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Affine-invariant quadruple systems

By

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§ 1. Introduction

Let t, v, k, λ be positive integers satisfying $v > k > t$. A t-(v, k, λ) design is an ordered pair $(V, \mathcal{B})$, where V is a finite set of v points, $\mathcal{B}$ is a collection of k-subsets of V, say blocks, such that every t-subset of V occurs in exactly λ blocks in $\mathcal{B}$. In what follows we simply write t-designs. A 3-$(v, 4, 1)$ design is called a Steiner quadruple system and denoted by SQS(v). It is known that an SQS(v) exists if and only if $v \equiv 2, 4 \pmod 6$ (see [9]). For $\lambda > 1$, a 3-$(v, 4, \lambda)$ design is called a λ-fold quadruple system and denoted by λ-fold QS(v) for short.

An automorphism group G of a t-design $(V, \mathcal{B})$ is a permutation group defined on V which leaves $\mathcal{B}$ invariant. For a fixed block $B \in \mathcal{B}$, the orbit of B under G is $\mathcal{O}_G(B) = \{B^g \mid g \in G\}$. Thus, $\mathcal{B}$ can be partitioned into orbits under G, say G-orbits. Moreover, if the cardinality of an orbit $\mathcal{O}$ equals to the order of G, then $\mathcal{O}$ is said to be full, otherwise, short. Any block in $\mathcal{O}$ can be regarded as a base block of the orbit.

In particular, a t-(v, k, λ)-design is said to be cyclic if it admits a cyclic group C_v of order v as its automorphism. A C_v-orbit is called a cyclic orbit. Without loss of generality, we identify the point set of a cyclic t-design with the additive group of $\mathbb{Z}_v = \mathbb{Z}/v\mathbb{Z}$, the integers modulo v. Furthermore, a cyclic t-design is said to be strictly cyclic, if all cyclic orbits are full. In what follows, we denote a cyclic SQS by CSQS, a strictly cyclic SQS by sSQS. The necessary conditions for the existence of a CSQS(v) and an sSQS(v) are $v \equiv 2, 4 \pmod 6$ and $v \equiv 2, 10 \pmod{24}$ respectively (see [12]).

The work on sSQS by Köhler [12] established a connection between sSQS and 1-factors of "Köhler graphs" named after him. Some approaches to Köhler's work by Siemon [23] [24] checked the existence of 1-factors of "Köhler graphs" for quite a few admissible parameters. Piotrowski [22] constructed sSQS$(2p)$ admitting the dihedral

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group D_{2p} as automorphism. For more information on CSQS and SQS with other specified automorphism groups, the reader may refer to Lindner and Rosa [17], Grannel and Griggs [8], Hartman and Phelps [10], Munemasa and Sawa [21].

Let $(\mathbb{Z}_v, \mathcal{B})$ be an sSQS. For any $B \in \mathcal{B}$ and $\tau : x \mapsto \alpha x$, $\alpha \in \mathbb{Z}_v^\times$, if $B^\tau \in \mathcal{B}$, then α is called a multiplier of $(\mathbb{Z}_v, \mathcal{B})$, where $\mathbb{Z}_v^\times$ is the multiplicative group of $\mathbb{Z}_v$, i.e. the group of all units of $\mathbb{Z}_v$.

Definition 1.1. For an sSQS $(\mathbb{Z}_v, \mathcal{B})$, if all the units of $\mathbb{Z}_v$ are multipliers, then $(\mathbb{Z}_v, \mathcal{B})$ is said to be affine-invariant.

In another words, an affine-invariant sSQS $(\mathbb{Z}_v, \mathcal{B})$ admits the affine group A as an automorphism, where A is defined by $A = \{(i, \alpha) \mid i \in \mathbb{Z}_v, \alpha \in \mathbb{Z}_v^\times\} \cong \mathbb{Z}_v \rtimes \mathbb{Z}_v^\times$. Given a quadruple B, denote the orbit of B under the affine group A by $\mathcal{O}_A(B)$, say an affine orbit.

Example 1.2. The unique (up to isomorphism) SQS(10) is affine-invariant strictly cyclic. Let $\mathbb{Z}_{10}$ be its point set. Let

$$B_1 = \{0, 1, 5, 9\}, \quad B_2 = \{0, 2, 5, 8\}, \quad B_3 = \{0, 1, 3, 4\}$$

be base blocks of the cyclic orbits. We have $B_1 \times 3 + 5 = \{0, 3, 5, 7\} + 5 = \{5, 8, 0, 2\} = B_2$ over $\mathbb{Z}_{10}$. Hence, the cyclic orbits of B_1 and B_2 are contained in the same affine orbit. In fact, there are two affine orbits having B_1 (or B_2) and B_3 as base blocks respectively.

In general, for 3-$(v, 4, \lambda)$ designs admitting the affine group, we also say they are affine-invariant. Affine-invariant 3-$(p, 4, \lambda)$ designs were first proposed by Köhler [14] for odd primes p and admissible λ by means of some graph KG(p). Along this direction, Brand and Sutinuntopas [4] generalized Köhler's results to finite fields. In particular, we denote a 2-fold quadruple system of order v by 2QS(v) for short.

Theorem 1.3 (Köhler [14]). *If the graph* KG(p) *has a 1-factor, then*

(i) an affine-invariant 3-$(p, 4, 2)$ *designs exists, for* $p \equiv 1, 5 \pmod{12}$ *and*

(ii) an affine-invariant 3-$(p, 4, 4)$ *designs exists, for* $p \equiv 7, 11 \pmod{12}$.

Approach on sSQS (i.e., $\lambda = 1$) is less known. Yoshikawa [29] presented the following results in his master thesis.

Theorem 1.4 (Yoshikawa [29]). *There exists an affine-invariant sSQS*$(2p)$*, for prime* $p \equiv 1, 5 \pmod{12}$ *and* $5 \leq p < 200$, $p \neq 13$, *i.e.,* $p \in \{5, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113, 137, 149, 157, 173, 181, 193, 197\}$.

§ 2. A family of graphs associated with PSL$(2,p)$

In this section, we introduce a family of graphs which play important roles in our constructions. Suppose p is a prime with $p \equiv 1, 5 \pmod{12}$. Let $\mathbb{F}_p$ denote the finite field of order p. Denote the 1-dimensional projective line by $\mathscr{P}(\mathbb{F}_p)$ which can be identified with $\mathbb{F}_p \cup \{\infty\}$.

Let

$$\sigma_A : x \mapsto 1 - x, \quad \sigma_B : x \mapsto \frac{1}{x}, \quad \sigma_C : x \mapsto \frac{1-x}{1-2x}$$

be mapping in PSL$(2,p)$. Let $x \in \mathscr{P}(\mathbb{F}_p)$. Denote the orbit of x under the subgroups $\langle \sigma_A, \sigma_B \rangle$ by $C(x)$, i.e.,

$$C(x) = \{x^\sigma \mid \sigma \in \langle \sigma_A, \sigma_B \rangle\} = \left\{x, \frac{1}{x}, \frac{x-1}{x}, \frac{x}{x-1}, \frac{1}{1-x}, 1-x\right\}.$$

Thus $\mathscr{P}(\mathbb{F}_p)$ can be partitioned into $\{C(x) \mid x \in \mathscr{P}(\mathbb{F}_p)\}$. In projective geometry, $C(x)$ is also called the cross-ratio class with respect to x. The cardinality of $C(x)$ is established as follows.

$$|C(x)| = \begin{cases} 3 & \text{if } x \in \{0,1,\infty\} \cup \{-1,2,2^{-1}\}; \\ 2 & \text{if } x \in \{\xi, 1-\xi\}; \\ 6 & \text{otherwise,} \end{cases}$$

where $\xi = \frac{1+\sqrt{-3}}{2}$ is a root of $x^2 - x + 1 = 0$, when $p \equiv 1 \pmod 3$.

Definition 2.1. Let CG$(\mathscr{P}(\mathbb{F}_p))$ be a graph (multigraph with loops) with vertex set $V = \{C(x) \mid x \in \mathscr{P}(\mathbb{F}_p)\}$. For any pair of vertices (not necessarily distinct) $C, C' \in V$, let C be adjacent to C' by $r_{C,C'}$ edges, where $r_{C,C'} = \frac{1}{2}\left|\{x \mid x \in C, x^{\sigma_C} \in C'\}\right|$.

Let $\Omega_p = \mathbb{F}_p \setminus \{0,1,-1,2,2^{-1}\}$. Let CG$(\Omega_p)$ denote the induced subgraph on $\{C(x) \mid x \in \Omega_p\}$ of CG$(\mathscr{P}(\mathbb{F}_p))$. In another word, by removing the vertices $C(0)$ and $C(2)$ from CG$(\mathscr{P}(\mathbb{F}_p))$, the resulting graph is CG(Ω_p). Let CG$^*(\Omega_p)$ denote the resulting graph (possibly having multiple edges) obtained by removing all loops from CG(Ω_p).

Lemma 2.2 ([19],[18]). *For $p > 17$, in* CG$^*(\Omega_p)$*, all the vertices have degree* 3 *except the following.*

(i) $C(3)$ *has degree* 2*;*

(ii) For $p \equiv 1 \pmod{12}$, $C(\xi)$ *has degree* 1*, where* $\xi = \frac{1+\sqrt{-3}}{2}$ *is a root of* $x^2 - x + 1 = 0$*;*

(iii) $C(\chi)$ *has degree* 2*, where* $\chi = \frac{1+\sqrt{-1}}{2}$ *is a root of* $2x^2 - 2x + 1 = 0$*;*

(iv) For $p \equiv 1, 29, 41, 49 \pmod{60}$, *$C(\mu)$ has degree 1, where* $\mu = \frac{3+\sqrt{5}}{2}$ *is a root of* $x^2 - 3x + 1 = 0$.

Theorem 2.3 ([19],[18]). *For* $p \equiv 1, 5 \pmod{12}$ *and* $p \not\equiv 1, 49 \pmod{60}$, $\mathrm{CG}(\Omega_p)$ *has a 1-factor if it has no bridge besides its pendant edge.*

§ 3. Direct Constructions of affine-invariant sSQS(2p)

Suppose $(\mathbb{Z}_{2p}, \mathcal{B})$ is an affine-invariant sSQS, where $p \equiv 1, 5 \pmod{12}$ is prime, which satisfies the necessary condition for the existence of an sSQS$(2p)$ (see [12]). Denote the set of nonzero elements of the finite field $\mathbb{Z}_p$ by $\mathbb{Z}_p^*$. We identify the point set $\mathbb{Z}_{2p}$ with $\mathbb{Z}_p \times \mathbb{Z}_2$, and denote the point (x, y) by x_y for convenience. Additions and multiplications over $\mathbb{Z}_p \times \mathbb{Z}_2$ are defined as follows:

$$\begin{aligned} x_y + x'_{y'} &= (x + x')_{(y+y')} \\ x_y x'_{y'} &= (xx')_{(yy')} \end{aligned}$$

where $x + x'$, xx' are addition and multiplication modulo p, and $y + y'$, yy' are addition and multiplication modulo 2. For an sSQS $(\mathbb{Z}_p \times \mathbb{Z}_2, \mathcal{B})$, we classify all blocks (quadruples) in $\mathcal{B}$ into three types.

Type I contains all the quadruples of form $\{a_0, b_0, c_1, d_1\}$, simply denoted by $\{a, b; c, d\}$, where $a \neq b$ and $c \neq d$.

Type II contains all the quadruples of form $\{a_0, b_0, c_0, d_1\}$ or $\{a_1, b_1, c_1, d_0\}$ simply denoted by $\{a, b, c; d\}$, where a, b, c are pairwise distinct.

Type III contains all the quadruples of form $\{a_0, b_0, c_0, d_0\}$ or $\{a_1, b_1, c_1, d_1\}$, simply denoted by $\{a, b, c, d\}$, where a, b, c, d are pairwise distinct.

Similarly, the triples of form $\{a_0, b_0, c_0\}$ or $\{a_1, b_1, c_1\}$ are called *pure triples*, simply denoted by $\{a, b, c\}$, and the triples of form $\{a_0, b_0, c_1\}$ or $\{a_1, b_1, c_0\}$ are called *mixed triples* simply denoted by $\{a, b; c\}$. Clearly, pure triples are contained in Type II and (or) III quadruples, and mixed triples are contained in Type I and (or) II quadruples.

Construction 3.1 ([19]). If $\mathrm{CG}(\Omega_p)$ has a 1-factor, let $a_1, a_2, \ldots, a_{\lfloor \frac{p}{12} \rfloor}$ be elements in Ω_p, such that

$$E(F) = \left\{ \{C(a_1), C(a_1^{\sigma c})\}, \{C(a_2), C(a_2^{\sigma c})\}, \ldots, \{C(a_{\lfloor \frac{p}{12} \rfloor}), C(a^{\sigma c}_{\lfloor \frac{p}{12} \rfloor})\} \right\},$$

is the edge set of F.

Let $b_1, b_2, \ldots, b_{\frac{p-1}{4}}$ be elements in $\mathbb{Z}_p \setminus \{0, 1, 2^{-1}\}$, such that

$$\left\{\mathrm{orb}_{\boldsymbol{AC}}(b_i) \mid i = 1, 2, \ldots, \tfrac{p-1}{4}\right\} = \left\{\mathrm{orb}_{\boldsymbol{AC}}(b) \mid b \in \mathbb{Z}_p \setminus \{0, 1, 2^{-1}\}\right\},$$

where $\mathrm{orb}_{\boldsymbol{AC}}(b) = \{b, 1-b, \frac{b}{2b-1}, \frac{1-b}{1-2b}\}$.

All base blocks of affine-invariant sSQS$(2p)$ are shown as follows.

(i) For $p \equiv 1 \pmod{12}$,

$$\begin{aligned}
&\text{Type I}, && \{0,1;b_i,1-b_i\}, \text{ for } i = 1,2,\ldots,\tfrac{p-1}{4},\\
&\text{Type II}', && \{0,1,-1;0\},\\
&\text{Type III}', && \{0,1,\xi,\xi^{\sigma c}\},\\
&\text{Type III}, && \{0,1,a_i,1-a_i\}, \text{ for } i = 1,2,\ldots,\tfrac{p-13}{12},\ a_i \notin C(\xi) \cup C(\xi^{\sigma c}),
\end{aligned}$$

where $\xi = \frac{1+\sqrt{-3}}{2}$ is a root of $x^2 - x + 1 = 0$ over $\mathbb{Z}_p$.

(ii) For $p \equiv 5 \pmod{12}$,

$$\begin{aligned}
&\text{Type I}, && \{0,1;b_i,1-b_i\}, \text{ for } i = 1,2,\ldots,\tfrac{p-1}{4},\\
&\text{Type II}', && \{0,1,-1;0\},\\
&\text{Type III}, && \{0,1,a_i,1-a_i\}, \text{ for } i = 1,2,\ldots,\tfrac{p-5}{12},
\end{aligned}$$

§ 4. Recursive Constructions of affine-invariant sSQS$(2p^m)$

Let $p \equiv 5 \pmod{12}$. We begin by giving the recursive construction of an affine-invariant sSQS over $\mathbb{Z}_{2p^2} \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_2$ from the affine-invariant sSQS over $\mathbb{Z}_{2p} \cong \mathbb{Z}_p \times \mathbb{Z}_2$.

Construction 4.1 ([20]). For prime $p \equiv 5 \pmod{12}$, the base blocks of the affine-invariant sSQS$(2p^2)$ are

Type I′ $\{0, 1; \alpha, \beta\}$

Type I $\{0, 1; b_i + sp, 1 - (b_i + sp)\}$, for $i = 1, 2, \ldots, \frac{p-5}{4}$, $s = 0, 1, \ldots, p-1$;

Type II′ $\{0, 1, -1 + sp; sp\}$, for $s = 0, 1, \ldots, \frac{p-1}{2}$;

Type III $\{0, 1, a_i + sp, 1 - (a_i + sp)\}$, for $i = 1, 2, \ldots, \frac{p-5}{12}$, $s = 0, 1, \ldots, p-1$;

Type IV $\{0, p, s; \alpha s + 2^{-1}\beta p\}$, for $s = g^0, g^1, \ldots, g^{\frac{p-3}{2}}$, g is a generator of $\mathbb{Z}_{p^2}^{\times}$;

Type V $pB \pmod{p^2}$, for all base blocks B of the affine-invariant sSQS$(2p)$,

where α, β are roots of $2x^2 - 2x + 1 = 0$ over $\mathbb{Z}_{p^2}$.

Furthermore, the recursive construction can be generalized to affine-invariant sSQS$(2p^m)$.

Construction 4.2 ([20]). For prime $p \equiv 5 \pmod{12}$, if the affine-invariant sSQS$(2p)$ and sSQS$(2p^{m-1})$ are constructed, then the base blocks of the affine-invariant sSQS$(2p^m)$ can be obtained as follows.

Type I′ $\{0, 1; \alpha, \beta\}$

Type I $\{0, 1; b_i + sp^{m-1}, 1 - (b_i + sp^{m-1})\}$, for $i = 1, 2, \ldots, \frac{p-5}{4}$, $s = 0, 1, \ldots, p-1$;

Type II′ $\{0, 1, -1 + sp^{m-1}; sp^{m-1}\}$, for $s = 0, 1, \ldots, \frac{p-1}{2}$;

Type III $\{0, 1, a_i + sp^{m-1}, 1 - (a_i + sp^{m-1})\}$, for $i = 1, 2, \ldots, \frac{p-5}{12}$, $s = 0, 1, \ldots, p-1$;

Type IV $\{0, p^t, s_t; \alpha s_t + (2s_t - p^t)^{-1}\beta p^t s_t\}$, for $t = 1, 2, \ldots, m-1$, $s_t = g_t^0, g_t^1, \ldots, g_t^{\frac{\varphi(p^{m-t})}{2}-1}$, g_t is a generator of $\mathbb{Z}^{\times}_{p^{2(m-t)}}$;

Type V $pB \pmod{p^m}$, for all base blocks B of the affine-invariant sSQS$(2p^{m-1})$,

where α, β are roots of $2x^2 - 2x + 1 = 0$ over $\mathbb{Z}_{p^m}$.

§ 5. Direct Constructions of affine-invariant 2QS(p)

We denote a 2-fold quadruple system of order v (3-$(v, 4, 2)$ design) by 2QS(v) for short. Suppose p is a prime with $p \equiv 5 \pmod{12}$. We can again use the graph CG(Ω_p) to obtain the base blocks. Roughly speaking, by removing a 1-factor from CG(Ω_p), the resulting graph leads to the base blocks of an affine-invariant 2QS(p).

Construction 5.1 ([18]). Suppose CG(Ω_p) has a 1-factor, say F. For every edge $e_i = \{C(x), C(x^{\sigma c})\}$ in CG$(\Omega_p) - F$ with $C(x) \neq C(x^{\sigma c})$, let $a_i = x$, where $i = 1, 2, \ldots, l_p$ and $l_p = \begin{cases} \frac{p-17}{6} & \text{if } p \equiv 29, 41 \pmod{60}, \\ \frac{p-11}{6} & \text{otherwise} \end{cases}$ denote the number of edges (excluding loops) in CG$(\Omega_p) - F$. Then the base blocks of the affine-invariant 2QS(p) are

Type I $\{0, 1, a_i, 1 - a_i\}$, for $i = 1, 2, \ldots, l_p$,

Type II $\{0, 1, a_{l_p+1}, 1 - a_{l_p+1}\}$, where $a_{l_p+1} = -1$;

Type III $\{0, 1, a_{l_p+2}, 1 - a_{l_p+2}\}$, where $a_{l_p+2} = \chi$;

Type IV $\{0, 1, a_{l_p+3}, 1 - a_{l_p+3}\}$, where $a_{l_p+3} = \mu$, if $p \equiv 29, 41 \pmod{60}$,

where $\mu = \frac{3+\sqrt{5}}{2}$ is a root of $x^2 - 3x + 1 = 0$ and $\chi = \frac{1+\sqrt{-1}}{2}$ is a root of $2x^2 - 2x + 1 = 0$.

It is remarkable that this construction can be naturally generalized to finite fields $\mathbb{F}_q$ for prime power $q \equiv 5 \pmod{12}$. Accordingly, the graph $\mathrm{CG}(\mathscr{P}(\mathbb{F}_q))$ is also a natural generalization of $\mathrm{CG}(\mathscr{P}(\mathbb{F}_p))$.

§ 6. Recursive Constructions of affine-invariant 2QS(p^m)

We begin by constructing an affine-invariant 2QS(p^2) from the affine-invariant 2QS(p). Let χ_1 and χ_2 denote a root of $2x^2 - 2x + 1 = 0$ over $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ respectively. Let μ_1 and μ_2 denote a root of $x^2 - 3x + 1 = 0$ over $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ respectively. Denote $B_s^{(1)}(a) = \{0, 1, a + sp, 1 - (a + sp)\}$.

Construction 6.1 ([18]). For prime $p \equiv 5 \pmod{12}$, by using the same notation with Construction 5.1, the base blocks of the affine-invariant 2QS(p^2) are

Type I $B_s^{(1)}(a_i)$, for $i = 1, \ldots, l_p$ and $s = 0, \ldots, p-1$;

Type II $B_s^{(1)}(-1)$, and $s = 0, 1, \ldots, p-1$;

Type III $B_s^{(1)}(\chi_2)$, and $s = 0, 1, \ldots, \frac{p-1}{2}$;

Type IV $B_s^{(1)}(\mu_2)$, and $s = 0, 1, \ldots, p-1$, if $p \equiv 29, 41 \pmod{60}$;

Type V $\{0, p, s, s+p\}$, for $s = g^0, g^1, \ldots, g^{\frac{p-3}{2}}$, g is a generator of $\mathbb{Z}_{p^2}^{\times}$;

Type VI $pB \pmod{p^2}$, for all base blocks B of the affine-invariant 2QS(p).

Construction 6.1 can be naturally generalized to construct affine-invariant 2QS(p^m) for any positive integer m. Generally, $|\mathbb{Z}_{p^m}^{\times}| = p(p^{m-1} - 1)$ and

$$\mathbb{Z}_{p^m}^{\times} = \mathbb{Z}_{p^m} \setminus p\mathbb{Z}_{p^{m-1}} = (\mathbb{Z}_p \setminus \{0\}) + p\mathbb{Z}_{p^{m-1}},$$

where

$$p\mathbb{Z}_{p^{m-1}} = p\mathbb{Z}/p^m\mathbb{Z} = \{p, 2p, \ldots, p(p^{m-1} - 1)\}.$$

Let χ_t and μ_t denote a root of $2x^2 - 2x + 1 = 0$ and $x^2 - 3x + 1 = 0$ respectively over $\mathbb{Z}_{p^t}$. Denote $B_s^{(1)}(a) = \{0, 1, a + sp^{m-1}, 1 - (a + sp^{m-1})\}$.

Construction 6.2 ([18]). For prime $p \equiv 5 \pmod{12}$, by using the same notation with Construction 5.1, the base blocks of the affine-invariant 2QS(p^m) are

Type I $B_s^{(1)}(a_i)$, for $i = 1, \ldots, l_p$ and $s = 0, \ldots, p-1$;

Type II $B_s^{(1)}(-1)$, and $s = 0, 1, \ldots, p-1$;

Type III $B_s^{(1)}(\chi_m)$, and $s = 0, 1, \ldots, \frac{p-1}{2}$;

Type IV $B_s^{(1)}(\mu_m)$, and $s = 0, 1, \ldots, p-1$, if $p \equiv 29, 41 \pmod{60}$;

Type V $\{0, p^t, s_t, s_t + p^t\}$, for $t = 1, \ldots, m-1$ and $s_t = g_t^0, g_t^1, \ldots, g_t^{\frac{\varphi(p^{m-t})}{2}-1}$, where g_t is a generator of $\mathbb{Z}_{p^{2(m-t)}}^{\times}$;

Type VI $pB \pmod{p^m}$, for all base blocks B of the affine-invariant 2QS(p^{m-1}).

§ 7. Related unsolved problems

The studies on designs admitting affine groups are less known. We present some natural problems related to affine-invariant designs.

Problem 7.1. Does there exist affine-invariant sSQS($2n$) or 2QS(n) when n is not a prime power?

Problem 7.2. If we relax the condition of block size, does there exist affine-invariant 3BD?

Problem 7.3. Does there exist affine-invariant t-(v, k, λ) design with larger k or t?

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