of heat kernel estimates under Feynman-Kac perturbations for symmetric Markov processes*, 2015 in preparation.

[15] D. Kim, K. Kuwae and Y. Tawara, *Large deviation principle for generalized Feynman-Kac functionals and its applications*, preprint, to appear in Tohoku Math. J. (2012).

[16] K. Kuwae, *Functional calculus for Dirichlet forms*, Osaka J. Math. **35** (1998), no. 3, 683–715.

[17] K. Kuwae and M. Takahashi, *Kato class functions of Markov processes under ultracontractivity*, Potential theory in Matsue, 193–202, Adv. Stud. Pure Math. **44**, Math. Soc. Japan, Tokyo, 2006.

[18] Z.-M. Ma and M. Röckner, *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Springer Universitext, 1992.

[19] P. Stollmann and J. Voigt, *Perturbation of Dirichlet forms by measures*, Potential Anal. **5** (1996), no. 2, 109–138.

[20] M. Takeda, *Gaussian bounds of heat kernels for Schrödinger operators on Riemannian manifolds*, Bull. Lond. Math. Soc. **39** (2007), no. 1, 85–94.

[21] Z. Zhao, *Subcriticality and gaugeability of the Schrödinger operator*, Trans. Amer. Math. Soc. **334** (1992), no. 1, 75–96.

Daehong Kim
Department of Mathematics and Engineering
Graduate School of Science and Technology
Kumamoto University
Kumamoto, 860-8555 JAPAN
E-mail address: daehong@gpo.kumamoto-u.ac.jp

Kazuhiro Kuwae
Department of Applied Mathematics
Faculty of Science
Fukuoka University
Fukuoka 814-0180 JAPAN
E-mail address: kuwae@fukuoka-u.ac.jp