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An invariance principle for stochastic heat equations with periodic coefficients

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Abstract

This is a shot note on the attempt to apply Kipnis-Varadhan's theory on central limit theorem for additive functionals of reversible Markov process to stochastic PDEs. We considered the mild solution to a stochastic heat equation with periodic coefficients, which is closely related to the dynamical sine-Gordon equation. We summarize the results on the invariant measure, ergodicity and generator, and prove a central limit theorem via the general method. We also obtain an invariance principle.

1 1-dimensional diffusion

A general theory of functional CLT for Markov processes is developed in [4], based on a martingale-decomposition of the targeted functional. This method is extended to non-reversible cases in many references, e.g. [6], [7], [8] and [10]. Combined with Itô's formula, it can be used to prove the central limit theorem for diffusion processes in $\mathbb{R}^d$ with periodic coefficients, as illustrated in [5, Chapter 9].

Heuristically, let $d=1$ and a, $U\in C_b^2(\mathbb{R})$ be two functions of period 1. Furthermore, suppose that a has a strictly positive infimum and let $X_t\in\mathbb{R}$ be the solution to the Itô stochastic differential equation

$$\begin{cases} dX_t = V(X_t)dt + a^{\frac{1}{2}}(X_t)dw_t \\ X_0 = x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $V(x)\triangleq -\frac{1}{2}[U'(x)a(x)+a'(x)]$ and w_t is a standard 1-dimensional Brownian motion. Under regularity assumptions made about the coefficients, X_t exists uniquely and has the Markov property. Let P_t be the transition probability semigroup generated by X_t, defined as

$$P_tf(x)\triangleq\mathbb{E}[f(X_t)|X_0=x]$$

for all $f\in C_b(\mathbb{R})$ and $x\in\mathbb{R}$. The generator $\mathcal{L}$ associated to X_t is

$$\mathcal{L}f(x)=\frac{1}{2}\left[-U'(x)a(x)+a'(x)\right]f'(x)+\frac{1}{2}a(x)f''(x)$$

for $f \in C^2(\mathbb{R})$ and $x \in \mathbb{R}$. Since $\int_{\mathbb{R}} \mathcal{L}f(x)e^{-U(x)}dx = 0$, $e^{-U(x)}dx$ is invariant under P_t.

To deal with the issue that $e^{-U(x)}dx$ is of infinite mass, let $\dot{X}_t$ be the diffusion induced by X_t on the 1-dimensional torus $\mathbb{T} \triangleq \mathbb{R}/\sim$, where the equivalence relation $x \sim y$ holds if and only if $x - y \in \mathbb{Z}$. It is clear that this process is well-defined because we have the periodic condition. Denote by $\dot{P}_t$, $\dot{\mathcal{L}}$ the Markov semigroup on and generator related to $\dot{X}_t$ respectively. It is clear that the probability measure

$$\pi(d\dot{x}) = \frac{1}{Z}e^{-U(\dot{x})}d\dot{x}$$

is invariant and ergodic under $\dot{P}_t$, where Z is a normalization constant.

In order to prove the CLT for X_t we consider the cell problem $-\dot{\mathcal{L}}f = V$. It does not take too much effort to solve it with

$$f(\dot{x}) = -\dot{x} + C_1 \int_0^x e^{U(y)}a^{-1}(y)dy + C_2$$

where $C_1^{-1} = \int_0^1 e^{U(y)}a^{-1}(y)dy$ and C_2 is an constant such that $\int_{\mathbb{T}} f(\dot{x})d\dot{x} = 0$.

Notice that the trajectory of particle can be decomposed to the additive functional $\int_0^t V(X_s)ds$ and the martingale $\int_0^t a^{\frac{1}{2}}(X_s)dw_s$. Applying the classical Itô's formula to f in the cell problem, we have

$$X_t = x + f(\dot{x}) - f(\dot{X}_t) + \int_0^t a^{\frac{1}{2}}(X_s)\left[f'(X_s) + 1\right]dw_s.$$

By the ergodic theorem and the martingale CLT (see in [11]), we obtain that as $t \to \infty$, $t^{-\frac{1}{2}}X_t$ converges weakly to a centered normal distribution with variance given by

$$\sigma^2 = \frac{1}{Z}\int_0^1 a(x)\left[C_1 e^{U(x)}a^{-1}(x) - 1\right]^2 e^{-U(x)}dx.$$

2 Stochastic heat equation

The following dynamical sine-Gordon equation is introduced in [3]

$$\partial_t u = \frac{1}{2}\Delta u + c\sin(\beta u + \theta) + \xi, \tag{2.1}$$

where c, β and θ are real constants and ξ denotes the space-time white noise. (2.1) is the natural dynamic associated to the usual quantum sine-Gordon model, whose solution can be constructed via classical Itô's theory on stochastic PDE when spatial dimension is 1, or via Hairer's theory of regularity structures when spatial dimension is 2, see in [3]. From a physical perspective, (2.1) describes globally neutral gas of interacting charges at different temperature β.

The strategy illustrated before is expected to be applicable to (2.1). We only deal with a 1-dimensional general model here, written as

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) - V_x'(u(t,x)) + \dot{W}(t,x), & t > 0, x \in (0,1), \\ \partial_x u(t,0) = \partial_x u(t,1) = 0, & t > 0, \\ u(0,x) = v(x), & x \in [0,1]. \end{cases} \tag{2.2}$$

In (2.2), $V_x'(u) \triangleq \frac{d}{du}V_x(u)$ where $V_x(\cdot) = V(x, \cdot)$ is a family of C^1 functions on $\mathbb{R}$ indexed by $x \in [0, 1]$, satisfying

(1) $\forall u \in \mathbb{R}$, $V_x(u)$ is Borel-measurable in x;

(2) $\sup_{x\in[0,1],u\in\mathbb{R}} \{|V_x(u)| + |V_x'(u)|\} < \infty$;

(3) $\forall x \in [0, 1]$, V_x' is global Lipschitz continuous with the same Lipschitz constant.

(4) $\forall x \in [0, 1]$, V_x is periodic in u: $V_x(u) = V_x(u + 1)$.

The stochastic PDE (2.2) is originally defined in [2] for the purpose of describing the motion of a flexible Brownian string in some potential field. The solution $u(t)$ uniquely exists in $C[0, 1]$ and forms a continuous Markov process. Furthermore, if $\{w_x\}_{x\in[0,1]}$ is a 1-dimensional Brownian motion whose initial distribution is the Lebesgue measure on $\mathbb{R}$, then the reversible measure of $u(t)$ is an infinite measure on $C[0, 1]$ given by

$$\mu(dv) = \exp\left\{-2\int_0^1 V_x(v(x))dx\right\}\mu_w(dv), \tag{2.3}$$

where μ_w stands for the measure induced by w_x (see in [2]).

Similar to the 1-dimensional diffusion, we consider an equivalence relation in $C[0, 1]$ such that $v_1 \sim v_2$ if and only if $v_1 - v_2$ equals to some integer-valued constant function. Let $\dot{E} = C[0, 1]/\sim$ and identify $\dot{v} \in \dot{E}$ with its representative $v \in C[0, 1]$ such that $v(0) \in [0, 1)$. A function f on $C[0, 1]$ can be automatically regarded as a function on $\dot{E}$ if it satisfies that $f(v + \mathbf{1}) = f(v)$. Let $\dot{u}(t)$ be the process induced by $u(t)$ on $\dot{E}$.

It is clear that $\dot{u}(t)$ inherits the Markov property and a finite reversible measure form $u(t)$. Precisely, suppose $\{w_x'\}_{x\in[0,1]}$ to be a 1-dimensional Brownian motion whose initial distribution is the Lebesgue measure on $[0, 1)$, then

$$\pi(d\dot{v}) = \frac{1}{Z}\exp\left\{-2\int_0^1 V_x(\dot{v}(x))dx\right\}\pi_w(d\dot{v}) \tag{2.4}$$

is a probability measure and is reversible for $\dot{u}(t)$, where π_w stands for the measure of w_x' and Z is a normalization constant. Let $\mathcal{H}$ be the Hilbert space $L^2(\dot{E}, \pi)$, with the inner product $\langle\cdot,\cdot\rangle_\pi$ and the norm $\|\cdot\|_\pi$. Denote by $\{\dot{\mathcal{P}}_t\}$ the Markov semigroup generated by $\dot{u}(t)$ on $\mathcal{H}$. Recall the results in [9] on the strong Feller property and irreducibility of $\{\dot{\mathcal{P}}_t\}$, we can conclude that π is the only one invariant measure, thus it is ergodic.

Let $\mathcal{E}_A(H)$ be the linear span of all real and imaginary parts of functions on H of the form $h \mapsto e^{i\langle l,h\rangle}$ where $l \in C^2[0, 1]$ such that $l'(0) = l'(1) = 0$. Moreover, suppose $\mathcal{E}_A(\dot{E})$ to be the collection of functions in $\mathcal{E}_A(H)$ such that $f(v) = f(v + \mathbf{1})$ for all $v \in E$. For $f \in \mathcal{E}_A(\dot{E})$, define

$$\dot{\mathcal{K}}_0 f(\dot{v}) = \frac{1}{2}\langle\partial_x^2 Df(\dot{v}), v\rangle + \frac{1}{2}Tr\left[D^2 f(\dot{v})\right] - \langle Df(\dot{v}), V'(v(\cdot))\rangle, \tag{2.5}$$

where D denotes the Fréchet derivative. The integration-by-part formula for Wiener measure suggests that

$$E_\pi\|Df\|^2 = 2\langle f, -\dot{\mathcal{K}}_0 f\rangle_\pi, \tag{2.6}$$

thus $\dot{\mathcal{K}}_0$ is dissipative on $\mathcal{H}$. Denote its closure by $(\mathcal{D}(\dot{\mathcal{K}}), \dot{\mathcal{K}})$. Along a similar strategy used in [1], we can conclude that $\dot{\mathcal{K}}$ generates $\{\dot{\mathcal{P}}_t\}$ on $\mathcal{H}$.

Proposition 2.1. *For all $f \in \mathcal{D}(\dot{\mathcal{K}})$, the following equation holds π-a.s. and in $\mathcal{H}$.*

$$f(\dot{u}(t)) = f(\dot{u}(0)) + \int_0^t \dot{\mathcal{K}} f(\dot{u}(r))dr + \int_0^t \langle Df(\dot{u}(r)), dW_r\rangle. \tag{2.7}$$

Proof. If $f \in \mathcal{E}_A(\dot{E})$, (2.7) follows from the classical Itô's formula easily.

For general f, pick $f_m \in \mathcal{E}_A(\dot{E})$ such that $f_m \to f$, $\dot{\mathcal{K}} f_m \to \dot{\mathcal{K}} f$ in $\mathcal{H}$. Then (2.6) suggests that $\|Df_m - Df\|$ also vanishes in $\mathcal{H}$ as $m \to \infty$. Therefore, (2.7) follows from the Itô isometry. □

3 CLT and invariance principle

Theorem 3.1. *Under an initial probability distribution ν such that $\nu \ll \mu$,*

$$\lim_{t\to\infty} E_\nu \left| \mathbb{E}\left[f\left(\frac{u(t)}{\sqrt{t}}\right) \middle| \mathcal{F}_0 \right] - \int_{\mathbb{R}} f(\mathbf{1}\cdot y) N_{\sigma^2}(dy) \right| = 0 \tag{3.1}$$

holds for all $f \in C_b(C[0,1])$, where σ is a constant introduced later and N_{σ^2} stands for a 1-dimensional centered Gaussian distribution on $\mathbb{R}$ with variance σ^2.

We first define two Hilbert spaces related to the operator $\dot{\mathcal{K}}$. For $f \in \mathcal{E}_A(\dot{E})$ let

$$\|f\|_1^2 = \langle -\dot{\mathcal{K}} f, f\rangle_\pi = \frac{1}{2} E_\pi \|Df\|^2.$$

Let $\mathcal{H}_1$ be completion of $\mathcal{E}_A(\dot{E})$ under $\|\cdot\|_1$, which turns to be a Hilbert space if all f such that $\|f\|_1 = 0$ are identified with 0. On the other hand, let

$$\mathcal{I} = \left\{ f \in \mathcal{H}; \|f\|_{-1} \triangleq \sup_{g\in\mathcal{E}_A(\dot{E}), \|g\|_1=1} \langle f, g\rangle_\pi < \infty \right\}$$

Let $\mathcal{H}_{-1}$ be the completion of $\mathcal{I}_{-1}$ under $\|\cdot\|_{-1}$, which also becomes a Hilbert space if all f with $\|f\|_{-1} = 0$ are identified with 0. Denote by $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_{-1}$ the inner products defined by polarization in $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ respectively.

Proof of Theorem 3.1. Pick $\varphi \in C^2[0,1]$ such that $\varphi'(0) = \varphi'(1) = 0$. The trajectory of $\langle u(t), \varphi\rangle$ can be written as a sum of the additive functional $\int_0^t V^\varphi(\dot{u}(r))dr$ plus a Brownian motion $\langle W_t, \varphi\rangle$, where V^φ is a functional on $\dot{E}$ defined as

$$V^\varphi(\dot{v}) \triangleq \frac{1}{2}\int_0^1 v(x)\varphi''(x)dx - \int_0^1 V_x'(v(x))\varphi(x)dx.$$

It is not hard to verify that $V^\varphi \in \mathcal{H}\cap\mathcal{H}_{-1}$ and $\|V^\varphi\|_{-1} \le \frac{\sqrt{2}}{2}\|\psi\|$, however to solve the cell problem in $\mathcal{H}$ turns to be not easy. Instead, for $\lambda > 0$ we consider the resolvent equation written as

$$\lambda f_\lambda^\varphi - \dot{\mathcal{K}} f_\lambda^\varphi = V^\varphi. \tag{3.2}$$

Taking inner product with f_λ^φ in (3.2), since $\dot{u}(t)$ is reversible under π we have

$$\sup_{\lambda>0} \|\dot{\mathcal{K}} f_\lambda^\varphi\|_{-1} = \sup_{\lambda>0} \|f_\lambda^\varphi\|_1 \le \|V^\varphi\|_{-1} < \infty. \tag{3.3}$$

Decompose the additive functional as $\int_0^t V^{\varphi}(\dot{u}(r))dr = M_\lambda^{\varphi}(t) + R_\lambda^{\varphi}(t)$, where M_λ^{φ} is the Dynkin's martingale and R_λ^{φ} is the residual term

$$M_\lambda^{\varphi}(t) = f_\lambda^{\varphi}(\dot{u}(t)) - f_\lambda^{\varphi}(\dot{u}(0)) - \int_0^t \dot{\mathcal{K}} f_\lambda^{\psi}(\dot{u}(r))dr,$$

$$R_\lambda^{\varphi}(t) = f_\lambda^{\varphi}(\dot{u}(0)) - f_\lambda^{\varphi}(\dot{u}(t)) + \lambda \int_0^t f_\lambda^{\psi}(\dot{u}(r))dr.$$

Applying (2.7) to f_λ^{φ}, combining it with this decomposition, we have

$$\langle u(t), \varphi \rangle = \langle u(0), \varphi \rangle + \int_0^t \langle Df_\lambda^{\varphi}(\dot{u}(r)) + \varphi, dW_r \rangle + R_\lambda^{\varphi}(t).$$

Condition (3.3) implies that (see in [5, Chapter 2]) there exists some $f^{\varphi} \in \mathcal{H}_1$ and an adapted process $R^{\varphi}(t)$ such that

$$\langle u(t), \varphi \rangle = \langle u(0), \varphi \rangle + \int_0^t \langle Df^{\varphi}(\dot{u}(r)) + \varphi, dW_r \rangle + R^{\varphi}(t).$$

Now the vanishment of $R^{\varphi}(t)$ (see in [5, Chapter 2]) and martingale CLT show that under initial distribution $\nu \ll \mu$,

$$\lim_{t\to\infty} E_\nu \left| \mathbb{E}\left[f\left(\frac{\langle u(t), \varphi \rangle}{\sqrt{t}} \right) \middle| \mathcal{F}_0 \right] - \int_{\mathbb{R}} f(y) N_{\sigma^2}(dy) \right| = 0 \tag{3.4}$$

for all $f \in C_b(\mathbb{R})$ and $\theta \in \mathbb{R}$, where $\sigma_\varphi^2 = E_\pi \|Df^{\varphi} + \varphi\|^2$.

Finally, to prove Theorem 3.1 we only need to pick $\varphi = e_j$ in (3.4) such that $\{e_j\}$ forms a CONS of $L^2[0,1]$ including the constant function $\mathbf{1}$ and sum them up. □

Now fix $T > 0$ and consider the $C[0,1]$-valued process $\{u^{(\epsilon)}(t) = \epsilon u(\epsilon^{-2}t)\}_{t\in[0,T]}$. By checking the tightness we can prove the next invariance principle.

Theorem 3.2. *Under initial distribution $\nu \ll \mu$, $\{\epsilon u(\epsilon^{-2}t), t \in [0,T]\}$ converges weakly to a Gaussian process $\{\sigma B_t \cdot \mathbf{1}, t \in [0,T]\}$ as $\epsilon \downarrow 0$, where B_t is a 1-dimensional Brownian motion on $[0,T]$ and σ is the same constant as in Theorem 3.1.*

Proof. It is sufficient to verify the tightness. Recall that $u(t)$ satisfies that

$$u(t) = S(t)v + \int_0^t S(t-r)[-V'(u(r,\cdot))]dr + \int_0^t S(t-r)dW_r$$

Denote the three terms in the right-hand side by $X(t)$, $Y(t)$ and $Z(t)$ respectively. Furthermore, let $X^{\perp}(t) \triangleq X(t) - \int_0^1 X(t,x)dx$ and define $Y^{\perp}$, $Z^{\perp}$ similarly. Then

$$\epsilon u(\epsilon^{-2}t) = \epsilon \int_0^1 u(\epsilon^{-2}t, x)dx + \epsilon X^{\perp}(\epsilon^{-2}t) + \epsilon Y^{\perp}(\epsilon^{-2}t) + \epsilon Z^{\perp}(\epsilon^{-2}t).$$

When $\epsilon \downarrow 0$, [5, Theorem 2.32] yields that the integral term is tight, while $\{\epsilon X^{\perp}(\epsilon^{-2}t), t \in [0,T]\}$ vanishes uniformly since the heat semigroup is contractive.

The tightness of the two terms about $Y^{\perp}$ and $Z^{\perp}$ follows from the following estimates. For all $p > 1$, there exists a finite constant C_p only depending on $\{V_x\}$ such that for all t_1, $t_2 \in [0, \infty)$ and $x_1, x_2 \in [0, 1]$,

$$E\left|Y^{\perp}(t_1, x_1) - Y^{\perp}(t_2, x_2)\right|^{2p} \leq C_p(|t_1 - t_2|^p + |x_1 - x_2|^p); \tag{3.5}$$

$$E\left|Z^{\perp}(t_1, x_1) - Z^{\perp}(t_2, x_2)\right|^{2p} \leq C_p(|t_1 - t_2|^{\frac{p}{2}} + |x_1 - x_2|^p). \tag{3.6}$$

(3.5) and (3.6) are standard estimates for stochastic heat equations and the proof only involves computations, so we omit them here. □

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