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Complex Hadamard matrices attached to some association schemes *

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1 Introduction

A complex Hadamard matrix is a square matrix H with complex entries of absolute value 1 satisfying $HH^{*}=nI$, where $*$ stands for the Hermitian transpose and I is the identity matrix of order n. They are the natural generalization of real Hadamard matrices. Complex Hadamard matrices appear frequently in various branches of mathematics and quantum physics.

A type-II matrix, or an inverse orthogonal matrix, is a square matrix W with nonzero complex entries satisfying $WW^{(-)^{\top}}=nI$, where (x, y)-entry of $W^{(-)}$ is defined by $W_{y,x}{}^{-1}$. Obviously, a complex Hadamard matrix is a type-II matrix.

Complete classifications of complex Hadamard matrices, and of type-II matrices are only available up to order $n=5$ (see [7, 14, 10]). Although it is shown by Craigen [7] that there are uncountably many equivalence classes of complex Hadamard matrices of order n whenever n is a composite number, some type-II matrices are more closely related to combinatorial objects than the others. Szollosi [16] used design theoretical methods to construct complex Hadamard matrices. Strongly regular graphs were used to construct type-II matrices in [5, 6]. See [15] for a generalization. In this paper, we construct type-II matrices and complex Hadamard matrices in the Bose–Mesner algebra of a certain 3-class symmetric association scheme. In particular, we recover the complex Hadamard matrices of order 15 found in [4].

The method of finding complex Hadamard matrices in the Bose–Mesner algebra of a symmetric association scheme generalizes the classical work of Goethals and

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Seidel [9]. Assuming that the association scheme is symmetric, the resulting complex Hadamard matrices are symmetric. It turns out that this assumption enables us to consider only the real parts of the entries of a complex Hadamard matrix, since the orthogonality can be expressed in terms of the real parts. Extending this reduction to type-II matrices, we are led to consider a rational map whose inverse is explicitly given in Section 2. In Section 3, we explain why only real parts come into play when we construct complex Hadamard matrices in the Bose–Mesner algebra of a symmetric association scheme. In Section 4, we consider a particular family of 3-class association schemes. This family was found after extensive computer experiment on the list of 3-class association schemes up to 100 vertices given in [8]. Surprisingly, most other association schemes up to 100 vertices, with the exceptions of amorphic or pseudocyclic schemes, do not admit a complex Hadamard matrix in their Bose–Mesner algebras. In Section 5, we compute the Haagerup set to show inequivalence of type-II matrices constructed in Section 4.

All the computer calculations in this paper were performed by Magma [2].

2 The image of a rational map

We define a polynomial in three indeterminates X, Y, Z as follows:

$$g(X,Y,Z) = X^2 + Y^2 + Z^2 - XYZ - 4.$$

Lemma 1.

$$g\left(\frac{X}{Y} + \frac{Y}{X}, \frac{X}{Z} + \frac{Z}{X}, \frac{Z}{Y} + \frac{Y}{Z}\right) = 0.$$

Lemma 2. *In the rational function field with four indeterminates X, Y, Z and z, the following identities hold:*

$$w + \frac{1}{w} = Y + \frac{z^2(z^2-1)g + c_1 f}{(z^2-1)(zZ-Y)(zX-2)}, \tag{1}$$

$$\frac{z}{w} + \frac{w}{z} = Z + \frac{z^2(z^2-1)g + c_2 f}{z(z^2-1)(zZ-Y)(zX-2)}, \tag{2}$$

$$ww' = 1 + \frac{z^2 g + (2zX - zYZ + f)f}{z(zZ-Y)(zY-Z)}, \tag{3}$$

where

$$\begin{aligned}
f &= z^2 - zX + 1, \\
g &= g(X,Y,Z), \\
c_1 &= (z^2-1)(zX - Z^2 + 2) - (zY - Z)^2, \\
c_2 &= (z^2-1)(zX - Y^2 + 2) - (zZ - Y)^2, \\
w &= \frac{z^2-1}{zZ-Y}, \\
w' &= \frac{z^{-2}-1}{z^{-1}Z - Y}.
\end{aligned}$$

We define a polynomial in six indeterminates $X_{0,1}, X_{0,2}, X_{0,3}, X_{1,2}, X_{1,3}, X_{2,3}$ as follows:

$$h(X_{0,1}, X_{0,2}, X_{0,3}, X_{1,2}, X_{1,3}, X_{2,3}) = \det \begin{bmatrix} 2 & X_{0,1} & X_{0,2} \\ X_{0,1} & 2 & X_{1,2} \\ X_{0,3} & X_{1,3} & X_{2,3} \end{bmatrix}.$$

Lemma 3. *In the rational function field with four indeterminates X_0, X_1, X_2, X_3, set*

$$x_{i,j} = \frac{X_i}{X_j} + \frac{X_j}{X_i} \quad (0 \le i < j \le 3).$$

Then $h(x_{0,1}, x_{0,2}, x_{0,3}, x_{1,2}, x_{1,3}, x_{2,3}) = 0$.

For a finite set N and a positive integer k, we denote by $\binom{N}{k}$ the collection of all k-element subsets of N.

Lemma 4. *Let $N = \{0, 1, \dots, d\}$, $N_3 = \binom{N}{3}$ and $N_4 = \binom{N}{4}$. Let $a_{i,j}$ $(0 \le i, j \le d,\ i \ne j)$ be complex numbers satisfying*

$$a_{i,j} = a_{j,i} \quad (0 \le i < j \le d), \tag{4}$$

$$g(a_{i,j}, a_{j,k}, a_{i,k}) = 0 \quad (\{i, j, k\} \in N_3), \tag{5}$$

$$h(a_{i,j}, a_{i,k}, a_{i,\ell}, a_{j,k}, a_{j,\ell}, a_{k,\ell}) = 0 \quad (\{i, j, k, \ell\} \in N_4). \tag{6}$$

Assume

$$a_{i_0,i_1} \ne \pm 2 \quad \text{for some } i_0, i_1 \text{ with } 0 \le i_0 < i_1 \le d. \tag{7}$$

Let w_{i_0}, w_{i_1} be nonzero complex numbers satisfying

$$\frac{w_{i_0}}{w_{i_1}} + \frac{w_{i_1}}{w_{i_0}} = a_{i_0,i_1}. \tag{8}$$

Define complex numbers w_i $(0 \le i \le d,\ i \ne i_0, i_1)$ by

$$w_i = \frac{w_{i_1}^2 - w_{i_0}^2}{a_{i_1,i} w_{i_1} - a_{i_0,i} w_{i_0}}. \tag{9}$$

Then

$$\frac{w_j}{w_i} + \frac{w_i}{w_j} = a_{i,j} \quad (0 \le i < j \le d). \tag{10}$$

Conversely, if complex numbers $\{w_i\}_{i=0}^d$ satisfy (10), *then* (9) *holds.*

Moreover, if $a_{i,j}$ $(0 \le i < j \le d)$ are all real and

$$-2 < a_{i_0,i_1} < 2, \tag{11}$$

then $|w_i| = |w_j|$ for $0 \le i < j \le d$.

Theorem 1. *Let d, N, N_3, N_4 be as in Lemma 4. Define $\phi : (\mathbb{C}^\times)^{d+1} \to \mathbb{C}^{d(d+1)/2}$ by*

$$\phi(w_0, \dots, w_d) = \left(\frac{w_i}{w_j} + \frac{w_j}{w_i} \right)_{0 \le i < j \le d}.$$

Then the image of ϕ coincides with the zeros of the ideal generated by the polynomials

$$g(X_{i,j}, X_{j,k}, X_{i,k}) = 0 \quad (\{i,j,k\} \in N_3), \tag{12}$$
$$h(X_{i,j}, X_{i,k}, X_{i,\ell}, X_{j,k}, X_{j,\ell}, X_{k,\ell}) = 0 \quad (\{i,j,k,\ell\} \in N_4), \tag{13}$$

where $X_{i,j} = X_{j,i}$.

The following lemma will be used in the proof of Theorem 2.

Lemma 5. *In the rational function field with three indeterminates* X_1, X_2, X_3*, set*

$$x_{i,j} = \frac{X_i}{X_j} + \frac{X_j}{X_i} \quad (0 \le i < j \le 3),$$

where $X_0 = 1$. *Then*

$$\begin{aligned}&(X_1X_2X_3 + 1)(x_{0,1}x_{0,2} + x_{0,3} - x_{1,2}) \\ &= (X_1X_2 + X_3)(x_{0,1}x_{0,2}x_{0,3} + 2 - \frac{1}{2}(x_{1,2}x_{0,3} + x_{1,3}x_{0,2} + x_{2,3}x_{0,1})).\end{aligned}$$

3 Type-II matrices contained in a Bose–Mesner algebra

Throughout this section, we let $\mathcal{A}$ denote a symmetric Bose–Mesner algebra with adjacency matrices $A_0 = I, A_1, \ldots, A_d$. Let n be the size of the matrices A_i, and we denote by

$$P = (P_{i,j})_{\substack{0\le i\le d\\ 0\le j\le d}}$$

the first eigenmatrix of $\mathcal{A}$. Then the adjacency matrices are expressed as

$$A_j = \sum_{i=0}^{d} P_{i,j}E_i \quad (j = 0, 1, \ldots, d),$$

where $E_0 = \frac{1}{n}J, E_1, \ldots, E_d$ are the primitive idempotents of $\mathcal{A}$. The second eigenmatrix

$$Q = (Q_{i,j})_{\substack{0\le i\le d\\ 0\le j\le d}}$$

is defined as $Q = nP^{-1}$, so that

$$E_j = \frac{1}{n}\sum_{i=0}^{d} Q_{i,j}A_i \quad (j = 0, 1, \ldots, d)$$

holds. Since $QP = nI$ and $Q_{i,0} = P_{i,0} = 1$ for $i = 0, 1, \ldots, d$, we have

$$\sum_{j=1}^{d} Q_{i,j} = n\delta_{i,0} - 1. \tag{14}$$

Lemma 6. *Let $w_0, w_1, \ldots, w_d$ be nonzero complex numbers, and set*

$$W = \sum_{j=0}^{d} w_j A_j \in \mathcal{A}, \tag{15}$$

Then the following are equivalent.

(i) *W is a type-II matrix,*

(ii)

$$\left(\sum_{j=0}^{d} w_j P_{k,j}\right)\left(\sum_{j=0}^{d} w_j^{-1} P_{k,j}\right) = n \quad (k = 1, \ldots, d). \tag{16}$$

Lemma 7. *Let e_k be the polynomial in the variables $X_{i,j}$ $(0 \le i < j \le d)$ defined by*

$$e_k = \sum_{0 \le i < j \le d} P_{k,i} P_{k,j} X_{i,j} + \sum_{i=0}^{d} P_{k,i}^2 - n \quad (k = 1, \ldots, d). \tag{17}$$

If the matrix W given by (15) *is a type-II matrix which is not equivalent to an ordinary Hadamard matrix, then the complex numbers $a_{i,j}$ defined by* (10) *are common zeros of the polynomials e_k $(1 \le k \le d)$ and satisfy* (4)–(7).

Conversely, if $a_{i,j}$ $(1 \le i, j \le d)$ are common zeros of the polynomials e_k $(1 \le k \le d)$ and satisfy (4)–(7)*, then there exist complex numbers $w_0, w_1, \ldots, w_d$ satisfying* (10) *such that the matrix W is a type-II matrix which is not equivalent to an ordinary Hadamard matrix.*

Moreover, the matrix W is a scalar multiple of a complex Hadamard matrix which is not equivalent to an ordinary Hadamard matrix if and only if $a_{i,j}$ defined by (10) *are common real zeros of the polynomials e_k $(1 \le k \le d)$, satisfy* (4)–(7) *and* (11).

4 Infinite families of complex Hadamard matrices

Let $q \ge 4$ be an integer, and $n = q^2 - 1$. We consider a three-class association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^3)$ with the first eigenmatrix:

$$P = \begin{bmatrix} 1 & \frac{q^2}{2} - q & \frac{q^2}{2} & q-2 \\ 1 & \frac{q}{2} & -\frac{q}{2} & -1 \\ 1 & -\frac{q}{2}+1 & -\frac{q}{2} & q-2 \\ 1 & -\frac{q}{2} & \frac{q}{2} & -1 \end{bmatrix}. \tag{18}$$

For $q = 2^s$ with an integer $s \ge 2$, there exists a 3-class association scheme with the first eigenmatix (18) (see [3, 12.1.1]).

Let $\mathcal{M} = \langle A_0, A_1, A_2, A_3 \rangle$ be the Bose–Mesner algebra of $\mathcal{X} = (X, \{R_i\}_{i=0}^3)$. Then, $\mathcal{X}$ has two non-trivial fusion schemes. One is an imprimitive scheme $\mathcal{X}_1 = (X, \{R_0, R_1 \cup R_2, R_3\})$ with the first eigenmatrix:

$$P_1 = \begin{bmatrix} 1 & q(q-1) & q-2 \\ 1 & 0 & -1 \\ 1 & -q+1 & q-2 \end{bmatrix}. \tag{19}$$

Another is a primitive scheme $\mathcal{X}_2 = (X, \{R_0, R_1 \cup R_3, R_2\})$ with the first eigenmatrix:

$$P_2 = \begin{bmatrix} 1 & \frac{q^2}{2} - 2 & \frac{q^2}{2} \\ 1 & \frac{q}{2} - 1 & -\frac{q}{2} \\ 1 & -\frac{q}{2} - 1 & \frac{q}{2} \end{bmatrix}. \tag{20}$$

Theorem 2. *Let w_1, w_2, w_3 be nonzero complex numbers. The matrix*

$$W = A_0 + w_1 A_1 + w_2 A_2 + w_3 A_3 \in \mathcal{M} \tag{21}$$

is a type-II matrix if and only if one of the following holds:

(i) $w_1 = w_2 = w_3$, *where*

$$w_3 + \frac{1}{w_3} + q^2 - 3 = 0,$$

(ii) w_3 *is as in* (i), *and*

$$w_1 = w_2 = \frac{-(q-3)w_3 + (q-1)}{q^2 - 2q - 1},$$

(iii)

$$w_1 + \frac{1}{w_1} = \frac{2(q^2-6)}{q^2-4}, \quad w_2 = -1, \quad w_3 = w_1,$$

(iv)

$$w_1 = w_3 = 1, \quad w_2 + \frac{1}{w_2} = \frac{-2(q^2-2)}{q^2},$$

(v)

$$w_1 + \frac{1}{w_1} = -\frac{2}{q}, \quad w_2 = \frac{1}{w_1}, \quad w_3 = 1,$$

(vi)

$$w_1 + \frac{1}{w_1} = a_{0,1},$$

and

$$w_i = \frac{w_1^2 - 1}{a_{1,i} w_1 - a_{0,i}} \quad (i = 2, 3),$$

where

$$\begin{aligned} a_{0,1} &= \frac{-(q-1)(q-2) + (q+2)r}{2q(q+1)}, \\ a_{0,2} &= \frac{(q+2)(q-1) - (q-2)r}{2q(q-3)}, \\ a_{0,3} &= \frac{5q^2 - 2q - 19 - (q-1)r}{2(q+1)(q-3)}, \\ a_{1,2} &= \frac{2(-q^4 + 2q^3 + 4q^2 - 10q + 1 + (q-1)r)}{q^2(q+1)(q-3)}, \\ a_{1,3} &= -a_{0,2}, \\ r^2 &= (17q-1)(q-1). \end{aligned}$$

Note that $w_1w_2 = -w_3$ *holds.*

Corollary 1. *Let* W *be a type-II matrix in* Theorem 2. *Then,* W *is a complex Hadamard matrix if and only if* W *is given in* (iii), (iv), (v), *or* (vi) *with* $r = \sqrt{(17q-1)(q-1)} > 0$.

Chan [4], found three complex Hadamard matrices on the line graph of the Petersen graph. This is the 3-class association scheme with the first eigenmatrix (18), where $q = 4$, and the three matrices can be described as the matrix W in (21) with w_1, w_2, w_3 given as follows.

$$w_1 = 1, \qquad w_2 = \frac{-7 \pm \sqrt{15}i}{8}, \qquad w_3 = 1, \tag{22}$$

$$w_1 = \frac{5 \pm \sqrt{11}i}{6}, \qquad w_2 = -1, \qquad w_3 = w_1, \tag{23}$$

$$w_1 = \frac{-1 \pm \sqrt{15}i}{4}, \qquad w_2 = w_1^{-1}, \qquad w_3 = 1. \tag{24}$$

The cases (22), (23) and (24) are given by (iv), (iii) and (v), respectively, of Theorem 2. Note that (22) is equivalent to the matrix U_{15} in [16].

The complex Hadamard matrix of order 15 constructed in Theorem 2 (vi) seems to be new. This is obtained by setting $q = 4$ and $r = \sqrt{201}$, and has coefficients $w_1, w_2, w_3 = -w_1w_2$, where

$$\begin{aligned} w_1 + \frac{1}{w_1} &= a_{0,1}, \\ a_{0,1} &= \frac{3}{20}(\sqrt{201} - 1), \\ w_2 &= \frac{a_{0,1}w_1 - 2}{a_{1,2}w_1 - a_{0,2}}, \\ a_{0,2} &= -\frac{1}{4}(\sqrt{201} - 9), \\ a_{1,2} &= \frac{3\sqrt{201} - 103}{40}. \end{aligned}$$

We have verified using the span condition [13, Proposition 4.1] that, this matrix, as well as the one given by (22) are isolated, while the two matrices given by (23) and (24) do not satisfy the span condition.

5 Equivalence

For a type-II matrix W of order n, the Haagerup set $H(W)$ (see [10]) is defined as

$$H(W) = \left\{ \frac{W_{i_1,j_1}W_{i_2,j_2}}{W_{i_1,j_2}W_{i_2,j_1}} \;\middle|\; 1 \le i_1, i_2, j_1, j_2 \le n \right\}.$$

We also define

$$K(W) = \{w + \frac{1}{w} \mid w \in H(W) \setminus \{1\}\}.$$

Two complex Hadamard matrices W_1 and W_2 are said to be equivalent if they are type-II equivalent. It is easy to see that, if W_1 and W_2 are equivalent, then $H(W_1) = H(W_2)$, and hence $K(W_1) = K(W_2)$. In this section, we compute the Haagerup sets of type-II matrices constructed in Theorem 2 to conclude that some of them are inequivalent to others.

We suppose that

$$W = \sum_{i=0}^{d} w_i A_i$$

is a complex Hadamard matrix, where $A_0, \ldots, A_d$ are the adjacency matrices of a symmetric Bose–Mesner algebra of an association scheme $(X, \{R_i\}_{i=0}^d)$, and $w_0 = 1$. Let $H(W)$ be the Haagerup set of W. Then

$$H(W) = \bigcup_{i=1}^{4} H_i(W),$$

where

$$H_i(W) = \left\{ \frac{W_{x_1,y_1} W_{x_2,y_2}}{W_{x_2,y_1} W_{x_1,y_2}} \;\middle|\; x_1, x_2, y_1, y_2 \in X,\ |\{x_1, x_2, y_1, y_2\}| = i \right\}$$

for $i = 1, 2, 3, 4$. Clearly,

$$\begin{aligned} H_1(W) &= \{1\}, \\ H_2(W) &= \{1\} \cup \{w_i^{\pm 2} \mid i = 1, \ldots, d\}. \end{aligned} \tag{25}$$

It should be remarked that, although $H(W)$ is an invariant, none of $H_i(W)$ $(i = 2, 3, 4)$ is.

Lemma 8. *If* $|X| \geq 3$, *then*

$$H_3(W) = \{1\} \cup \left\{ \left(\frac{w_i w_j}{w_k} \right)^{\pm 1} \;\middle|\; 1 \leq i, j, k \leq d,\ p_{ij}^k > 0 \right\}.$$

Lemma 9. *Let* Δ *be a subset of* $\{1, \ldots, d\}$. *Suppose that there exists* $i \in \{1, \ldots, d\}$ *such that* $p_{i_1,j_1}^i > 0$ *for any* $i_1, j_1 \in \Delta$. *Then*

$$H_4(W) \supset \left\{ \frac{w_{i_1} w_{i_2}}{w_{j_1} w_{j_2}} \;\middle|\; i_1, i_2, j_1, j_2 \in \Delta \right\} \setminus \{1\}.$$

In particular, if there exists $i \in \{1, \ldots, d\}$ *such that* $p_{i_1,j_1}^i > 0$ *for any* $i_1, j_1 \in \{1, \ldots, d\}$, *then*

$$H_4(W) \setminus \{1\} = \left\{ \frac{w_{i_1} w_{i_2}}{w_{j_1} w_{j_2}} \;\middle|\; i_1, i_2, j_1, j_2 \in \{1, \ldots, d\} \right\} \setminus \{1\}.$$

Lemma 10. *Suppose that there exists $i \in \{1, \dots, d-1\}$ such that $p^i_{i_1,j_1} > 0$ for any $i_1, j_1 \in \{1, \dots, d-1\}$. Moreover, suppose $p^d_{i,j} > 0$ for any $j \in \{1, \dots, d-1\}$. Then*

$$H_4(W) \setminus \{1\} = \left\{ \frac{w_{i_1} w_{i_2}}{w_{j_1} w_{j_2}} \;\middle|\; i_1, i_2, j_1, j_2 \in \{1, \dots, d\} \right\} \setminus \{1\}.$$

Below, we determine the Haagerup set of the type-II matrices given in Theorem 2. In what follows, let $\mathfrak{X} = (X, \{R_i\}_{i=0}^3)$ be an association scheme with the first eigenmatrix (18), where q is an even positive integer with $q \geq 4$. The intersection numbers of $\mathfrak{X}$ are given by

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{q^2}{2} - q & \frac{(q-2)^2}{4} & \frac{(q-2)^2}{4} & \frac{q(q-4)}{4} \\ 0 & \frac{q(q-2)}{4} & \frac{q(q-2)}{4} & \frac{q^2}{4} \\ 0 & \frac{q-4}{2} & \frac{q-2}{2} & 0 \end{bmatrix}, \tag{26}$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{q(q-2)}{4} & \frac{q(q-2)}{4} & \frac{q^2}{4} \\ \frac{q^2}{2} & \frac{q^2}{4} & \frac{q^2}{4} & \frac{q^2}{4} \\ 0 & \frac{q}{2} & \frac{q-2}{2} & 0 \end{bmatrix}, \tag{27}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{q-4}{2} & \frac{q-2}{2} & 0 \\ 0 & \frac{q}{2} & \frac{q-2}{2} & 0 \\ q-2 & 0 & 0 & q-3 \end{bmatrix}, \tag{28}$$

where B_h has (i,j)-entry p^j_{hi} $(0 \leq i, j \leq 3)$.

Lemma 11. *Let $W = I + \sum_{i=1}^3 w_i A_i$ be a type-II matrix belonging to the Bose–Mesner algebra of $\mathfrak{X}$. Then*

$$\begin{aligned} H(W) &= \{w_i^{\pm 2} \mid i = 1,2,3\} \\ &\cup \left\{ \left(\frac{w_{i_1} w_{i_2}}{w_{i_3}} \right)^{\pm 1} \;\middle|\; 1 \leq i_1, i_2, i_3 \leq 3,\ p^{i_1}_{i_2,i_3} > 0 \right\} \\ &\cup \left\{ \frac{w_{i_1} w_{i_2}}{w_{j_1} w_{j_2}} \;\middle|\; i_1, i_2, j_1, j_2 \in \{1,2,3\} \right\}. \end{aligned}$$

Using Lemma 11, we can determine the Haagerup set $H(W)$ for each type-II matrix given in Theorem 2. Note that the description of $H(W)$ in Table 1 is valid for all even $q \geq 4$, even though $p^3_{11} = 0$ for $q = 4$.

The elements of $H(W)$ given in Table 1 can be found as follows:

As for the Case (i), $K(W)$ has two elements

$$w_1 + \frac{1}{w_1} = -q^2 + 3, \quad w_1^2 + \frac{1}{w_1^2} = q^4 - 6q^2 + 7.$$

As for (ii), setting

$$A = -\frac{q-3}{q^2 - 2q - 1}, \quad B = \frac{q-1}{q^2 - 2q - 1},$$

	$H(W) \setminus \{1\}$	$K(W)$
(i)	$\{w_1^{\pm 1}, w_1^{\pm 2}\}$	$-q^2+3,\ q^4-6q^2+7$
(ii)	$\{w_1^{\pm 1}, w_1^{\pm 2}, w_3^{\pm 1}, w_3^{\pm 2}, (\frac{w_1^2}{w_3})^{\pm 1}, (\frac{w_3}{w_1})^{\pm 1}, (\frac{w_3}{w_1})^{\pm 2}\}$	$-q^2+3,\ q^4-6q^2+7, \frac{q^3-3q^2-q+7}{q^2-2q-1}, \ldots$
(iii)	$\{-1, \pm w_1^{\pm 1}, \pm w_1^{\pm 2}\}$	$-2,\ \pm\frac{2(q^2-6)}{q^2-4},\ \pm\frac{2(q^4-16q^2+56)}{(q^2-4)^2}$
(iv)	$\{w_2^{\pm 1}, w_2^{\pm 2}\}$	$-\frac{2(q^2-2)}{q^2},\ \frac{2(q^4-8q^2+8)}{q^4}$
(v)	$\{w_1^{\pm 1}, w_1^{\pm 2}, w_1^{\pm 3}, w_1^{\pm 4}\}$	$-\frac{2}{q}, \ldots$
(vi)	$\{-1, \pm w_1^{\pm 1}, \pm w_2^{\pm 1}, \pm w_1^{\pm 2}, \pm w_2^{\pm 2}, (w_1^{\pm 1} w_2^{\pm 1})^2, \pm w_1^{\pm 1} w_2^{\pm 1}, \pm (w_1^2 w_2)^{\pm 1}, \pm (w_1 w_2^2)^{\pm 1}\}$	$-2, \ldots$

Table 1: Haagerup sets

we have

$$\begin{aligned} w_1 &= w_3 A + B, \\ \frac{A^2+B^2-1}{AB} &= q^2-3. \end{aligned}$$

This implies

$$\frac{1}{w_1} = \frac{1}{w_3} A + B$$

so that

$$\begin{aligned} w_1 + \frac{1}{w_1} &= \left(w_3 + \frac{1}{w_3}\right) A + 2B \\ &= \frac{q^3-3q^2-q+7}{q^2-2q-1}. \end{aligned}$$

The Cases (iii) and (iv) are immediate. Finally, it is clear that $K(W)$ contains $-\frac{2}{q}$ and -2, in the Cases (v) and (vi), respectively. We do not need the remaining elements of $K(W)$ to prove the following propositions.

Proposition 1. *Let $W_1, \ldots, W_6$ be type-II matrices given in* (i)–(vi) *of Theorem* 2, *respectively. Then $W_1, \ldots, W_6$ are pairwise inequivalent.*

Proposition 2. *Let W_+ and W_- be type-II matrices given in Theorem* 2 (vi) *with $r > 0$ and $r < 0$, respectively. Then W_+ and W_- are inequivalent.*

We were able to use the Haagerup set to distinguish some of the complex Hadamard matrices in Theorem 2. This is because the Haagerup set can be described by the intersection numbers of the association scheme, and is independent of the isomorphism class. In general, if $q \geq 8$ is a power of 2, there may be many non-isomorphic association schemes with the eigenmatrix (18). We do not know whether complex Hadamard matrices having the same coefficients are equivalent if they belong to Bose–Mesner algebras of non-isomorphic association schemes.

Note that there are two type-II matrices described in Theorem 2(i), since $w_1 = w_2 = w_3$ is either of the two zeros of a quadratic equation. Similarly, there are two type-II matrices in each of (ii)–(v) in Theorem 2. Moreover, there are four type-II matrices in (vi), since there are two choices for r and $a_{0,1}^2 - 4 \neq 0$. The following lemma shows that the two type-II matrices in Theorem 2(i) are inequivalent, and so are those in Theorem 2(ii).

Lemma 12. *Let W and W' be type-II matrices belonging to the Bose–Mesner algebra of an association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Suppose that each of W and W' has $d+1$ distinct entries, the valencies of $\mathcal{X}$ are pairwise distinct, and $\min\{p_{11}^i \mid 0 < i \leq d\} > \frac{|X|}{2}$. If W and W' are type-II equivalent, then W is a scalar multiple of W'.*

For a matrix W with nonzero complex entries, we denote its entrywise inverse by $W^{(-)}$.

Proposition 3. *Let W be a type-II matrix given in* (i) *of Theorem* 2. *Then W and $W^{(-)}$ are inequivalent. The same conclusion holds if W is a type-II matrix given in* (ii) *of Theorem* 2.

We do not know whether the two type-II matrices in each of (iii)–(v) in Theorem 2 are equivalent or not, and whether the two type-II matrices in Theorem 2(vi) with a given sign for r are equivalent or not.

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A Verification by Magma

Isolation

```
n:=15;
A0:=ScalarMatrix(n,1);
J:=Parent(A0)![1:i in [1..n^2]];
LO3:=LineGraph(OddGraph(3));
A1:=AdjacencyMatrix(LO3);
A2:=A1^2-A1-4*A0;
A3:=J-A0-A1-A2;
DM:=DistanceMatrix(LO3);
DM eq A1+2*A2+3*A3;

hermitianConjugate:=
 func<H|Parent(H)![ComplexConjugate(x):x in Eltseq(Transpose(H))]>;

complexHadamard:=function(xyz)
 AA:=[ChangeRing(A,Parent(xyz[1])):A in [A1,A2,A3]];
 return A0+xyz[1]*AA[1]+xyz[2]*AA[2]+xyz[3]*AA[3];
end function;

spanCondition:=function(H)
 F:=Parent(H[1,1]);
 MnF:=Parent(H);
 n:=Nrows(H);
 Es:=[MnF|0:i in [1..n]];
 for i in [1..n] do
  Es[i][i,i]:=1;
 end for;
 EsF:=[MnF|e:e in Es];
 Hs:=hermitianConjugate(H);
 Vn:=VectorSpace(F,n^2);
 bracket:=sub<Vn|[Vn|Eltseq(v*Hs*w*H-Hs*w*H*v):v,w in EsF]>;
 return Dimension(bracket) eq n^2-2*n+1;
end function;

F<s>:=QuadraticField(-15);
y:=(-7+s)/8;
H:=complexHadamard([1,y,1]);
H*hermitianConjugate(H) eq n*A0;
spanCondition(H);

F<s>:=QuadraticField(-11);
x:=(5+s)/6;
```

```
H:=complexHadamard([x,-1,x]);
H*hermitianConjugate(H) eq n*A0;
not spanCondition(H);

F<s>:=QuadraticField(-15);
x:=(-1+s)/4;
H:=complexHadamard([x,x^(-1),1]);
H*hermitianConjugate(H) eq n*A0;
not spanCondition(H);

F<s>:=QuadraticField(201);
Z:=(53-3*s)/10;
R<T>:=PolynomialRing(F);
K<z>:=ext<F|T^2-Z*T+1>;
z+1/z eq Z;
x:=1/144*((-5*Z+31)*z-25*Z+155);
xb:=1/144*((-5*Z+31)*z^(-1)-25*Z+155);
y:=1/144*((25*Z-155)*z+5*Z-31);
yb:=1/144*((25*Z-155)*z^(-1)+5*Z-31);
x*xb eq 1;
y*yb eq 1;
H:=complexHadamard([x,y,z]);
H*hermitianConjugate(H) eq n*A0;
spanCondition(H);
```

Table 1

```
HWminus1:=function(w)
 I3:={1..3};
 H3q:={w[i1]*w[i2]/w[i3]:i1,i2,i3 in I3
   |#[i:i in [i1,i2,i3]|i eq 3] ne 2};
 H3q4:={w[i1]*w[i2]/w[i3]:i1,i2,i3 in I3
   |#[i:i in [i1,i2,i3]|i eq 3] ne 2 and {i1,i2,i3} ne {1,3}};
 plus:=[{w[i]^2:i in I3} join H:H in [H3q,H3q4]];
 return {(p join {x^(-1):x in p} join
  {w[i1]*w[i2]/(w[j1]*w[j2]):i1,i2,j1,j2 in I3})
  diff {1}:p in plus};
end function;
Rw<w1,w2,w3>:=FunctionField(Rationals(),3);
HWminus1([w1,w1,w1]) eq
 {&join{{w1^s,w1^(s*2)}:s in {1,-1}}};
HWminus1([w1,w1,w3]) eq
 {&join{{w^s,w^(s*2)}:s in {1,-1},w in {w1,w3}} join
 &join{{(w1^2/w3)^s,(w3/w1)^s,(w3/w1)^(s*2)}:s in {1,-1}}};
HWminus1([w1,-1,w1]) eq {{-1} join
```

```
 &join{{s1*w1^s,s1*w1^(s*2)}:s,s1 in {1,-1}}};
HWminus1([1,w2,1]) eq {&join{{w2^s,w2^(s*2)}:s in {1,-1}}};
HWminus1([w1,w1^(-1),1]) eq {{w1^(s*k):s in {1,-1},k in {1..4}}};
HWminus1([w1,w2,-w1*w2]) eq {{-1} join
 {s0*w^(s*k):w in {w1,w2},s,s0 in {1,-1},k in {1,2}} join
 &join{{s0*w1^s1*w2^s2,(w1^s1*w2^s2)^2}:s0,s1,s2 in {1,-1}}
 join &join{{s0*(w1^2*w2^(-1))^s,s0*(w1^(-1)*w2^2)^s}
  :s,s0 in {1,-1}}};
// (i)
Rq<q>:=FunctionField(Rationals());
(-q^2+3)^2-2 eq q^4-6*q^2+7;
// (ii)
Rw3<w3>:=FunctionField(Rq);
A:=-(q-3)/(q^2-2*q-1);
B:=(q-1)/(q^2-2*q-1);
(A^2+B^2-1)/(A*B) eq q^2-3;
w1:=A*w3+B;
(A/w3+B)-1/w1 eq 1/w1*A*B*(w3+1/w3+(q^2-3));
// (iii)
(2*(q^2-6)/(q^2-4))^2-2 eq 2*(q^4-16*q^2+56)/(q^2-4)^2;
// (iv)
(-2*(q^2-2)/q^2)^2-2 eq 2*(q^4-8*q^2+8)/q^4;
```

Proof of Proposition 1

(iii)$\not\cong$(vi)

```
Rq<q>:=FunctionField(Rationals());
k3a:=(q^2-6)/(q^2-4);
k3b:=2*(q^4-16*q^2+56)/(q^2-4)^2;
ra1:=(k3a+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(ra1^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
ra2:=(-k3a+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(ra2^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
rb1:=(k3b+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(rb1^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
rb2:=(-k3b+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(rb2^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
```

(i)$\not\cong$(ii)

```
n:=q^2-1;
```

```
Numerator((2-n)-(q^3-3*q^2-q+7)/(q^2-2*q-1))
 eq -(q-2)*(q+1)*(q^2-5);
Numerator((n^2-4*n+2)-(q^3-3*q^2-q+7)/(q^2-2*q-1))
 eq (q-2)*(q+1)^2*(q^3-2*q^2-4*q+7);
```

(iv)$\ncong$(v)

```
Numerator(-2*(n-1)/(n+1)-(-2)/q)
 eq -2*(q-2)*(q+1);
Numerator(2*(n^2-6*n+1)/(n+1)^2-(-2)/q)
 eq 2*(q-2)*(q+1)*(q^2+2*q-4);
```

Proof of Proposition 2

```
ra1:=(2+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(ra1^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
ra2:=(-2+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(ra2^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
```