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Numerical outflow boundary conditions for the Navier-Stokes equations

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This short note is an announcement and simplified version of our recent paper [20]. We state only our motivation, problems and some results. For the complete proof, we refer to [20].

1 Motivation

In numerical simulation of real-world flow problems, we often encounter some issues concerning artificial boundary conditions. A typical and important example is the blood flow problem in the large arteries, where the blood is assumed to be a viscous incompressible fluid (cf. [7], [17]). The blood vessel is modeled by a branched pipe as illustrated, for example, by Fig. 1. We are able to give a velocity profile at the *inflow boundary* S and the flow is supposed to be a perfect non-slip on the wall C. Then, the blood flow simulation is highly dependent on the choice of artificial boundary conditions posed on the *outflow boundary* Γ.

In order to state the problem more specifically, let $\Omega\subset\mathbb{R}^d$, $d=2,3$, be a bounded domain and let the boundary $\partial\Omega$ be composed of three parts S, C and Γ. Those S, C and Γ are assumed to be smooth surfaces, although the whole boundary $\partial\Omega$ itself is not smooth. Then, for $T>0$, we consider the Navier-Stokes equations

$$u_t+(u\cdot\nabla)u=\nabla\cdot\sigma(u,p)+f \qquad \text{in } \Omega\times(0,T), \tag{1a}$$
$$\nabla\cdot u=0 \qquad \text{in } \Omega\times(0,T), \tag{1b}$$
$$u=b \qquad \text{on } S\times(0,T), \tag{1c}$$
$$u=0 \qquad \text{on } C\times(0,T), \tag{1d}$$
$$u|_{t=0}=u_0 \qquad \text{on } \Omega \tag{1e}$$

for the velocity $u=(u_1,\ldots,u_d)$ and the pressure p with the density $\rho=1$ and the kinematic viscosity ν of the viscous incompressible fluid under consideration. Therein, $\sigma(u,p)=(\sigma_{i,j}(u,p))=-pI+2\nu D(u)$ denotes the stress tensor, where $D(u)=(D_{i,j}(u))=\left(\frac{1}{2}\left(\nabla u+\nabla u^{\mathrm{T}}\right)\right)$ the deformation-rate tensor and I the identity. The prescribed functions $f=f(x,t)$ and

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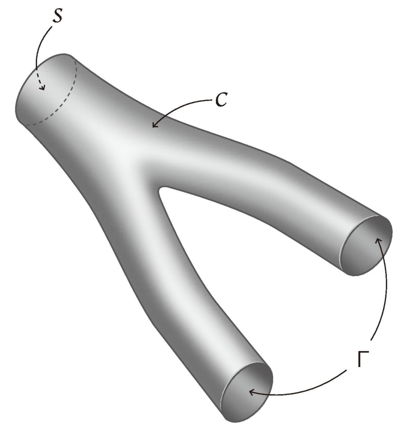

Figure 1: A branched pipe

$u_0 = u_0(x)$ denote the external force and initial velocity, respectively. Moreover, $b = b(x,t)$ denotes the prescribed inflow velocity with $b|_{\partial S} = 0$.

A setting of the boundary condition on Γ is not a trivial task, since the flow distribution and pressure field are unknown and cannot be prescribed in many simulations. As a common outflow boundary condition, the free-traction condition or the so-called do-nothing condition

$$\tau(u,p) = 0 \quad \text{on} \quad \Gamma \tag{2}$$

is still frequently used so far (cf. [10], [9]), where

$$\tau(u,p) = \sigma(u,p)n \tag{3}$$

denotes the traction vector on $\partial\Omega$ and n the outward normal vector to $\partial\Omega$. Though this condition is enough for many problems, it sometimes causes serious numerical instability near Γ (cf. [5, Remark 4.1], [18]). Actually, from the view-point of mathematics, the energy inequality is not guaranteed under (2) and it is a drawback of employing (2). To describe this issue, we take a *reference flow* (g,π) which is the solution of the Stokes system

$$\nabla \cdot \sigma(g,\pi) = 0, \quad \nabla \cdot g = 0 \quad \text{in } \Omega, \tag{4a}$$

$$g = b \text{ on } S, \quad g = 0 \text{ on } C, \quad g = g_0(x)\beta(t) \text{ on } \Gamma \tag{4b}$$

for all $t \in [0,T]$, where $g_0 = g_0(x) \in C_0^\infty(\Gamma)^d$ is a prescribed function satisfying

$$\int_\Gamma g_0 \cdot n \, d\Gamma = 1, \quad g_0 \cdot n \geq 0 \text{ on } \Gamma \tag{5}$$

and we set

$$\beta(t) = -\int_S b \cdot n \, dS.$$

(The function g is nothing but a lifting function of b.) By using this, we will find (u,p) of the form

$$u = U + g, \qquad p = P + \pi.$$

Then, the energy inequality for (1) reads as

$$\sup_{t\in[0,T]} \|U\|^2_{L^2(\Omega)^d} + 2\nu \int_0^T D_{ij}(U)D_{ij}(U) \le C, \tag{6}$$

where C denotes a positive constant depending only on f, u_0, b and T. This inequality is of use. It plays a crucial role in the construction of a solution of the Navier-Stokes equations as is just discussed in this paper. Furthermore, it is connected with the stability of numerical schemes from the view-point of numerical computation. That is, it is preferred that the energy inequality does not spoiled after discretizations. For example, we often take some kinds of approximation to

$$\int_\Omega (u\cdot\nabla)v\cdot w\ dx$$

to ensure the energy inequality under time discretizations. However, it is not certain the energy inequality (6) to hold under (2) even for the continuous case. In fact, assuming (1) admits a smooth solution $(u,p) = (U+g, P+\pi)$ in $0 \le t \le T$ and multiplying the both sides of (1a) by U, we have by the integration by parts

$$\frac{d}{dt}\|U\|^2_{L^2(\Omega)^d} + \underbrace{2\nu \int_\Omega D_{ij}(U)D_{ij}(U)\ dx + \frac{1}{2}\int_\Gamma u_n|U|^2\ d\Gamma - \int_\Gamma \tau(u,p)\cdot U\ d\Gamma}_{=I}$$
$$= \int_\Omega [f - g_t - (g\cdot\nabla)g]\cdot U\ dx - \int_\Omega (U\cdot\nabla)g\cdot U\ dx. \tag{7}$$

If $I \ge 0$, we can derive (6); However, it is impossible to get $I \ge 0$ since we have no information about u_n on Γ under (2). (Bothe et al. [6] recently studied the well-posedness of the Navier-Stokes equations under a class of energy preserving boundary conditions; However, the common one (2) was discussed only in the case of the Stokes equations.)

With this connection, F. Boyer, F. Bruneau and P. Fabrie proposed and studied a class of nonlinear boundary conditions that ensure the energy inequality (cf. [1], [2], [3], [4]). A typical outflow condition they proposed is given as

$$\tau(u,p) = -\frac{1}{2}[u_n]_- U + 2\nu D(g)n \quad \text{on } \Gamma, \tag{8}$$

where

$$[s]_\pm = \max\{0, \pm s\}, \qquad s = [s]_+ - [s]_-.$$

Under the boundary condition (8), the identity (7) implies

$$\frac{d}{dt}\|U\|^2_{L^2(\Omega)^d} + 2\nu \int_\Omega D_{ij}(U)D_{ij}(U)\ dx + \frac{1}{2}\int_\Gamma [u_n]_+|U|^2\ d\Gamma$$
$$= \int_\Omega [f - g_t - (g\cdot\nabla)g]\cdot U\ dx - \int_\Omega (U\cdot\nabla)g\cdot U\ dx + \int_\Gamma 2\nu D(g)n\cdot U\ d\Gamma.$$

Then, after some calculations, we obtain the energy inequality (6). Actually, they established the well-posedness of (1) with a class of boundary conditions, including (8), by Galerkin's method based on (6).

As a matter of fact, a similar boundary condition is successfully applied in actual computations, that is, in blood flow simulation for thoracic arteries. In Bazilevs et al. [5, §4], they employed the following condition. First, they introduced a regularized traction vector

$$\tilde{\tau}(u,p) = \tau(u,p) + [u_n]_- u$$

and considered the resistance boundary condition

$$\tilde{\tau}(u,p) \cdot n + R \int_\Gamma u_n \, d\Gamma + p_0 = 0, \quad \tilde{\tau}(u,p) - [\tilde{\tau}(u,p) \cdot n]n = 0 \quad \text{on } \Gamma,$$

where R and p_0 are prescribed constants that control the average of the flow rate across Γ (cf. [19], [8]). This condition is equivalently written as

$$\tau(u,p) = -[u_n]_- u - \left(R \int_\Gamma u_n \, d\Gamma + p_0 \right) n. \tag{9}$$

If $b = 0$ (then we can take $g = 0$ and $\pi = 0$), we derive the energy inequality under this condition. They offered several numerical results for medical problems and did not give any mathematical considerations. On the other hand, Labeur and Wells [14] considered essentially the same condition as (9) with $R = p_0 = 0$, where they studied energy stable hybrid discontinuous finite element method but did not discuss about the well-posedness of the continuous problem.

Those previous works suggest us that it is important to control the flow-direction near the outflow boundary for stable numerical computations and that the energy inequality is a key property to check whether the flow-direction is suitable or not. Therefore, it is worth-while considering flow-direction boundary conditions, such as (8) and (9), from the view-point of numerical analysis. Furthermore, it seems that there are little works devoted to those boundary conditions from the view-point of pure analysis.

The condition (8) is useful, but it has a few difficulties. Thus, a non-trivial relationship is assumed between the traction $\tau(u,p)$ and the velocity u in (8) and we have to determine the reference velocity g before computation. On the other hand, it is not obvious that the condition (9) is suitable for the case $b \neq 0$.

In the present paper, we propose a new boundary condition. That is, in order to control the flow direction at Γ, we pose a unilateral boundary condition of Signorini's type

$$\begin{cases} u_n \geq 0, \\ \tau_n(u,p) \geq 0, \ u_n \tau_n(u,p) = 0, \ \tau_T(u) = 0 \end{cases} \text{on } \Gamma, \tag{10}$$

where

$$\tau_n(u,p) = \tau(u,p)n, \qquad \tau_T(u) = \tau(u,p) - \tau_n(u,p)n.$$

This is an analogy to Signorini's condition in the theory of elasticity (cf. [12]). Under this condition, the solution of (1) satisfies the energy inequality (cf. Theorem 4) and it is indeed a generalization of the free-traction condition (2). Namely,

$$\text{if } u_n > 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(u,p) = 0 \text{ on } \omega;$$
$$\text{if } u_n = 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(u,p) \geq 0 \text{ on } \omega.$$

The condition (10) is described in terms of inequalities so that it cannot be directly applied to numerical calculations. However, we can utilize its penalty approximation

$$\tau_n(u,p) = \frac{1}{\varepsilon}[u_n]_-, \quad \tau_T(u) = 0 \quad \text{on } \Gamma, \tag{11}$$

where $0 < \varepsilon \ll 1$ is the penalty parameter. After introducing a C^1 regularization of $[\cdot]_-$ (for example, ρ_δ in (??)), we can solve (1) with (11) by using, for example, Newton's iteration. We do not need to introduce the reference velocity g for computation. (For mathematical analysis below, we need g.) It is indeed an approximation of (10); Thus, we have

$$(u_\varepsilon, p_\varepsilon) \to (u,p) \quad \text{as} \quad \varepsilon \to 0$$

in a certain sense or other, where (u,p) and $(u_\varepsilon, p_\varepsilon)$ denote solutions of (1) with, respectively, (10) and (11). Moreover, the condition (11) is closely related with (9) in a certain sense. Namely, although ε is originally defined as a positive constant, we set it as a function;

$$\frac{1}{\varepsilon} = [u_n]_-.$$

Then, (11) implies

$$\tau_n(u,p) = [u_n]_-^2 = -[u_n]_- u_n, \quad \tau_T(u) = 0.$$

Hence, as for the normal component, (11) and (9) are equivalent in the case $R = p_0 = 0$. This suggests that (9) is of use for the case $b \neq 0$. This is another motivation for studying (11).

Our ultimate aim is to develop the theory for the initial-boundary value problems for the Navier-Stokes equations (1) with (10) or with (11) from the standpoint both of analysis and numerical computations. In this short note, we mention results on the well-posedness of these problems; We refer to [20] for the complete proof. We postpone a study on time discretizations and the finite element approximation in future works; a partial result on the finite element approximation for a model (stationary) Stokes problem will be reported in Saito et al. [15].

2 Summary

We shall give the precise statement of our main results in §6, after having described the variational interpretation of our target problems. However, let us summarize our results here for the reader's convenience.

First, we assume that the prescribed inflow velocity $b = b(x,t)$ satisfies $b|_{\partial S} = 0$ and

$$\beta(t) = -\int_S b \cdot n \, dS > 0, \quad t \in [0,T].$$

Consequently, we will have

$$\int_\Gamma u \cdot n \, d\Gamma = \beta(t) > 0, \quad t \in [0,T].$$

As is clearly stated in Introduction of [11], weak solutions of Leray-Hopf's class is not suitable for the purpose of application to numerical analysis. We are strongly motivated by [11] and interested in constructing of strong solutions of Ladyzhenskaya's class (cf. [13]), that is, we will find functions

$$u \in L^\infty(0,T;H^1(\Omega)^d), \quad u_t \in L^2(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^2(\Omega)^d),$$
$$p \in L^\infty(0,T;L^2(\Omega))$$

that satisfy the Navier-Stokes equation (1) with the unilateral boundary condition (10) in the sense of distributions.

To this end, it suffices to find (U,P) satisfying the following perturbed Navier-Stokes problem.

(NS) For $t \in (0,T)$, find (U,P) such that

$$\begin{aligned}
&U_t + ((U+g)\cdot\nabla)U + (U\cdot\nabla)g - \nabla\cdot\sigma(U,P) = F && \text{in } \Omega, && (12a)\\
&\nabla\cdot U = 0 && \text{in } \Omega, && (12b)\\
&U = 0 && \text{on } S\cup C, && (12c)\\
&U_n + g_n \ge 0, \quad \tau_n(U+g,P+\pi) \ge 0 && \text{on } \Gamma, && (12d)\\
&(U_n+g_n)\tau_n(U+g,P+\pi) = 0, \quad \tau_T(U) = -\tau_T(g) && \text{on } \Gamma, && (12e)\\
&U(x,0) = U_0 && \text{on } \Omega, && (12f)
\end{aligned}$$

where

$$F = f - g_t - (g\cdot\nabla)g,$$
$$U_0 = u_0 - g(0).$$

Actually, under some appropriate assumptions on F, U_0, and (g,π) (cf. (A1)–(A4) below), we will prove (cf. Theorem 2 in §6), there exists a unique solution of (NS) satisfying

$$U \in L^\infty(0,T;H^1(\Omega)^d), \quad U_t \in L^2(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^2(\Omega)^d),$$
$$P \in L^\infty(0,T;L^2(\Omega)).$$

For the penalty problem (1) with (11), we consider the following perturbed problem.

(NS$_\varepsilon$) Let $0<\varepsilon\ll 1$. For all $t\in(0,T)$, find $(U_\varepsilon,P_\varepsilon)$ such that

$$\begin{aligned}
&U_{\varepsilon,t} + (U_\varepsilon + g\cdot\nabla)U_\varepsilon + (U_\varepsilon\cdot\nabla)g - \frac{1}{\rho}\nabla\cdot\sigma(U_\varepsilon,P_\varepsilon) = F && \text{in } \Omega,\\
&\nabla\cdot U_\varepsilon = 0 && \text{in } \Omega,\\
&U_\varepsilon = 0 && \text{on } S\cup C,\\
&\tau_n(U_\varepsilon+g,P_\varepsilon+\pi) = \frac{1}{\varepsilon}[U_{\varepsilon n}+g_n]_-, \quad \tau_T(U_\varepsilon) = -\tau_T(g) && \text{on } \Gamma,\\
&U_\varepsilon(x,0) = u_0 - g(0) && \text{on } \Omega,
\end{aligned}$$

Then,

$$u_\varepsilon = U_\varepsilon + g, \qquad p_\varepsilon = P_\varepsilon + \pi$$

solve (1) with (11). Under the same assumptions on F, U_0, and (g,π), we will prove (cf. Theorem 3 in §6), there exists a unique solution of (NS$_\varepsilon$) satisfying

$$U_\varepsilon \in L^\infty(0,T;H^1(\Omega)^d), \quad U_{\varepsilon,t} \in L^2(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^2(\Omega)^d),$$
$$P_\varepsilon \in L^\infty(0,T;L^2(\Omega))$$

for a sufficiently small ε.

The plan of our analysis is as follows. We firstly give variational formulations (NS-E) and (NS_ε-E) of (NS) and (NS_ε) in §5, after having described the variational interpretations of traction vectors $\tau(u,p)$, $\tau_n(u,p)$ and $\tau_T(u)$ in §4. Further, (NS-E) is converted into the variational inequality problem (NS-I). We also introduce the solenoidal (divergence-free) versions (NS-I^σ) and (NS_ε-E^σ) of (NS-I) and (NS_ε-E), respectively.

Theorems 2 and 3 are divided into several propositions (See [20] for the statements of these propositions):

- Proposition 1 (The unique existence of U of a solution of (NS-I^σ));
- Proposition 2 (The existence of an associating pressure P with U of (NS-I));
- Proposition 3 (The uniqueness of (NS-I));
- Proposition 4 (The unique existence of U_ε of a solution of (NS_ε-E^σ));
- Proposition 5 (The existence of an associating pressure P_ε with U_ε of (NS_ε-E));
- Proposition 6 (The uniqueness of (NS_ε-E)).

We use a C^1 regularization ρ_δ of $[\cdot]_-$ and the standard Galerkin's method to prove Proposition 4. Therein, several a priori estimates including

$$\|[U_{\varepsilon,n} + g_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \leq C\sqrt{\varepsilon}$$

play important role. Then, we prove Proposition 1 by compactness. Usually, we apply a version of De Rham's theorem (cf. [16, Lemma IV.1.4.3] for example) to deduce a pressure of the Navier-Stokes equations, after a velociy has been obtained. Unfortunately, it is not enough to deduce pressures P and P_ε for our problems. Actually, we have to choose suitable constants k and k_ε such that $(U, \mathring{P} + k)$ and $(U_\varepsilon, \mathring{P}_\varepsilon + k_\varepsilon)$ satisfy (NS-I) and (NS_ε-E), respectively, where $\mathring{P}$ and $\mathring{P}_\varepsilon$ are associating pressures with U and U_ε, respectively. (P and P_ε are L^2 functions in Ω with zero mean values.)

(NS-E) and (NS_ε-E) admit energy inequalities of the form (6).

3 Notation

We recall that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain and the boundary $\partial\Omega$ is composed of three parts S, C and Γ.

We follow the standard notation, for example, of [?] and [16] as for function spaces and their norms. We employ the abbreviations:

$$\begin{aligned} \|u\| = \|u\|_\Omega = \|u\|_{0,\Omega} &= \|u\|_{L^2(\Omega)^d} \text{ or } \|u\|_{L^2(\Omega)}; \\ \|u\|_1 = \|u\|_{1,\Omega} &= \|u\|_{H^1(\Omega)^d} \text{ or } \|u\|_{H^1(\Omega)}; \\ \|u\|_\Gamma = \|u\|_{0,\Gamma} &= \|u\|_{L^2(\Gamma)^d} \text{ or } \|u\|_{L^2(\Gamma)}; \\ (u,v) &= (u,v)_{L^2(\Omega)^d} \text{ or } (u,v)_{L^2(\Omega)}. \end{aligned}$$

We frequently use the following function spaces:

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } C \cap S\}, \qquad V^\sigma = \{v \in v \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$
$$V_0 = H_0^1(\Omega)^d, \qquad V_0^\sigma = \{v \in V_0 \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$
$$K = \{v \in V \mid v_n + g_n \geq 0 \text{ on } \Gamma\}, \qquad K^\sigma = \{v \in K \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$
$$Q = L^2(\Omega), \qquad Q_0 = L_0^2(\Omega) = \left\{q \in Q \mid \int_\Omega q\ dx = 0\right\}, \qquad M = H_{00}^{\frac{1}{2}}(\Gamma).$$

The spaces V and V^σ are closed subspaces of $H^1(\Omega)^d$ and are equipped with norm $\|\cdot\|_1$. The spaces V_0 and V_0^σ are also closed subspaces of $H^1(\Omega)^d$ and are equipped with norm $\|\cdot\|_1$ by virtue of the Poincaré inequality.

In general, X' denotes the dual space of a Banach space X.

We use the following forms (the summation convection is employed):

$$\begin{aligned}
&a(u,v) = 2\nu \int_\Omega D_{ij}(u) D_{ij}(v)\ dx \quad (u, v \in H^1(\Omega)^d);\\
&a_1(u,v,w) = \int_\Omega [(u \cdot \nabla)v] w\ dx \quad (u, v, w \in H^1(\Omega)^d);\\
&b(v,p) = -\int_\Omega (\nabla \cdot v) p\ dx \quad (v \in H^1(\Omega)^d,\ p \in L^2(\Omega));\\
&[\lambda, \eta] = \text{ the duality pairing between } \lambda \in M' \text{ and } \eta \in M;\\
&[[\lambda, \eta]] = \text{ the duality pairing between } \lambda \in (M^d)' \text{ and } \eta \in M^d.
\end{aligned}$$

As usual, we write C to express various positive constants that depend only on Ω.

For a vector-valued function v defined on $\partial\Omega$, it's normal and tangential components are denoted, respectively, by

$$v_n = v \cdot n, \qquad v_T = v - (v_n)n.$$

4 The re-definition of traction vectors

For $(U, P) \in V \times Q$, we cannot define $\tau(U, P)$ as a function on Γ because of the lack of regularity. However, if (U, P) is smooth and satisfies (12a), it also satisfies, for $t \in (0, T)$

$$\begin{aligned}
\int_\Gamma \tau(U,P) \cdot v\ d\Gamma = (U_t, v) + a(U,v) + a_1(U+g, U, v)&\\
+ a_1(U, g, v) + b(v, P) - (F, v) \qquad (v \in V),&
\end{aligned}$$

where $\tau(U, p)$ is understood as a usual function on Γ.

Based on this identity, we re-define the traction vector $\tau(U, P)$ as a *functional* over M^d for a solution $(U, P) \in V \times Q$ of (NS) in the sense of distributions (More precisely, for (U, P) satisfying (18a) below). We recall the following result.

Lemma 4.1. *There exists an extension operator $\mathcal{E} : M^d \to V$ such that $\mathcal{E}\eta = \eta$ on Γ and $\|\mathcal{E}\eta\|_V \leq C\|\eta\|_{M^d}$ for all $\eta \in M^d$. Conversely, for any $w \in V$, we have $\eta = w|_\Gamma \in M^d$ and $\|\eta\|_{M^d} \leq C\|w\|_V$.*

As a consequence, we obtain an extension operator $\mathcal{E}_n : M \to V$ such that

$$(\mathcal{E}_n\eta)_n = \eta \ , (\mathcal{E}_n\eta)_T = 0 \ \text{ on } \Gamma, \qquad \|\mathcal{E}_n\eta\|_V \le C\|\eta\|_M$$

for any $\eta \in M$. Now we propose the re-definition of $\tau(U,P)$ as follows:

$$\begin{aligned}[][[\tau(U,P),\eta]] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U+g, U, w_\eta)\\ + a_1(U, g, w_\eta) + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M^d), \quad (13)\end{aligned}$$

where $w_\eta = \mathcal{E}\eta \in V$. Actually, the right-hand side of (13) does not depend on the way of extension; Hence, this definition is well-defined. Similarly, we re-define as

$$\begin{aligned}[][[\tau_T(U),\eta]] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U+g, U, w_\eta) + a_1(U, g, w_\eta)\\ + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M^d \text{ with } \eta_n = 0;\ w_\eta = \mathcal{E}\eta) \quad (14)\end{aligned}$$

and

$$\begin{aligned}[][\tau_n(U,P),\eta] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U+g, U, w_\eta)\\ + a_1(U, g, w_\eta) + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M;\ w_\eta = \mathcal{E}_n\eta). \quad (15)\end{aligned}$$

Then,

$$[[\tau(U,P),\eta]] = [\tau_n(U,P),\eta_n] + [[\tau_T(U),\eta_T]] \qquad (\eta \in M^d). \qquad (16)$$

For a solution $(U_\varepsilon, P_\varepsilon)$ of (NS$_\varepsilon$), we propose the similar re-definition. For example,

$$\begin{aligned}[][\tau_n(U_\varepsilon,P_\varepsilon),\eta] = (U_{\varepsilon,t}, w_\eta) + a(U_\varepsilon, w_\eta) + a_1(U_\varepsilon+g, U_\varepsilon, w_\eta)\\ + a_1(U_\varepsilon, g, w_\eta) + b(w_\eta, P_\varepsilon) - (F, w_\eta) \qquad (\eta \in M;\ w_\eta = \mathcal{E}_n\eta). \quad (17)\end{aligned}$$

On the other hand, we will assume that $\tau(g,\pi) \in H^1(0,T; L^2(\Gamma)^d)$ (see, (A1) below) so that we have

$$[[\tau(g,\pi),\eta]] = \int_\Gamma \tau(g,\pi)\cdot\eta \ d\Gamma \quad (\eta \in M^d).$$

5 Problems

Under those re-definitions presented in the previous section, we precisely interpret (NS) as follows.

(NS-E) For a.e. $t \in (0,T)$, find $(U(t), P(t)) \in V \times Q$ with $U_t(t) \in Q^d$ such that

$$\begin{aligned}
&(U_t, v) + a(U,v) && && \\
&\quad + a_1(U+g, U, v) + a_1(U, g, v) + b(v, P) = (F, v) && \forall v \in V_0, && (18a)\\
&b(U,q) = 0 && \forall q \in Q, && (18b)\\
&U_n + g_n \ge 0 && \text{on } \Gamma, && (18c)\\
&[\tau_n(U,P) + \tau_n(g,\pi), \eta] \ge 0 && \forall \eta \in M,\ \eta \ge 0, && (18d)\\
&[\tau_n(U,P) + \tau_n(g,\pi), U_n + g_n] = 0, && && (18e)\\
&[[\tau_T(U) + \tau_T(g), \eta]] = 0 && \forall \eta \in M, && (18f)\\
&U(x,0) = U_0 && \text{on } \Omega. && (18g)
\end{aligned}$$

Remark 5.1. If $(U, P) \in V \times Q$ satisfies (18a), then $[\tau(U, P), \eta]$ and $[[\tau_T(U), \eta]]$ are well-defined by (15) and (14).

(NS-E) can be converted into the following variational inequality problem.

(NS-I) For a.e. $t \in (0, T)$, find $(U(t), P(t)) \in K \times Q$ with $U_t(t) \in Q^d$ such that

$$\begin{aligned} &(U_t, v - U) + a(U, v - U) + a_1(U + g, U, v - U) \\ &\quad + a_1(U, g, v - U) + b(v - U, P) \geq (F, v - U) - [[\tau(g, \pi), v - U]] && \forall v \in K, && (19a) \\ &b(U, q) = 0 && \forall q \in Q, && (19b) \\ &U(x, 0) = U_0 && \text{on } \Omega. && (19c) \end{aligned}$$

The following solenoidal version of (NS-I) will be of use later.

(NS-I$^\sigma$) For a.e. $t \in (0, T)$, find $U(t) \in K^\sigma$ with $U_t(t) \in Q^d$ such that

$$\begin{aligned} &(U_t, v - U) + a(U, v - U) + a_1(U + g, U, v - U) \\ &\quad + a_1(U, g, v - U) \geq (F, v - U) - [[\tau(g, \pi), v - U]] && \forall v \in K^\sigma, && (20a) \\ &U(x, 0) = U_0 && \text{on } \Omega. && (20b) \end{aligned}$$

We state the following variational formulations of (NS$_\varepsilon$).

(NS$_\varepsilon$E) For a.e. $t \in (0, T)$, find $(U_\varepsilon(t), P_\varepsilon(t)) \in V \times Q$ with $U_{\varepsilon,t}(t) \in Q^d$ such that

$$\begin{aligned} &(U_{\varepsilon,t}, v) + a(U_\varepsilon, v) + a_1(U_\varepsilon + g, U_\varepsilon, v) + a_1(U_\varepsilon, g, v) \\ &\quad + b(v, P_\varepsilon) - \frac{1}{\varepsilon} \int_\Gamma [U_{\varepsilon n} + g_n]_- v_n \, d\Gamma = (F, v) - [[\tau(g, \pi), v]] && \forall v \in V, && (21a) \\ &b(U_\varepsilon, q) = 0 && \forall q \in Q, && (21b) \\ &U_\varepsilon(x, 0) = U_0 && \text{on } \Omega. && (21c) \end{aligned}$$

(NS$_\varepsilon$E$^\sigma$) For a.e. $t \in (0, T)$, find $U_\varepsilon(t) \in V^\sigma$ with $U_{\varepsilon,t}(t) \in Q^d$ such that

$$\begin{aligned} &(U_{\varepsilon,t}, v) + a(U_\varepsilon, v) + a_1(U_\varepsilon + g, U_\varepsilon, v) \\ &\quad + a_1(U_\varepsilon, g, v) - \frac{1}{\varepsilon} \int_\Gamma [U_{\varepsilon n} + g_n]_- v_n \, ds = (F, v) - [[\tau(g, \pi), v]] && \forall v \in V^\sigma, && (22a) \\ &U_\varepsilon(x, 0) = U_0, && \text{on } \Omega. && (22b) \end{aligned}$$

6 Main results

We are now in a position to state the main results of this paper. Recall that (g, π) is the solution of the Stokes system (4) and $g_0 \in C_0^\infty(\Gamma)^d$ is defined by (5). We make the following assumptions.

(A1) $f \in H^1(0, T; Q^d)$ and $\tau(g, \pi)|_\Gamma \in H^1(0, T; L^2(\Gamma)^d)$.

(A2) $g \in H^2(0, T; Q^d) \cap L^\infty(0, T; V^\sigma)$ and $g_t \in L^2(0, T; V^\sigma)$.

(A3) $\beta(t) \geq \beta_0 > 0$ for $t \in [0, T]$ with some $\beta_0 > 0$ and $\beta(t) \in C^2(0, T)$.

(A4) $U_0 \in V_0^\sigma \cap H^2(\Omega)^d$ and it satisfies

$$-(\nu\Delta U_0, v) = a(U_0, v) + \int_\Gamma \tau(g,\pi)|_{t=0} v \, d\Gamma \quad (v \in V^\sigma).$$

Remark 6.1. Conditions (A1) and (A2) implies $F \in H^1(0,T;Q^d)$ and $F \in L^\infty(0,T;Q^d)$.

Remark 6.2. Condition (A2) leads to $g \in L^\infty(0,T;Q^d)$.

Remark 6.3. On Γ, $\tau(g,\pi)|_{t=0}$ is well-defined by (A1).

Theorem 1. *Problems* (NS-E) *and* (NS-I) *are equivalent.*

Theorem 2. *Assume that* (A1)–(A4) *are satisfied. When $d=2$, there exists a unique*

$$U \in L^\infty(0,T;V^\sigma), \quad U_t \in L^2(0,T;V^\sigma) \cap L^\infty(0,T;Q^d), \tag{23a}$$

$$P \in L^\infty(0,T;Q) \tag{23b}$$

satisfying (NS-I) *for any $T \in (0,\infty)$. In particular, (U,P) is the unique solution of* (1) *with* (10) *in the sense of distributions. When $d=3$, the same conclusion holds for a smaller time interval $(0,T_*]$, where T_* denotes a positive constant depending on U_0.*

Theorem 3. *Assume that* (A1)–(A4) *are satisfied. When $d=2$, there exists a unique*

$$U_\varepsilon \in L^\infty(0,T;V^\sigma), \quad U_{\varepsilon,t} \in L^2(0,T;V^\sigma) \cap L^\infty(0,T;Q^d), \tag{24a}$$

$$P_\varepsilon \in L^\infty(0,T;Q) \tag{24b}$$

satisfying (NS$_\varepsilon$-E) *for any $T \in (0,\infty)$ and a sufficiently small ε. More precisely, there exists $\varepsilon_0 > 0$, which depends only on F, g, U_0, Ω and T, such that* (NS$_\varepsilon$-E) *admits a unique solution $(U_\varepsilon, P_\varepsilon)$ satisfying* (24) *for any $\varepsilon \in (0,\varepsilon_0]$. When $d=3$, the same conclusion holds for a smaller time interval $(0,T_*]$, where T_* denotes a positive constant depending on U_0.*

Theorem 4 (Energy inequality for (NS-E)). *If there exists*

$$U \in L^\infty(0,T;Q^d) \cap L^2(0,T;V^\sigma), \quad U_t \in L^2(0,T;Q^d) \tag{25}$$

that satisfy (NS-E) *in $0 \le t \le T$ with some $P(t) \in Q$, then we have*

$$\sup_{0\le t\le T} \|U(t)\|^2 + \int_0^T a(U(t),U(t)) \, dt \le C(T), \tag{26}$$

where $C(T)$ denotes a positive constant depending on F, g, U_0, Ω and T.

Theorem 5 (Energy inequality for (NS$_\varepsilon$-E)). *Let $\varepsilon > 0$. Suppose that there exists*

$$U_\varepsilon \in L^\infty(0,T;Q^d) \cap L^2(0,T;V^\sigma), \quad U_{\varepsilon,t} \in L^2(0,T;Q^d)$$

that satisfy (NS$_\varepsilon$-E) *in $0 \le t \le T$ with some $P_\varepsilon(t) \in Q$. Moreover, assume that*

$$\|[U_{\varepsilon n} + g_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \le C_2(T)\sqrt{\varepsilon} \quad (t \in [0,T]). \tag{27}$$

Then, there exists $\varepsilon_1 > 0$ such that we have, for $\varepsilon \in (0,\varepsilon_1]$,

$$\sup_{0\le t\le T} \|U_\varepsilon(t)\|^2 + \int_0^T a(U_\varepsilon(t),U_\varepsilon(t)) \, dt \le C(T). \tag{28}$$

Therein, ε_1, $C_2(T)$ and $C(T)$ denote positive constants depending on F, g, U_0, Ω and T.

Remark 6.4. As is described in [20, §4], the existence proof of U_ε depends on the inequality (27). Hence, it is not restrictive that we assume (27) as long as the solution exists.

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