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# Integration by parts formula for Feynman path integrals

By

Daisuke FUJIWARA*

## Abstract

The aim of this paper is to present

1. Review of time slicing approximation method of Feynman path integrals introduced by Feynman [4].
2. An integration by parts formula for Feynman path integrals under suitable assumption:

$$\int_{\Omega_{x,y}} DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) = -\int_{\Omega_{x,y}} F(\gamma)\mathrm{Div}\, p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) - i\nu\int_{\Omega_{x,y}} F(\gamma)DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma).$$

   This formula is an analogy to Elworthy's integration by parts formula for Wiener integrals. cf. [3]

3. An application of integration by parts formula to semiclassical asymptotic formula which holds in the case of $F(\gamma^*) = 0$. Here $\gamma^*$ is the stationary point of the phase $S(\gamma)$, i.e., $\delta S(\gamma^*) = 0$.

## Contents



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*Department of Mathematics, Gakushuin University, 1-5-1 Mejiro Toshima-ku, Tokyo 171-8588, Japan.

## § 1. Path integral defined by Feynman

For simplicity we restrict ourselves to the case where the configuration space is $\mathbf{R}^1$. In this case Lagrangian function with potential $V(t, x)$ is

$$L(t, \dot{x}, x) = \frac{1}{2}\dot{x}^2 - V(t, x).$$

The case where non zero magnetic potential is present is discussed in [11]. Action of path $\gamma : [s, s'] \to \mathbf{R}$ is

$$S(\gamma) = \int_s^{s'} L(t, \dot{\gamma}(t), \gamma(t))dt. \tag{1.1}$$

We assume throughout this paper the following assumption for potential $V(t, x)$ cf. W.Pauli [14]:

**Assumption 1.1.** 1. $V(t, x)$ is a real continuous function of $(t, x)$. If $t$ is fixed, then it is a function of class $C^\infty$ in $x$.

2. For $\forall m \geq 0$ there exists $v_m \geq 0$ such that

$$\max_{|\alpha|=m} \sup_{(t,x)\in[s,s']\times\mathbf{R}^d} \left|\partial_x^\alpha V(t, x)\right| \leq v_m(1 + |x|)^{\max\{2-m,0\}}.$$

We write $\mathcal{H}$ for the $L^2$-Sobolev space $H^1(s, s')$ of order 1 in $[s, s']$. For any $x, y \in \mathbf{R}$ we write $\mathcal{H}_{x,y}$ for the closed subset $\{\gamma \in H^1(s, s'); \gamma(s) = y, \gamma(s') = x\}$ of $\mathcal{H}$. If $x = 0$ and $y = 0$ we write $\mathcal{H}_0$ for $\mathcal{H}_{0,0}$. It is clear that action $S(\gamma)$ (1.1) is well defined for $\gamma \in \mathcal{H}$ under the Assumption 1.1.

**Proposition 1.2.** *Let $\delta_0 > 0$ be so small that*

$$\frac{\delta_0^2 v_2}{8} < 1. \tag{1.2}$$

*If $|s' - s| \leq \delta_0$, then for any $x, y \in \mathbf{R}$ there exists one and only path $\gamma^* \in \mathcal{H}_{x,y}$ such that*

$$S(\gamma^*) = \min_{\gamma\in\mathcal{H}_{x,y}} S(\gamma).$$

$\gamma^*$ is the classical path, i.e. the first variation $\delta S(\gamma^*)$ of $S(\gamma)$ at $\gamma^*$ vanishes:

$$\delta S(\gamma^*) = 0, \quad \gamma_0(s) = y, \quad \gamma_0(s') = x.$$

It is of class $C^2[s, s']$ and satisfies Euler equation:

$$\frac{d^2}{dt^2}\gamma(t) + \partial_x V(t, \gamma(t)) = 0,$$
$$\gamma(s') = x, \quad \gamma(s) = y.$$

We define

$$S(s', s, x, y) = S(\gamma^*). \tag{1.3}$$

This is called classical action.

Let $\Delta$ be an arbitrary division of the interval $[s, s']$ such that

$$\Delta : s = T_0 < T_1 < \cdots < T_J < T_{J+1} = s'. \tag{1.4}$$

We set $\tau_j = T_j - T_{j-1}, j = 1, 2, \ldots, J+1$ and $|\Delta| = \max_{1\leq j\leq J+1} \tau_j$.

Suppose that $|\Delta| \leq \delta$. We set $x_0 = y, x_{J+1} = x$. For all $x_j \in \mathbf{R}$, $j = 1, 2, \ldots, J$, there exists one and only one piecewise classical path $\gamma_\Delta(t)$ which is the classical path for $T_{j-1} \leq t \leq T_j$ and satisfies

$$\gamma_\Delta(T_j) = x_j, \quad (j = 0, 1, 2, \ldots, J+1). \tag{1.5}$$

$\gamma_\Delta$ may have edges at $T_j$. We use the symbol $\gamma_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ to express the piecewise classical path satisfying (1.5), when we want to express explicitly its dependence on $(x_{J+1}, x_J, \ldots, x_1, x_0)$.

If $\Delta$ and $x, y$ are given then we write $\Gamma_{x,y}(\Delta)$ for the totality of all piecewise classical path $\gamma_\Delta \in \mathcal{H}_{x,y}$. We write $\Gamma_0(\Delta)$ for $\Gamma_{0,0}(\Delta)$. By the map

$$\Gamma(\Delta) \ni \gamma_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) \to (x_{J+1}, x_J, \ldots, x_1, x_0) \in \mathbf{R}^{J+2} \tag{1.6}$$

we can identify $\Gamma(\Delta)$ and $\mathbf{R}^{J+2}$. Similarly $\Gamma_{x,y}(\Delta)$ is identified with $\mathbf{R}^J$.

Given a functional $F(\gamma)$, we often abbreviate $F(\gamma_\Delta)$ as $F_\Delta$. Once $\Delta$ is fixed, it is a function of $(x_{J+1}, x_J, \ldots, x_1, x_0)$ and we denote the dependence of $F(\gamma_\Delta)$ on $(x_{J+1}, x_J, \ldots, x_1, x_0)$ by writing $F(\gamma_\Delta) = F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$.

**Feynman's formulation of path integral.** Let $\nu = 2\pi h^{-1}$, where h is Planck's constant. And let $\Omega_{xy}$ be the space[1] of paths starting $y$ at time $s$ and reaching $x$ at time $s'$. Given a functional $F(\gamma)$ of $\gamma \in \Omega_{xy}$, Feynman [4] considered the following integral on finite dimensional space:

$$\begin{aligned} &I[F_\Delta](\Delta; \nu, s', s, x, y) \\ &= \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j}\right)^{1/2} \int_{\mathbf{R}^J} F(\gamma_\Delta)(x, x_J, \ldots, x_1, y) \times e^{i\nu S(\gamma_\Delta)(x, x_J, \ldots, x_1, y)} \prod_{j=1}^{J} dx_j. \end{aligned} \tag{1.7}$$

[1] In this note $\Omega$ is a symbol which expresses vaguely notion of path space.

Feynman defined his path integral by the formula:

$$\int_{\Omega_{xy}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}[\gamma] = \lim_{|\Delta|\to 0} I[F_\Delta](\Delta;\nu,s',s,x,y). \tag{1.8}$$

The integral $I[F_\Delta](\Delta;\nu,s',s,x,y)$ of (1.7) [2] is called time slicing approximation of Feynman path integral (1.8). We say $F(\gamma)$ is "*F-integrable*", if the limit on the right hand side of (1.8) exists.

The main aim of Feynman's paper [4] is the statement that the path integral (1.8) with $F \equiv 1$ and $s'$ replaced by $t$ is the fundamental solution of Schrödinger's equation

$$\frac{i}{\nu}\partial_t u(t,x) = H(t)u(t,x) \quad (t \in (s,s')), \tag{1.9}$$

where $H(t) = \frac{1}{2}(-\frac{i}{\nu}\partial_x)^2 + V(t,x)$ is the Hamiltonian operator.

## § 2. Some properties of classical action

From now on we always assume

$$|s'-s| \le \delta_0. \tag{2.1}$$

Calculation shows:

**Proposition 2.1.** *If $|s'-s| \le \delta$, $S(s',s,x,y)$ is of the following form:*

$$S(s',s,x,y) = \frac{|x-y|^2}{2(s'-s)} + (s'-s)\phi(s',s,x,y).$$

*The function $\phi(s',s,x,y)$ is a function of $(s',s,x,y)$ of class $C^1$ and $\exists C > 0$ such that*

$$|\phi(s',s,x,y)| \le C(1+|x|^2+|y|^2).$$

*Moreover, if $s'$ and $s$ are fixed, $\phi(s',s,x,y)$ is a $C^\infty$ function of $(x,y)$ and for $\forall m \ge 2$ we have*

$$\max_{2\le|\alpha|+|\beta|\le m} \sup_{(x,y)\in\mathbf{R}^2} \left|\partial_x^\alpha\partial_y^\beta \phi(s',s,x,y)\right| = \kappa_m < \infty.$$

*In particular,*

$$\kappa_2 \le \frac{v_2}{2}\left(1-\frac{v_2\delta_0^2}{8}\right)^{-1}.$$

Let $\Delta$ be the division of time interval $[s,s']$ as (1.4).
We discuss time slicing approximation of path integral.

$$\begin{aligned} &I[F_\Delta](\Delta;\nu,s',s,x,y) \\ &= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2} \int_{\mathbf{R}^J} F_\Delta(x_{J+1},x_J,\dots,x_1,x_0)e^{i\nu S_\Delta(x_{J+1},x_J,\dots,x_1,x_0)}\prod_{j=1}^{J} dx_j. \end{aligned} \tag{2.2}$$

[2]For fixed $\Delta$ the integral (1.7) does not converge absolutely even in the case $F(\gamma) \equiv 1$. We regard (1.7) as an oscillatory integral.

Here $S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$ is an abbreviation of $S(\gamma_\Delta)(x_{J+1}, x_J, \dots, x_1, x_0)$. We also abbreviate $S(T_j, T_{j-1}, x_j, x_{j-1})$ to $S_j(x_j, x_{j-1})$ and $\phi(T_j, T_{j-1}, x_j, x_{j-1})$ to $\phi_j(x_j, x_{j-1})$. Thus

$$S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) = \sum_{j=1}^{J+1} S_j(x_j, x_{j-1}) = \sum_{j=1}^{J+1} \left( \frac{|x_j - x_{j-1}|^2}{2\tau_j} + \tau_j \phi_j(x_j, x_{j-1}) \right).$$

Consider $J \times J$ matrix $\Psi$ whose $(j,k)$ element is

$$\Psi_{jk} = \partial_{x_j} \partial_{x_k} S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) \qquad (j, k = 1, 2, \dots, J).$$

Then we divide the matrix $\Psi$ into two parts.

$$\Psi = H_\Delta + W_\Delta,$$

where

$$H_\Delta = \begin{pmatrix} \frac{1}{\tau_1} + \frac{1}{\tau_2} & -\frac{1}{\tau_2} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\tau_2} & \frac{1}{\tau_2} + \frac{1}{\tau_3} & -\frac{1}{\tau_3} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\tau_3} & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & -\frac{1}{\tau_J} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\tau_J} & \frac{1}{\tau_J} + \frac{1}{\tau_{J+1}} \end{pmatrix}$$

and $W_\Delta$ is the matrix whose $(j,k)$ element is

$$w_{jk} = \begin{cases} \partial_{x_j}^2 (\tau_j \phi_j + \tau_{j+1} \phi_{j+1}) & \text{if} \quad j = k \\ \partial_{x_k} \partial_{x_j} \tau_j \phi_j & \text{if} \quad k = j - 1 \\ \partial_{x_k} \partial_{x_j} \tau_{j+1} \phi_{j+1} & \text{if} \quad k = j + 1 \\ 0 & \text{if} \quad |j - k| \geq 2. \end{cases} \tag{2.3}$$

The matrix $H_\Delta$ is a positive definite constant matrix with determinant

$$\det H_\Delta = \frac{\tau_1 + \tau_2 + \cdots + \tau_{J+1}}{\tau_1 \tau_2 \dots \tau_{J+1}} = \frac{(s' - s)}{\tau_1 \tau_2 \dots \tau_{J+1}}.$$

It has its inverse $H_\Delta^{-1}$. Regarding $W_\Delta$ as an perturbation, we write

$$\Psi = H_\Delta (I + H_\Delta^{-1} W_\Delta).$$

**Proposition 2.2.** *Let $0 < \delta_1$ be so small that $\delta_1 \leq \delta_0$ and $\kappa_2 \delta_1^2 < 1$. Let $|s' - s| \leq \delta_1$. Then $\forall (x_{J+1}, x_J, \dots, x_1, x_0) \in \mathbf{R}^{J+2}$*

$$(1 - \kappa_2 \delta_1^2)^J \leq \det(I + H_\Delta^{-1} W_\Delta) \leq (1 + \kappa_2 \delta_1^2)^J,$$

*and*

$$(1 - \kappa_2 \delta_1^2)^J \frac{(s' - s)}{\tau_1 \tau_2 \dots \tau_{J+1}} \leq \det \Psi = \det(H_\Delta + W_\Delta) \leq (1 + \kappa_2 \delta_1^2)^J \frac{(s' - s)}{\tau_1 \tau_2 \dots \tau_{J+1}}.$$

Assume $|s'-s| \le \delta_1$. Let $\gamma^*$ be the unique classical path in $\mathcal{H}_{x,y}$ and let $x_j^* = \gamma^*(T_j)$ for $j = 0, 1, 2, \ldots, J+1$ and $W_\Delta^* = W_\Delta\Big|_{x_j = x_j^*, 1\le j \le J}$. We set

$$D(\Delta; s', s, x, y) = \det(I + H_\Delta^{-1} W_\Delta^*)$$
$$= \left(\frac{\tau_1\tau_2\ldots\tau_{J+1}}{s'-s}\right) \det Hess_{x_J^*, x_{J-1}^*, \ldots x_1^*} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0).$$

Here $Hess_{x_J^*,\ldots x_1^*} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ is the Hessian matrix at $(x_J^*, \ldots x_1^*)$ of $S_\Delta$.

**Proposition 2.3.** *Suppose that* $0 < |s'-s| \le \delta_1$. *Define* $d(\Delta; s', s, x, y)$ *by*

(2.4) $$D(\Delta; s', s, x, y) = 1 + (s'-s)^2 d(\Delta; s', s, x, y).$$

*Then for any* $k \ge 0$

(2.5) $$\sup_{|s'-s|\le\delta_1} \sup_{\Delta} \max_{|\alpha|+|\beta|\le k} \sup_{(x,y)\in\mathbf{R}^2} |\partial_x^\alpha \partial_y^\beta d(\Delta; s', s, x, y)| < \infty.$$

**Proposition 2.4.** *If* $|t-s| \le \delta_1$, *then there exists the limit*

(2.6) $$\lim_{|\Delta|\to 0} D(\Delta; t, s, x, y) = D(t, s, x, y).$$

*Define*

(2.7) $$e(t, s, x, y) = \left(\frac{1}{2\pi(t-s)}\right)^{1/2} D(t, s, x, y)^{-1/2}.$$

*Then this satisfies the transport equation cf. [2]:*

$$\partial_t e(t, s, x, y) + \partial_x S(t, s, x, y)\partial_x e(t, s, x, y) + \frac{1}{2}\partial_x^2 S(t, s, x, y) e(t, s, x, y) = 0.$$

*Let* $(-\frac{d^2}{dt^2})^{-1}$ *be the Green operator of Dirichlet boundary problem. Then* $D(t, s, x, y)$ *equals the following infinite dimensional determinant:*

(2.8) $$D(t, s, x, y) = \det\left(-\frac{d^2}{dt^2} - \partial_x^2 V(t, \gamma^*)\right)\left(-\frac{d^2}{dt^2}\right)^{-1}.$$

## § 3. Stationary phase method for integrals over a space of large dimension

Let $f(x_{J+1}, x_J, \ldots, x_1, x_0)$ be a function of $(x_{J+1}, x_J, \ldots, x_1, x_0) \in \mathbf{R}^{J+2}$. Let $\Delta$ be a division of interval $[s, s']$ as (1.4). Then we can regard the function $f$ as a function defined on $\Gamma(\Delta)$, because $\mathbf{R}^{J+2}$ is identified with $\Gamma(\Delta)$. Let $0 = j_0 < j_1 < \cdots < j_p < j_{p+1} = J+1$ be a subsequence of $\{0, 1, \ldots, J, J+1\}$. Then

(3.1) $$\Delta' : s = T_{j_0} < T_{j_1} < \cdots < T_{j_p} < T_{j_{p+1}} = s'$$

is a division of the interval $[s, s']$ of which $\Delta$ is a refinement. We call a division $\Delta'$ of the interval $[s, s']$ coarser than the division $\Delta$ if $\Delta$ is a refinement of $\Delta'$. There exists a natural embedding map $\Gamma(\Delta') \subset \Gamma(\Delta)$ and $\Gamma_{x,y}(\Delta') \subset \Gamma_{x,y}(\Delta)$. We shall write $\iota_{\Delta'}^{\Delta} f : \Gamma(\Delta') \to \mathbf{C}$ for the pull back of a function $f : \Gamma(\Delta) \to \mathbf{C}$ by this embedding. If $f$ is a function defined on

$\mathbf{R}^{J+2}$, we can define a function $\iota^{\Delta}_{\Delta'} f$ defined on $\mathbf{R}^{p+2}$ using the identifications $\mathbf{R}^{J+2} \cong \Gamma(\Delta)$ and $\mathbf{R}^{p+2} \cong \Gamma(\Delta')$.

For integers $1 \le k < l \le J+1$ we define

$$S_{l,j}(x_l, \dots, x_{j-1}) = S_l(x_l, x_{l-1}) + S_{l-1}(x_{l-1}, x_{l-2}) + \cdots + S_j(x_j, x_{j-1}). \tag{3.2}$$

Note that $S_{J+1,1}(x_{J+1}, \dots, x_0) = S(x_{J+1}, \dots, x_0)$. We understand that $S_{j,j}(x_j, x_{j-1}) = S_j(x_j, x_{j-1})$. Suppose $1 \le k < l \le J+1$. For any fixed $(x_l, x_{j-1}) \in \mathbf{R}^2$ let $(x^*_{l-1}, x^*_{l-2}, \dots, x^*_j)$ be the stationary point of the function $S_{l,j}(x_l, \dots, x_{j-1})$ of (3.2). We shall write $x^*_k(x_l, x_{j-1})$ for $x^*_k$ when we wish to express that $x^*_k$ depends on $(x_l, x_{j-1})$.

Suppose $\Delta'$ is the division given by (3.1) coarser than $\Delta$. Then for any $f(x_{J+1}, x_J, \dots, x_0) \in \Gamma_{x,y}(\Delta)$ it is clear by definition of $\iota^{\Delta}_{\Delta'}$ that

$$\iota^{\Delta}_{\Delta'} f(x_{J+1}, x_{j_p}, \dots, x_{j_1}, x_0) = f(x_{J+1}, x_J, \dots, x_1, x_0)\Big|_{\substack{x_k = x^*_k(x_{j_n}, x_{j_{n-1}}), \\ j_{n-1} < k < j_n, n=1,2,\dots,p+1}}.$$

We write $S^*_{l,j}(x_l, x_{j-1})$ for the stationary value of $S_{l,j}(x_l, \dots, x_{j-1})$. i.e.,

$$\begin{aligned} S^*_{l,j}(x_l, x_{j-1}) =& S_l(x_l, x^*_{l-1}(x_l, x_{j-1})) \\ &+ \sum_{k=j+1}^{l-1} S_k(x^*_k(x_l, x_{j-1}), x^*_{k-1}(x_l, x_{j-1})) + S_j(x^*_j(x_l, x_{j-1}), x_{j-1}). \end{aligned}$$

As $S_j(x_j, x_{j-1}) = S(T_j, T_{j-1}, x_j, x_{j-1})$ is a classical action, it turns out that

$$S^*_{l,j}(x_l, x_{j-1}) = S_{l,j}(x_l, x_j). \tag{3.3}$$

Thus

$$\iota^{\Delta}_{\Delta'} S_\Delta(x_{J+1}, x_{j_p}, \dots, x_{j_1}, x_0) = \sum_{n=1}^{p+1} S_{j_n, j_{n-1}+1}(x_{j_n}, x_{j_{n-1}}).$$

The interval $[s, s']$ itself is a particular division of $[s, s']$, which we write $\Delta(J+1)$. Then $\iota^{\Delta}_{\Delta(J+1)} S_\Delta(x_{J+1}, x_0) = S(s', s, x, y)$.

Given a function $a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0)$ of $(x_{J+1}, x_J, \dots, x_1, x_0) \in \mathbf{R}^{J+2}$ with parameter $\lambda$ and a fixed division $\Delta$, we discuss

$$\begin{aligned} &I(\Delta, S_\Delta, a_\lambda, \nu)(x_{J+1}, x_0) \\ &= \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j}\right)^{1/2} \int_{\mathbf{R}^J} a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0) e^{i\nu S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)} \prod_{j=1}^{J} dx_j. \end{aligned} \tag{3.4}$$

Assumption for the amplitude $a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0)$ is the following:

**Assumption 3.1.** Let $m \ge 0$ be a constant. The amplitude $a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0)$ is defined on $\mathbf{R}^{J+2}$ and may depend on a parameter $\lambda$. For any integer $K \ge 0$ there exist constants $A_K > 0$ and $X_K \ge 1$ such that

1. If $|\alpha_j| \le K$ for all $j = 0, 1, \dots, J+1$, then $\forall (x_{J+1}, x_J, \dots, x_1, x_0) \in \mathbf{R}^{J+2}$

$$\Big|\Big(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j}\Big) a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0)\Big| \le A_K X_K^{J+2} (1 + |\lambda| + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m,$$

2. Let $\Delta'$ be any division defined by (3.1) coarser than $\Delta$ and let $\{\alpha_{j_k}\}$ be a sequence of indices each of which satisfies $|\alpha_{j_k}| \leq K$ for $k = 0, 1, \ldots, p+1$. Then for any $(x_0, x_{j_1}, \ldots, x_{j_p}, x_{J+1}) \in \mathbf{R}^{p+2}$

$$|\partial_{x_0}^{\alpha_0} \partial_{x_{J+1}}^{\alpha_{J+1}} \left( \prod_{k=1}^{p} \partial_{x_{j_k}}^{\alpha_{j_k}} \right) (\iota_{\Delta'}^{\Delta} a_\lambda)(x_{J+1}, x_{j_p}, \ldots, x_{j_1}, x_0)|$$
$$\leq A_K X_K^{p+2} (1 + |\lambda| + |x_{J+1}| + |x_{j_p}| + \ldots + |x_{j_1}| + |x_0|)^m.$$

Let $(x_{l-1}^*, \ldots, x_j^*)$ be the critical point of (3.2). And let $Hess$ mean the Hessian of $S_{l,j}$ at the critical point. We define

$$D_{x_{l-1}^*, \ldots, x_j^*}(S_{l,j}; x_l, x_{j-1}) = \left( \frac{\tau_l + \cdots + \tau_j}{\tau_l \cdots \tau_j} \right) \det Hess(\sum_{k=j}^{l} S_k(x_k, x_{k-1})) \Big|_{x_k = x_k^*, j \leq k \leq l-1}. \tag{3.5}$$

For any $k = 1, 2, \ldots, J+1$ we define the division

$$\Delta(k) : s = T_0 < T_k < T_{k+1} < \cdots < T_{J+1} = s'. \tag{3.6}$$

$\Delta(1) = \Delta$ and $\Delta(J+1)$ is the interval itself without any intermediate dividing point. The following theorem is known [7], [10].

**Theorem 3.2.** *Suppose that $|s' - s| \leq \delta_1$ and $a_\lambda(x_{J+1}, x_J, \ldots, x_1, x_0)$ satisfies Assumption 3.1. We further assume that $|\Delta||s' - s| \leq 1$. Then*

$$I(\Delta; S_\Delta, a_\lambda, \nu)(x_{J+1}, x_0) = \left( \frac{\nu}{2\pi i T} \right)^{1/2} e^{i\nu S(s', s, x_{J+1}, x_0)} k(\Delta; a_\lambda, \nu, s', s, x_{J+1}, x_0) \tag{3.7}$$

*with*

$$\begin{aligned} &k(\Delta; a_\lambda, \nu, s', s, x_{J+1}, x_0) \\ &= D_{x_J^*, \ldots, x_1^*}(S_{J+1,1}; x_{J+1}, x_0)^{-1/2} \Big( \iota_{\Delta(J+1)}^{\Delta} a_\lambda(x_{J+1}, x_0) + \nu^{-1}(s' - s) p(\Delta, x_{J+1}, x_0) \Big) \\ &\quad + \nu^{-1}(s' - s)^2 |\Delta| q(\Delta, x_{J+1}, x_0) + \nu^{-2}(s' - s)^2 r(\Delta, \nu, x_{J+1}, x_0). \end{aligned} \tag{3.8}$$

*Here*

$$\begin{aligned} &p(\Delta, x_{J+1}, x_0) \\ &= -\frac{i}{2(s' - s)} \sum_{j=1}^{J} \frac{(T_j - s)\tau_{j+1}}{(T_{j+1} - s)} (\iota_{\Delta(J+1)}^{\Delta(j)} \Big[ D_{x_{j-1}^*, \ldots, x_1^*}(S_{j,1}; x_j, x_0)^{1/2} \\ &\quad \times \partial_{x_j}^2 (D_{x_{j-1}^*, \ldots, x_1^*}(S_{j,1}, x_j, x_0)^{-1/2} \iota_{\Delta(j)}^{\Delta} a_\lambda)) \Big] (x_{J+1}, x_0). \end{aligned} \tag{3.9}$$

*$q(\Delta, x_{J+1}, x_0)$ is independent of $\nu$. And functions $q(\Delta, x_{J+1}, x_0)$ and $r(\Delta, \nu, x_{J+1}, x_0)$ satisfies the following estimate. For any $K \geq 0$ there exists an integer $M(K) \geq 0$ and a constant $C_K > 0$ independent of $\Delta$ such that*

$$(1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m} |\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} q(\Delta, x_{J+1}, x_0)| \leq C_K A_{M(K)} X_{M(K)}^2 \tag{3.10}$$

$$(1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m} |\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} r(\Delta, \nu, x_{J+1}, x_0)| \leq C_K A_{M(K)} X_{M(K)}^2, \tag{3.11}$$

*if multi-indices $\alpha_0, \alpha_{J+1}$ satisfies $|\alpha_0| \leq K$ and $|\alpha_{J+1}| \leq K$.*

Since

$$(s'-s)^{-1}\sum_{j=1}^{J}\frac{(T_j-s)\tau_{j+1}}{(T_{j+1}-s)}\le 1,$$

we have from Theorem 3.2

**Corollary 3.3.** *If* $|\alpha_{J+1}|\le K$ *and* $|\alpha_0|\le K$, *then*

$$(1+|\lambda|+|x_{J+1}|+|x_0|)^{-m}|\partial_{x_{J+1}}^{\alpha_{J+1}}\partial_{x_0}^{\alpha_0}k(\Delta;a_\lambda,\nu,s',s,x_{J+1},x_0)|\le C_K X_{M(K)}^2 A_{M(K)}. \tag{3.12}$$

**Remark 3.4.** Tsuchida [16] treated the case of non-zero vector potential.

**Definition 3.5.** Let $p\ge 0$ and $k\ge 0$ be integers. For any function $f:\mathbf{R}^n\ni x\to\mathbf{C}$ we define a norm

$$\|f\|_{\{p,k\}}=\sum_{|\alpha|\le k}\sup_{x\in\mathbf{R}^n}(1+|x|)^{-p}|\partial_x^\alpha f(x)|.$$

We write

$$\mathcal{B}_p(\mathbf{R}^n)=\{f\in C^\infty(\mathbf{R}^n):\|f\|_{\{p,k\}}<\infty,\quad\forall k\ge 0\}.$$

$\mathcal{B}_p(\mathbf{R}^n)$ is a Fréchet space. If $p=0$, we abbreviate $\mathcal{B}_0(\mathbf{R}^n)$ to $\mathcal{B}(\mathbf{R}^n)$.

**Definition 3.6.** Let $m\ge 0$ be a constant. Let $\{f_\lambda(x)\}_\lambda$ be a family of functions in $\mathcal{B}_p(\mathbf{R}^n)$. If this is a bounded set in $\mathcal{B}_p(\mathbf{R}^n)$, we write

$$f_\lambda=\mathcal{O}_{\mathcal{B}_p(\mathbf{R}^n)}(1).$$

And we write $f_\lambda=\mathcal{O}_{\mathcal{B}_p(\mathbf{R}^n)}(g)$ if $f_\lambda/g=\mathcal{O}_{\mathcal{B}_p(\mathbf{R}^n)}(1)$.

**Remark 3.7.** It follows from Theorem 3.2 that

$$\begin{aligned}&k(\Delta;a_\lambda,\nu,s',s,x_{J+1},x_0)\\&=D_{x_J^*,\dots,x_1^*}(S_{J+1,1};x_{J+1},x_0)^{-1/2}\\&\times\Big(\iota^{\Delta}_{\Delta(J+1)}a_\lambda(x_{J+1},x_0)+\nu^{-1}(s'-s)p(\Delta,x_{J+1},x_0)\\&+\nu^{-1}\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}((s'-s)^2|\Delta|)+\nu^{-2}\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}((s'-s)^2)\Big).\end{aligned}\tag{3.13}$$

**Assumption 3.8** (N.Kumano-go's assumption). Suppose $a_\lambda(x_{J+1},x_J,\dots,x_1,x_0)$ satisfies Assumption 3.1. Moreover, there exists a bounded Borel measure $\rho\ge 0$ on $[s,s']$ such that as far as $|\alpha_k|\le K$ for $k=0,1,2,\dots,J+1$

$$\begin{aligned}&\Big|\Big(\prod_{k=0}^{J+1}\partial_{x_k}^{\alpha_k}\Big)\partial_{x_j}a_\lambda(x_{J+1},x_J,\dots,x_1,x_0)\Big|\\&\quad\le A_KX_K^{J+2}\rho([T_{j-1},T_{j+1}])(1+|\lambda|+|x_{J+1}|+|x_J|+\dots+|x_1|+|x_0|)^m\quad(0\le\forall j\le J+1)\end{aligned}$$

and that as far as $|\alpha_{j_k}|\le K$ for $k=0,1,\dots,p+1$

$$\begin{aligned}&|\partial_{x_0}^{\alpha_0}\partial_{x_{J+1}}^{\alpha_{J+1}}\left(\prod_{k=1}^{p}\partial_{x_{j_k}}^{\alpha_{j_k}}\right)\partial_{x_{j_k}}(\iota^{\Delta}_{\Delta'}a_\lambda)(x_{J+1},x_{j_p},\dots,x_{j_1},x_0)|\\&\le A_KX_K^{p+2}\rho([T_{j_{k-1}},T_{j_{k+1}}])(1+|\lambda|+|x_{J+1}|+|x_{j_p}|+\dots+|x_{j_1}|+|x_0|)^m\quad(0\le\forall k\le p+1).\end{aligned}$$

**Proposition 3.9.** *Suppose* $a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0)$ *satisfies Kumano-go's assumption. Then the function* $k(\Delta; a_\lambda, \nu, s', s, x_{J+1}, x_0)$ *of (3.7) is of the form*

$$k(\Delta; a_\lambda, \nu, s', s, x_{J+1}, x_0) = D_{x_J^*,\dots,x_1^*}(S_{J+1,1}; x_{J+1}, x_0)^{-1/2}\Big(\iota^{\Delta}_{\Delta(J+1)} a_\lambda(x_{J+1}, x_0) + \nu^{-1} R(\Delta, x_{J+1}, x_0)\Big). \tag{3.14}$$

*And for any integer* $K \geq 0$ *there exist* $C_K$ *and* $M(K)$ *independent of* $\Delta$ *and* $\nu$ *such that as far as* $|\alpha| \leq K, \beta \leq K$

$$(1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m} |\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} R(\Delta, x_{J+1}, x_0)| \leq C_K A_{M(K)} |s' - s|(|s' - s| + \rho([s, s'])).$$

*i.e.,*

$$R(\Delta, x_{J+1}, x_0) = \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(|s' - s|(|s' - s| + \rho([s, s']))). \tag{3.15}$$

## § 4. Convergence of Feynman path integral

We discuss convergence of Feynman path integral. Our discussion is valid only for those $F(\gamma)$ that have rather restrictive properties.

**Assumption 4.1** (N.Kumano-go's condition). Let $m$ be a non-negative constant and $\rho$ be a bounded Borel measure $\rho \geq 0$ on $[s, s']$. Suppose $F(\gamma)$ is a functional defined for all piecewise classical path $\gamma \in \cup_\Delta \Gamma(\Delta)$. For any integer $K \geq 0$ there exist constants $A_K > 0$ and $X_K \geq 1$ such that for any division $\Delta$ defined by (1.4) and for any indices $\alpha_j$, $j = 0, 1, 2, \dots, J+1$ satisfying $|\alpha_j| \leq K$ there hold the following inequalities:

$$\left|\left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j}\right) F(\gamma_\Delta(x_{J+1}, x_J, \dots, x_1, x_0))\right| \leq A_K X_K^{J+2}(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m,$$

$$\left|\left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j}\right) \partial_{x_k} F(\gamma_\Delta(x_{J+1}, \dots, x_{k+1}, x_k, x_{k-1}, \dots, x_0)\right|$$
$$\leq A_K X_K^{J+2} \rho([T_{k-1}, T_{k+1}])(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m.$$

**Remark 4.2.** $F(\gamma) \equiv 1$ clearly satisfies this assumption.

**Example 4.3.** Let $\rho(t)$ be a function of bounded-variation on $[s, s']$ and $f(t, x)$ be a continuous function of $(t, x) \in [s, s'] \times \mathbf{R}$ and infinitely differentiable in $x$. Suppose that for any $\alpha$ there exists a positive constant $C_\alpha$ such that

$$|\partial_x^\alpha f(t, x)| \leq C_\alpha (1 + |x|)^m$$

with some $m \geq 0$ independent of $\alpha$ and $(t, x)$. Then the following functional satisfies Assumptions 4.1.

$$F(\gamma) = \int_s^{s'} f(t, \gamma(t))\, d\rho(t).$$

The next theorem was proved by N.Kumano-go [12], while the case $F(\gamma) \equiv 1$ had been known. [8], [11] and [6].

**Theorem 4.4.** *Suppose that $F(\gamma)$ satisfies Assumption 4.1 above and $|s'-s| \le \delta_1$. Let $I[F_\Delta](\Delta;\nu,s',s,x,y)$ be the time slicing approximation defined by (2.2). We write*

$$I[F_\Delta](\Delta;\nu,s',s,x,y) = \left(\frac{\nu}{2\pi i(s'-s)}\right)^{1/2} e^{i\nu S(s',s,x,y)} k(\Delta;F_\Delta,\nu,s',s,x,y). \tag{4.1}$$

*Then $k(F;\nu,s',s,x,y)) = \lim_{|\Delta|\to 0} k(\Delta;F_\Delta,\nu,s',s,x,y)$ exists in the space $\mathcal{B}_m(\mathbf{R}^2)$. More precisely, for any $K \ge 0$ there exists $C_K > 0$ such that if $|\alpha| \le K$ and $|\beta| \le K$*

$$\begin{aligned} &\sup_{(x,y)\in\mathbf{R}^2} (1+|x|+|y|)^{-m} |\partial_x^\alpha \partial_y^\beta (k(\Delta;F_\Delta,\nu,s',s,x,y) - k(F;\nu,s',s,x,y))| \\ &\qquad \le C_K A_{M(K)} X_{M(K)}^4 |\Delta| (\rho([s,s']) + |s'-s|). \end{aligned} \tag{4.2}$$

*$k(F;\nu,s',s,x,y)$ can be written as*

$$k(F;\nu,s',s,x,y) = D(s',s,x,y)^{-1/2}\left(F(\gamma^*) + \nu^{-1} R[F](\nu,s',s,x,y)\right) \tag{4.3}$$

*and for $|\alpha| \le K$ and $|\beta| \le K$*

$$|\partial_x^\alpha \partial_y^\beta R[F](\nu,s',s,x,y)| \le C_K A_{M(K)} |s-s'|(|s-s'| + \rho([s,s']))(1+|x|+|y|)^m. \tag{4.4}$$

*Set*

$$K[F](\nu,s',s,x,y) = \left(\frac{\nu}{2\pi i(s'-s)}\right)^{1/2} e^{i\nu S(s',s,x,y)} k(F;\nu,s',s,x,y). \tag{4.5}$$

*Then*

$$K[F](\nu,s',s,x,y) = \lim_{|\Delta|\to 0} I[F_\Delta](\Delta;\nu,s',s,x,y). \tag{4.6}$$

**Remark 4.5.** In short, $F(\gamma)$ is "F-integrable" if $F$ satisfies Assumption 4.1. We may write

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] = K[F](\nu,s',s,x,y). \tag{4.7}$$

**Remark 4.6.** Equality (4.3) together with (4.4) imply semiclassical asymptotic formula.

Theorem 4.4 follows from the next proposition.

**Proposition 4.7.** *Let $\Delta^*$ be an arbitrary refinement of $\Delta$. For any integer $K \ge 0$ there exist a constant $C_K$ and an integer $M(K)$ independent of $\Delta, \Delta^*$ and $\nu$ such that*

$$\begin{aligned} &|\partial_x^\alpha \partial_y^\beta (k(\Delta^*;F_{\Delta^*},\nu,s',s,x,y) - k(\Delta;F_\Delta,\nu,s',s,x,y))| \\ &\le C_k A_{M(K)} X_{M(K)}^4 |\Delta| \big(\rho([s,s']) + |\Delta|\big)(1+|x|+|y|)^m \end{aligned} \tag{4.8}$$

*if $|\alpha| \le k, |\beta| \le k$.*

We indicate the idea to prove Proposition 4.7. The division points of $\Delta^*$ that lies in the first subinterval $[T_0, T_1]$ of $\Delta$ make a division $\delta$ of $[T_0, T_1]$

$$\delta : s = T_0 = T_{1,0} < T_{1,1} < \cdots < T_{1,p_1+1} = T_1. \tag{4.9}$$

Let $\Delta_1$ be the division of $[s, s']$ defined by all division points of $\Delta$ and division points of $\Delta^*$ that lies in $[T_0, T_1]$. In other words

$$\Delta_1 : s = T_0 = T_{1,0} < T_{1,1} < \cdots < T_{1,p_1+1} = T_1 < T_2 < T_3 < \cdots < T_J < T_{J+1} = s'. \tag{4.10}$$

$\Delta_1$ is a refinement of $\Delta$. Let $(x, y) \in \mathbf{R}^2$. For arbitrary $(y_1, \ldots, y_{p_1}) \in \mathbf{R}^{p_1}$ and $(x_1, \ldots, x_J)$ there exists one and only one piecewise classical path $\gamma_{\Delta_1} \in \Gamma_{x,y}(\Delta_1)$ such that

$$\begin{aligned} y_k &= \gamma_{\Delta_1}(T_{1,k}), \quad \text{for} \quad 0 \le k \le p_1 + 1, \\ x_j &= \gamma_{\Delta_1}(T_j), \quad \text{for} \quad 0 \le j \le J + 1, \end{aligned}$$

where we set $y_0 = x_0$ and $y_{p_1+1} = x_1$ as well as $x_{J+1} = x$, $x_0 = y$.

**Proposition 4.8.**

$$\begin{aligned} &k(\Delta_1; F_{\Delta_1}, \nu, s', s, x_{J+1}, x_0) - k(\Delta; F_\Delta, \nu, s', s, x_{J+1}, x_0) \\ &= \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_1^2) + \nu^{-1}\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_1^2 + \tau_1 \rho([T_0, T_1])). \end{aligned}$$

Admitting this proposition as true for the moment, we proceed in the following way. We add to dividing points of division $\Delta_1$ all the division points of $\Delta^*$ that lie in $[T_1, T_2]$. Then we obtain a new division $\Delta_2$ of $[s, s']$. $\Delta_2$ is the same as $\Delta^*$ in $[T_0, T_2]$ and it is the same as $\Delta$ in $[T_2, T_{J+1}]$. We have in this case, corresponding to Proposition 4.8,

$$\begin{aligned} &k(\Delta_2; F_{\Delta_2}, \nu, s', s, x, y) - k(\Delta_1; F_{\Delta_1}, \nu, s', s, x, y) \\ &\quad = \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_2^2) + \nu^{-1}\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_2^2 + \tau_2 \rho([T_1, T_2])). \end{aligned} \tag{4.11}$$

Similarly, we make $\Delta_3$ from $\Delta_2$. Continuing this process $J+1$ times, we finally obtain $\Delta_{J+1} = \Delta^*$. Therefore,

$$\begin{aligned} &k(\Delta^*; F_{\Delta^*}, \nu, s', s, x, y) - k(\Delta; F_\Delta, \nu, s', s, x, y) \\ &= \sum_{j=1}^{J+1} \left( k(\Delta_j; F_{\Delta_j}, \nu, s', s, x, y) - k(\Delta_{j-1}; F_{\Delta_{j-1}}, \nu, s', s, x, y) \right) \\ &= \sum_{j=1}^{J+1} \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_j^2) + \nu^{-1}\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_j^2 + \tau_j \rho([T_{j-1}, T_j])) \\ &= \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(|\Delta|(s' - s)) + \nu^{-1}\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(|\Delta|(s' - s) + |\Delta|\rho([s, s'])). \end{aligned} \tag{4.12}$$

This proves Proposition 4.7.

We suggest how to prove Proposition 4.8. We define

$$S_\delta(x_1, y_{p_n}, \ldots, y_1, x_0) = \sum_{k=1}^{p_1+1} S(T_{1,k}, T_{1,k-1}; y_k, y_{k-1}).$$

Then

$$S_{\Delta_1}(x_{J+1},\dots,x_1,y_{p_1},\dots,y_1,x_0)=\left(\sum_{j=2}^{J+1}S_j(x_j,x_{j-1})\right)+S_\delta(x_1,y_{p_n},\dots,y_1,x_0).$$

By definition

(4.13)

$$\begin{aligned}&I[F_{\Delta_1}](\Delta_1;\nu,s',s,x,y)\\&=\prod_{j=2}^{J+1}\left(\frac{\nu}{2\pi i\tau_j}\right)^{1/2}\int_{\mathbf{R}^J}e^{i\nu\sum_{j=2}^{J+1}S_j(x_j,x_{j-1})}\prod_{j=1}^{J}dx_j\\&\times\prod_{k=1}^{p_1+1}\left(\frac{\nu}{2i\pi\sigma_k}\right)^{1/2}\int_{\mathbf{R}^{p_1}}e^{i\nu S_\delta(x_1,y_{p_1},\dots,y_1,x_0)}F_{\Delta_1}(x_{J+1},\dots,x_1,y_{p_1},\dots,y_1,x_0)\prod_{k=1}^{p_1}dy_k.\end{aligned}$$

We perform integration by the variables $(y_{p_1},\dots,y_1)$ prior to integration by variables $(x_J,\dots,x_1)$. Set

(4.14)

$$\begin{aligned}&\left(\frac{\nu}{2\pi i\tau_1}\right)^{1/2}e^{i\nu S_1(x_1,x_0)}F_{\Delta/\Delta_1}(x_{J+1},x_J,\dots,x_1,x_0)\\&=\prod_{k=1}^{p_1+1}\left(\frac{\nu}{2i\pi\sigma_k}\right)^{1/2}\int_{\mathbf{R}^{p_1}}e^{i\nu S_\delta(x_1,y_{p_1},\dots,y_1,x_0)}F_{\Delta_1}(x_{J+1},\dots,x_1,y_{p_1},\dots,y_1,x_0)\prod_{k=1}^{p_1}dy_k.\end{aligned}$$

Then (4.13) means that

$$I[F_{\Delta_1}](\Delta_1;\nu,s',s,x,y)=I[F_{\Delta/\Delta_1}](\Delta;\nu,s',s,x,y). \tag{4.15}$$

We apply Proposition 3.9 to the integration by $(y_{p_1},\dots,y_1)$ in (4.14). Then

$$\begin{aligned}&F_{\Delta/\Delta_1}(x_{J+1},x_J,\dots,x_1,x_0)\\&=D(\delta;x_1,x_0)^{-1/2}\Big(F_\Delta(x_{J+1},x_J,\dots,x_1,x_0)+\nu^{-1}R_\delta[F_{\Delta_1}](\nu,x_{J+1},x_J,\dots,x_1,x_0)\Big),\end{aligned} \tag{4.16}$$

here

$$R_\delta[F_{\Delta_1}](\nu,x_{J+1},x_J,\dots,x_1,x_0)=\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^{J+1})}(\tau_1^2+\tau_1\rho([T_0,T_1])). \tag{4.17}$$

On the other hand, it follows from Proposition 2.3 that $D(\delta;x_1,x_0)^{-1/2}=1+\mathcal{O}_{\mathcal{B}_0(\mathbf{R}^2)}(\tau_1^2)$. Combining these, we have

$$\begin{aligned}&F_{\Delta/\Delta_1}(x_{J+1},x_J,\dots,x_1,x_0)\\&=F_\Delta(x_{J+1},x_J,\dots,x_1,x_0)+\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^{J+2})}(\tau_1^2+\nu^{-1}(\tau_1^2+\tau_1\rho([T_0,T_1]))).\end{aligned}$$

We can show that we can apply Corollary 3.3 to the right hand side of (4.15) and that

$$\begin{aligned}&k(\Delta_1;F_{\Delta_1},\nu,s',s,x,y)\\&=k(\Delta;F_\Delta,\nu,s',s,x,y)+\mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_1^2+\nu^{-1}(\tau_1^2+\tau_1\rho([T_0,T_1]))).\end{aligned}$$

This shows Proposition 4.8.

In the case $F(\gamma)\equiv 1$ we discuss the integral transformation with the kernel $K[1](\nu,t,s,x,y)$.

**Definition 4.9.** We define for any $\varphi \in C_0^\infty(\mathbf{R})$

$$I(\Delta;\nu,t,s)\varphi(x) = \int_{\mathbf{R}} I[1](\Delta;\nu,t,s,x,y)\varphi(y)\,dy, \tag{4.18}$$

$$K(\nu,t,s)\varphi(x) = \int_{\mathbf{R}} K[1](\nu,t,s,x,y)\varphi(y)\,dy. \tag{4.19}$$

We write $\|A\|$ for the operator norm of a linear operator $A$ on $L^2(\mathbf{R})$. It turns out from $L^2$-boundedness theorem in [1] that the following facts hold:

**Proposition 4.10.** *Suppose that $|t-s| \le \delta_0$. Then there exists a positive constant $C$ independent of $\nu$, $t$ and $s$ such that*

$$\|I(\Delta;\nu,t,s)\| \le C, \quad \|K(\nu,t,s)\| \le C. \tag{4.20}$$

**Theorem 4.11.** *Suppose that $|t-s| \le \delta_0$. Then there exists a positive constant $C$ independent of $\nu$, $t$ ,$s$ such that*

$$\|I(\Delta;\nu,t,s) - K(\nu,t,s)\| \le C(s'-s)|\Delta|. \tag{4.21}$$

Next we shall discuss the relation between Feynman path integral and propagator of Schrödinger equation.

Let $H(t)$ be the Hamiltonian operator:

$$H(t) = \frac{1}{2}\left(-i\nu^{-1}\partial_x\right)^2 + V(t,x). \tag{4.22}$$

**Theorem 4.12.** *Suppose that $|t-s| \le \delta_0$. For any $f \in C_0^\infty(\mathbf{R})$ the $L^2(\mathbf{R})$-valued function $t \to K(\nu,t,s)f$ is strongly differentiable. It satisfies*

$$i\nu^{-1}\frac{d}{dt}K(\nu,t,s)f = H(t)K(\nu,t,s)f, \tag{4.23}$$

$$s - \lim_{|t-s|\to 0} K(\nu,t,s)f = f. \tag{4.24}$$

**Corollary 4.13.** *$K(\nu,t,s)f(x)$ is the classical solution of Scrödinger equation*

$$i\nu^{-1}\frac{\partial}{\partial t}K(\nu,t,s)f = \left[\frac{1}{2}\left(-i\nu^{-1}\frac{\partial}{\partial x}\right)^2 + V(t,x)\right]K(\nu,t,s)f(x), \tag{4.25}$$

*if $f \in C_0^\infty$.*

**Remark 4.14.** In the case $F(\gamma) \equiv 1$, $K[1](\nu,s',s,x,y) = \int_{\Omega_{x,y}} e^{i\nu S(\gamma)}\mathcal{D}[\gamma]$ is in fact the fundamental solution of Schrödinger equation (1.9). And it has semiclassical asymptotic formula given by (4.3) and (4.4) with $F(\gamma^*) = 1$. The principal term enjoys the property shown by Proposition 2.4. cf. [2]

These main statement of Feynman's paper [4] were verified rigorously in [5], [6], [11], [8].

## § 5. An integration by parts formula

### § 5.1. Some operators of trace class

We set $s = 0$ and $s' = T$ for simplicity. Let $\mathcal{X} = L^2([0,T])$ and $\mathcal{H} = H^1([0,T])$ be the real $L^2$-Sobolev space of order 1. For any $x, y \in \mathbf{R}$, we write $\mathcal{H}_{x,y} = \{\gamma \in \mathcal{H} : \gamma(0) = y, \gamma(T) =$

$x\}$. $\mathcal{H}_{x,y}$ is an infinite dimensional differentiable manifold. Its tangent space at $\gamma \in \mathcal{H}_{x,y}$ is identified with the space $\mathcal{H}_0 = H_0^1([0,T]) = \{\gamma \in \mathcal{H}; \gamma(0) = \gamma(T) = 0\}$

Let $\tilde{\rho} : \mathcal{H} \to \mathcal{X}$ be the natural embedding and $\rho : \mathcal{H}_0 \to \mathcal{X}$ be its restriction to $\mathcal{H}_0$ and $\rho^* : \mathcal{X} \to \mathcal{H}_0$ be its adjoint.

We write $( \ , \ )_{\mathcal{X}}$ for the inner product of $\mathcal{X}$. We write $\mathcal{L}(\mathcal{X})$ for the Banach space of all bounded linear operators in $\mathcal{X}$ equipped with operator norm $\| \ \|_{\mathcal{L}(\mathcal{X})}$. We adopt the following inner product of $\mathcal{H}_0$:

$$(h_1, h_2)_{\mathcal{H}_0} = \int_0^T \frac{d}{dt}h_1(t)\frac{d}{dt}h_2(t)\,dt \qquad (h_1, h_2 \in \mathcal{H}_0).$$

We write $\|h\|_{\mathcal{H}_0}$ for the norm of $h \in \mathcal{H}_0$ in $\mathcal{H}_0$. The cotangent vector $DF(\gamma)$ is identified with an element, which we also write $DF(\gamma) \in \mathcal{H}_0$, via the inner product of $\mathcal{H}_0$ by the equation $DF(\gamma)[h] = (DF(\gamma), h)_{\mathcal{H}_0}$.

Let $\omega = \pi T^{-1}$ and let $e_n(t) = \sqrt{\frac{2}{T}} \sin n\omega t$. Then $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal system of $\mathcal{X}$. We can choose a complete orthogonal system $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{H}_0$ such that $\rho\varphi_n = (n\omega)^{-1}e_n$, i.e., $\rho\varphi_n(t) = (n\omega)^{-1}\sqrt{\frac{2}{T}} \sin n\omega t$. It is clear that $\rho^* e_n = (n\omega)^{-1}\varphi_n$. Therefore, $\rho$ and $\rho^*$ are Hilbert Schmidt operators and

$$\rho\rho^* e_n = (n\omega)^{-2}e_n, \qquad \rho^*\rho\varphi_n = (n\omega)^{-2}\varphi_n \qquad (n = 1, 2, 3, \dots). \tag{5.1}$$

It turns out that

$$-\frac{d^2}{dt^2}\rho\rho^* e_n(t) = e_n(t), \qquad e_n(0) = e_n(T) = 0 \qquad (n = 1, 2, \dots). \tag{5.2}$$

**Proposition 5.1.** *cf. Kato [15]. Suppose that $B : \mathcal{X} \to \mathcal{X}$ is a bounded linear operator with operator norm $\|B\|_{\mathcal{L}(\mathcal{X})}$. Both of linear operators $\rho^* B\rho : \mathcal{H}_0 \to \mathcal{H}_0$ and $\rho\rho^* B : \mathcal{X} \to \mathcal{X}$ are of trace class. Their traces are equal:*

$$tr\rho^* B\rho = tr\rho\rho^* B.$$

Since $\rho\rho^* B$ is in trace class, it has the kernel function $\exists k(s,t) \in L^2([0,T] \times [0,T])$,i.e.,

$$\rho\rho^* Bf(s) = \int_0^T k(s,t)f(t)dt \qquad (\forall f \in \mathcal{X}). \tag{5.3}$$

In particular, the kernel function of $\rho\rho^*$ is the Green operator for the Dirichlet boundary value problem.

**Proposition 5.2.** *$k(s,t)$ has the properties:*

1. *If each $s \in [0,T]$ is fixed, then the function $k_s : [0,T] \ni t \to k(s,t)$ is a well-defined function in $\mathcal{X}$ of $t$.*
2. *$[0,T] \ni s \to k_s \in \mathcal{X}$ is a strongly continuous mapping from $[0,T]$ to $\mathcal{X}$.*
3. *The function $[0,T] \ni s \to k(s,t)$ regarded as a function of $s$ is in the image of the map $\rho$ if $t$ is fixed for almost all $t \in [0,T]$.*

**Proposition 5.3.** *The value $k(t,t)$ is well-defined for almost all $t \in [0,T]$ and*

$$\int_0^T |k(t,t)|^2\, dt < \infty.$$

$$tr\rho\rho^* B = \int_0^T k(t,t)\, dt.$$

## § 5.2. Admissible vector field

Let $p$ be a $C^1$ map $p : \mathcal{H}_{xy} \ni \gamma \to p(\gamma) \in \mathcal{H}_0$. Then $p(\gamma)$ is a tangent vector field on $\mathcal{H}_{xy}$. We write as usual $p(\gamma, s) = \rho p(\gamma)(s)$. We have $\partial_s p(\gamma, s) \in \mathcal{X}$.

**Definition 5.4** (Admissible vector field). We say that $p(\gamma)$ is an admissible vector field if $p(\gamma)$ has the following properties:

1. There exits a $C^1$ map $q : \mathcal{H} \to \mathcal{X}$ such that

$$p(\gamma) = \rho^* q(\gamma), \quad (\gamma \in \mathcal{H}_{x,y}). \tag{5.4}$$

2. If $\gamma \in \mathcal{H}_{x,y}$, then there exists a bounded linear map $B(\gamma) \in \mathcal{L}(\mathcal{X})$ such that the Fréchet differential $Dq(\gamma) : \mathcal{H}_0 \ni h \to Dq(\gamma)[h] \in \mathcal{X}$ is given by

$$Dq(\gamma)[h] = B(\gamma)\rho h \qquad (h \in \mathcal{H}_0). \tag{5.5}$$

**Remark 5.5.** Suppose $p(\gamma)$ is an admissible vector field. Then we often write $\dfrac{\delta q(\gamma)}{\delta \gamma}$ for $B(\gamma)$. It follows from (5.4) and (5.5) that

$$Dp(\gamma)[h] = \rho^* B(\gamma)\rho h \qquad (\gamma \in \mathcal{H}_{x,y}, h \in \mathcal{H}_0).$$

That is, for all $\gamma \in \mathcal{H}_{x,y}$ and $h_1, h_2 \in \mathcal{H}_0$,

$$(Dp(\gamma)[h_1], h_2)_{\mathcal{H}_0} = \Big(B(\gamma)\rho h_1, \rho h_2\Big)_{\mathcal{X}}.$$

**Definition 5.6** (Divergence of a vector field). Suppose that $p(\gamma)$ is an admissible vector field. We define its divergence $\operatorname{Div} p(\gamma)$ at $\gamma \in \mathcal{H}_{x,y}$ by the following equality:

$$\operatorname{Div} p(\gamma) = \operatorname{tr}\rho^* B(\gamma)\rho = \operatorname{tr}\rho^* \frac{\delta q(\gamma)}{\delta \gamma}\rho.$$

**Remark 5.7** (Another expression of divergence). Let $p(\gamma)$ be an admissible vector field. Since $p(\gamma, s) = (\rho p(\gamma))(s)$ for $s \in [0,T]$, $p(\gamma, s) = (\rho\rho^* q(\gamma))(s)$. Therefore, it follows from (5.5) that

$$Dp(\gamma, s)[h] = (\rho\rho^* Dq(\gamma)[h])(s) = (\rho\rho^* B(\gamma)\rho h)(s).$$

Let $k_\gamma(s,t)$ be the integral kernel function of the trace class operator $\rho\rho^* B(\gamma)$. Then

$$Dp(\gamma, s)[h] = \int_0^T k_\gamma(s,t)(\rho h)(t)\, dt. \tag{5.6}$$

We often write $\dfrac{\delta p(\gamma, s)}{\delta \gamma(t)}$ for $k_\gamma(s,t)$, i.e.,

$$Dp(\gamma, s)[h] = \int_0^T \frac{\delta p(\gamma, s)}{\delta \gamma(t)} \rho h(t)\, dt. \tag{5.7}$$

The next Proposition follows from Proposition 5.3.

**Proposition 5.8.** *Assume $p(\gamma)$ is an admissible vector field. Then*

$$\operatorname{Div} p(\gamma) = \int_0^T \frac{\delta p(\gamma,t)}{\delta\gamma(t)}dt.$$

The notion of admissible vector field defined above is an analogy to infinitesimal version of "admissible transformation" in the case of Wiener integral. cf.[13].

## § 5.3. m-smooth functional

We use the following notation : Let $\mathcal{Y}$ be a Banach space with norm $\| \ \|_{\mathcal{Y}}$. Let $\Delta$ be a division of $[0,T]$, $\gamma_\Delta$ and $\{x_{J+1}, x_J, \ldots, x_1, x_0\}$ be as before. Assume that $F(\gamma_\Delta)$ is a map $F : \Gamma(\Delta) \ni \gamma_\Delta \to F(\gamma_\Delta) \in \mathcal{Y}$ and is infinitely differentiable with respect to $(x_{J+1}, \ldots, x_0)$. Let $K$ be a nonnegative integer, $m$ be a nonnegative constant and $X \geq 1$ be a constant. Then we define a norm of $F(\gamma_\Delta)$ defined on $\Gamma(\Delta)$:

(5.8) $\|F(\gamma_\Delta)\|_{\{\mathcal{Y};\Delta,m,K,X,\}}$

$$= \max_{\substack{0\leq\alpha_j\leq K,\\ j=0,1,\ldots,J+1}} \sup_{(x_{J+1},\ldots,x_0)\in\mathbf{R}^{J+1}} (1+|x_{J+1}|+\cdots+|x_0|)^{-m} \left\| \prod_{j=0}^{J+1} X^{-|\alpha_j|}\partial_{x_j}^{\alpha_j} F(\gamma_\Delta)\right\|_{\mathcal{Y}} .$$

Moreover if $F(\gamma)$ is defined on $\mathcal{H}$, then we define

$$\|F\|_{\{\mathcal{Y};m,K,X\}} = \sup_\Delta \|F\|_{\{\mathcal{Y};\Delta,m,K,X\}}, \tag{5.9}$$

where sup is taken over all divisions $\Delta$ of $[0,T]$. If $\mathcal{Y} = \mathbf{R}$ or $\mathbf{C}$, we simply write $\|F\|_{\{\Delta,m,K,X\}}$ and $\|F\|_{\{m,K,X\}}$.

Suppose that a functional $F(\gamma) : \mathcal{H}_{x,y} \to \mathbf{C}$ is Fréchet differentiable at $\gamma$. Then $DF(\gamma)$ denotes its differential. For $h \in \mathcal{H}_0$,

$$DF(\gamma)[h] = (DF(\gamma), h)_{\mathcal{H}_0} \qquad (h \in \mathcal{H}_0).$$

Moreover, if there exists a density $f_\gamma(s) \in \mathcal{X}$ such that $DF(\gamma) = \rho^* f_\gamma$,i.e.,

$$DF(\gamma)[h] = \int_0^T f_\gamma(s)\rho h(s)\, ds \qquad (h \in \mathcal{H}_0), \tag{5.10}$$

then we often write $\dfrac{\delta F(\gamma)}{\delta\gamma(s)}$ or $\delta F(\gamma)(s)$ for $f_\gamma(s)$.

**Definition 5.9.** Let $m \geq 0$ be a constant. We call $F(\gamma)$ an $m$-smooth functional if $F(\gamma)$ satisfies the following conditions.

**F-I** $F(\gamma)$ is an infinitely differentiable map from $\mathcal{H}$ to $\mathbf{C}$.

**F-2** $\forall x, \forall y \in \mathbf{R}$ and $\gamma \in \mathcal{H}_{xy}$ the differential $DF(\gamma)$ has its density $\dfrac{\delta F(\gamma)}{\delta\gamma(s)}$, that is, $\forall\gamma \in \mathcal{H}_{x,y} \forall h \in \mathcal{H}_0$

$$DF(\gamma)[h] = \int_0^T \frac{\delta F(\gamma)}{\delta\gamma(s)}\rho h(s) ds,$$

**F-3** Functional $\dfrac{\delta F(\gamma)}{\delta\gamma(s)}$ is a continuous functional of $\mathcal{H}\times[0,T]\ni(\gamma,s)\longrightarrow\mathbf{C}$. It is infinitely differentiable with respect to $\gamma\in\mathcal{H}_{x,y}$ if $s$ is fixed.

**F-4** For any integer $K\geq 0$ there are constants $A_K>0$ and $X_K\geq 1$ such that $\forall K=0,1,2,\ldots,$,

$$A_K=\sup_{\gamma\in\mathcal{H}}\left(\|F(\gamma)\|_{\{m,K,X_K\}}+\sup_{s\in[0,T]}\left\|\frac{\delta F(\gamma)}{\delta\gamma(s)}\right\|_{\{m,K,X_K\}}\right)<\infty. \tag{5.11}$$

**Remark 5.10.** Let $\delta_2$ be so small that $v_2\delta_2^2<4$ and $v_2\delta_2<1$. If $T\leq\delta_2$, then a $m$-smooth functional satisfies condition of N. Kumano-go 4.1 and it is "F-integrable".

## § 5.4. An integration by parts formula

**Definition 5.11.** Let $m$ be a nonnegative number. We say that the vector field $p(\gamma)$ is an $m$-admissible vector field if it has all the following properties:

**P1** $p$ is an infinitely differentiable map $p:\mathcal{H}\ni\gamma\to p(\gamma)\in\mathcal{H}_0$ of which the restriction to $\mathcal{H}_{xy}$ is an admissible vector field for any fixed $x,y\in\mathbf{R}$, that is, there are $C^\infty$ maps $q:\mathcal{H}\to\mathcal{X}$ and $B:\mathcal{H}\to\mathcal{L}(\mathcal{X})$ such that $p(\gamma)=\rho^*q(\gamma)$ and that for $\gamma\in\mathcal{H}_{x,y}$ and all $h\in\mathcal{H}_0$, $Dq(\gamma)[h]=B(\gamma)\rho h$.

**P2** The map $\mathcal{H}\ni\gamma\to B(\gamma)\in\mathcal{L}(\mathcal{X})$ is infinitely differentiable. For any integer $K\geq 0$ there exists a constant $Y_K\geq 1$ such that

$$B_K=\sup_{\gamma\in\mathcal{H}}\left(\|q(\gamma)\|_{\{\mathcal{X},m,K,Y_K\}}+\|B(\gamma)\|_{\{\mathcal{L}(\mathcal{X});m,K,Y_K\}}\right)<\infty. \tag{5.12}$$

We often write $\dfrac{\delta q(\gamma)}{\delta\gamma}$ for $B(\gamma)$.

Let $\delta_0$ be as in (1.2). Our main theorem is the following cf.[9]:

**Theorem 5.12** (Integration by parts). *Let $T\leq\delta_0$. Suppose that $F(\gamma)$ is an $m$-smooth functional and that $p(\gamma)$ is an $m$'-admissible vector field. We further assume that two of $DF(\gamma)[p(\gamma)]$, $F(\gamma)\mathrm{Div}p(\gamma)$ and $F(\gamma)DS(\gamma)[p(\gamma)]$ are F-integrable. Then the rest is also F-integrable and the following equality holds.*

$$\begin{aligned}\int_{\Omega_{xy}}&DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma)\\&=-\int_{\Omega_{xy}}F(\gamma)\mathrm{Div}\,p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma)-i\nu\int_{\Omega_{xy}}F(\gamma)DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma).\end{aligned} \tag{5.13}$$

**Remark 5.13.** cf. N.Kumano-go [12]. If $p(\gamma,s)$ is independent of $\gamma$, i.e., $p(\gamma,s)=h(s)$ then $\mathrm{Divp}(\gamma)=0$ and the above formula reduces to

$$\int_{\Omega_{x,y}}DF(\gamma)[h]e^{i\nu S(\gamma)}\mathcal{D}(\gamma)=-i\nu\int_{\Omega_{x,y}}F(\gamma)DS(\gamma)[h]e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \tag{5.14}$$

**We explain the idea of proof.** We use the abbreviation:

$$N(\Delta) = \prod_{j=1}^{J+1} \left( \frac{\nu}{2\pi i \tau_j} \right)^{1/2},$$

and set $y_{\Delta,j} = p(\gamma_\Delta, T_j) = \rho p(\gamma_\Delta)(T_j)$ for $j = 0, 1, \ldots, J+1$, in particular $y_0 = 0 = y_{J+1}$. It is clear from definition of oscillatory integrals on $\mathbf{R}^J$ that

$$\int_{\mathbf{R}^J} \sum_{j=1}^{J} \frac{\partial}{\partial x_j} \left( F(\gamma_\Delta) y_{\Delta,j} e^{i\nu S(\gamma_\Delta)} \right) \prod_{j=1}^{J} dx_j = 0.$$

It follows from this that

$$\begin{aligned} (5.15) \qquad & N(\Delta) \int_{\mathbf{R}^J} \sum_{j=1}^{J} \partial_{x_j} (F(\gamma_\Delta)) y_{\Delta,j} e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ &= -N(\Delta) \int_{\mathbf{R}^J} F(\gamma_\Delta) \sum_{j=1}^{J} \partial_{x_j} (y_{\Delta,j}) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ &\quad - i\nu N(\Delta) \int_{\mathbf{R}^J} F(\gamma_\Delta) \sum_{j=1}^{J} y_{\Delta,j} \partial_{x_j} S(\gamma_\Delta) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j. \end{aligned}$$

Theorem 5.12 follows from the next Proposition.

**Proposition 5.14.**

$$\begin{aligned} (5.16) \qquad & \lim_{\Delta\to 0} N(\Delta) \int_{\mathbf{R}^J} F(\gamma_\Delta) \sum_{j=1}^{J} y_{\Delta,j} \partial_{x_j} S(\gamma_\Delta) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ &= \int_\Omega F(\gamma) DS(\gamma)[p(\gamma)] e^{i\nu S(\gamma)} \mathcal{D}(\gamma), \end{aligned}$$

$$\begin{aligned} (5.17) \qquad & \lim_{\Delta\to 0} N(\Delta) \int_{\mathbf{R}^J} \sum_{j=1}^{J} \partial_{x_j} (F(\gamma_\Delta)) y_{\Delta,j} e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ &= \int_\Omega DF(\gamma)[p(\gamma)] e^{i\nu S(\gamma)} \mathcal{D}(\gamma), \end{aligned}$$

$$\begin{aligned} (5.18) \qquad & \lim_{\Delta\to 0} N(\Delta) \int_{\mathbf{R}^J} F(\gamma_\Delta) \sum_{j=1}^{J} \partial_{x_j} (y_{\Delta,j}) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^{J} dx_j \\ &= \int_\Omega F(\gamma) \mathrm{Div}\, p(\gamma) e^{i\nu S(\gamma)} \mathcal{D}(\gamma). \end{aligned}$$

Proof of (5.16). Since $\gamma_\Delta(t)$ is a piecewise classical path with edges at $t = T_j$ for $j = 1, 2, \ldots, J$, integration by parts gives

$$\begin{aligned} (5.19) \qquad DS(\gamma_\Delta)[p(\gamma_\Delta)] &= \int_0^T \left( \frac{d}{dt}\gamma_\Delta(t) \frac{d}{dt} p(\gamma_\Delta, t) - \partial_x V(t, \gamma_\Delta(t)) p(\gamma_\Delta, t) \right) dt \\ &= \sum_{j=1}^{J+1} \frac{d}{dt}\gamma_\Delta(T_j - 0) p(\gamma_\Delta, T_j) - \frac{d}{dt}\gamma_\Delta(T_{j-1} + 0) p(\gamma_\Delta, T_{j-1}) = \sum_{j=1}^{J} \partial_{x_j} S(\gamma_\Delta) y_{\Delta,j}. \end{aligned}$$

(5.16) is proved.

**Idea of proof of (5.17).** Since $F(\gamma)$ is $m$-smooth, $\delta F(\gamma) = \delta F(\gamma)/\delta\gamma \in \mathcal{X}$. We express the right hand side of (5.17) as a limit of time slicing approximation. Then we have only to prove that

$$\lim_{\Delta\to 0} N(\Delta)\int_{\mathbf{R}^J}\Big(\sum_{j=1}^{J}\partial x_j(F(\gamma_\Delta))y_{\Delta,j} - DF(\gamma_\Delta)[p(\gamma_\Delta)]\Big)e^{i\nu S(\gamma_\Delta)}\prod_{j=1}^{J}dx_j = 0. \tag{5.20}$$

Let $\zeta_{\Delta,j}(t) = \partial_{x_j}\gamma_\Delta(t)$ for $t\in[0,T]$. Then $\partial_{x_j}F(\gamma_\Delta) = (\delta F(\gamma_\Delta),\zeta_{\Delta,j})_{\mathcal{X}}$. It is clear that $\zeta_{\Delta,j}(t)=0$ if $t\notin[T_{j-1},T_{j+1}]$ and that

$$\frac{d^2}{dt^2}\zeta_{\Delta,j}(t) + \partial_x^2 V(t,\gamma_\Delta(t))\zeta_{\Delta,j}(t) = 0 \qquad (t\in(T_{j-1},T_j)\cup(T_j,T_{j+1})), \tag{5.21}$$

and that $\zeta_{\Delta,j}(T_{j-1}) = 0 = \zeta_{\Delta,j}(T_{j+1})$ and $\zeta_{\Delta,j}(T_j) = 1$. It is a piecewise $C^1$ continuous function.

We compare $\zeta_{\Delta,j}(t)$ with the piecewise linear function $e_{\Delta,j}(t)$ such that for $1\le j\le J$

$$e_{\Delta,j}(t) = \begin{cases} 0 & \text{if } t\notin[T_{j-1},T_{j+1}],\\ (t-T_{j-1})\tau_j^{-1} & \text{if } t\in[T_{j-1},T_j],\\ (T_{j+1}-t)\tau_{j+1}^{-1} & \text{if } t\in[T_j,T_{j+1}].\end{cases} \tag{5.22}$$

$e_{\Delta,0}(t)$ and $e_{\Delta,J+1}(t)$ are defined in such a way that

$$\sum_{j=0}^{J+1} e_{\Delta,j}(t) = 1 \qquad (t\in[0,T]). \tag{5.23}$$

Then it turns out that for any $\alpha$, $\beta$

$$|\partial_{x_{j-1}}^{\alpha}\partial_{x_j}^{\beta}(\zeta_{\Delta,j}(t)-e_{\Delta,j}(t))| = \mathcal{O}(\tau_j^2) \quad (t\in[T_{j-1},T_j]) \tag{5.24}$$

$$|\partial_{x_j}^{\alpha}\partial_{x_{j+1}}^{\beta}(\zeta_{\Delta,j}(t)-e_{\Delta,j}(t))| = \mathcal{O}(\tau_{j+1}^2) \quad (t\in[T_j,T_{j+1}]). \tag{5.25}$$

Therefore,

$$\begin{aligned} DF(\gamma_\Delta)[p(\gamma_\Delta)] - \sum_j \partial_{x_j}F(\gamma_\Delta)y_{\Delta,j} &= DF(\gamma_\Delta)[p(\gamma_\Delta)] - \sum_j y_{\Delta,j}(\delta F(\gamma_\Delta),\zeta_{\Delta,j})_{\mathcal{X}}\\ = \sum_j\big(\delta F(\gamma_\Delta),(\rho p(\gamma_\Delta)-y_{\Delta,j})e_{\Delta,j}\big)_{\mathcal{X}} &- \sum_j y_{\Delta,j}\big(\delta F(\gamma_\Delta),(e_{\Delta,j}-\zeta_{\Delta,j})\big)_{\mathcal{X}},\end{aligned}$$

Using (5.24), we can show

$$\sum_j y_{\Delta,j}\big(\delta F(\gamma_\Delta),(e_{\Delta,j}-\zeta_{\Delta,j})\big)_{\mathcal{X}} = \mathcal{O}(|\Delta|T). \tag{5.26}$$

Since $p(\gamma)$ is $m'$-admissible, $\rho p(\gamma_\Delta)(t) = \rho\rho^* q(\gamma_\Delta)$ is in $C^1([0,T])$. As $\rho p(\gamma_\Delta)(T_j) = y_{\Delta,j}$ and $e_{\Delta,j}$ vanishes outside $[T_{j-1},T_{j+1}]$, we can show

$$(\rho p(\gamma_\Delta)(t) - y_{\Delta,j})e_{\Delta,j}(t) = \mathcal{O}(\tau_j+\tau_{j+1}) \qquad (t\in[0,T]).$$

Hence

$$\sum_j\big(\delta F(\gamma_\Delta),(\rho p(\gamma_\Delta)(t)-y_{\Delta,j})e_{\Delta,j})\big)_{\mathcal{X}} = \mathcal{O}(|\Delta|T). \tag{5.27}$$

It follows from (5.26),(5.27) and Theorem 3.2 that

$$N(\Delta)\int_{\mathbf{R}^J}\Big(\sum_{j=1}^{J}\partial x_j(F(\gamma_\Delta))y_{\Delta,j}-DF(\gamma_\Delta)[p(\gamma_\Delta)]\Big)e^{i\nu S(\gamma_\Delta)}\prod_{j=1}^{J}dx_j=\mathcal{O}(T|\Delta|). \tag{5.28}$$

This shows (5.20).

Similarly, we can show (5.18).

## § 6. Application to semiclassical asymptotic behaviour of Feynman path integrals

We always assume $T<\delta$. Let $F(\gamma)$ be an $m$-smooth functional. Then semiclassical asymptotic formula (4.3) was proved by Kumano-go [12]. The principal part of (4.3) is $F(\gamma^*)$, the value of $F$ at the classical path $\gamma^*$.

What happens if $F(\gamma^*)=0$ ? Integration by parts formula enables us to get a sharper information even in this case.

**Assumption 6.1.** 1. $F(\gamma)$ is a real valued $m$-smooth functional. For fixed $\gamma\in\mathcal{H}_{x,y}$, $\dfrac{\delta F(\gamma)}{\delta\gamma(s)}$ is a $\mathcal{X}$-valued function, which we write $\dfrac{\delta F(\gamma)}{\delta\gamma}$. The map $\mathcal{H}_{x,y}\ni\gamma\to\dfrac{\delta F(\gamma)}{\delta\gamma}\in\mathcal{X}$ is a $C^\infty$ map. There exists a $C^\infty$ map $\mathcal{H}_{x,y}\ni\gamma\to A(\gamma)\in B(\mathcal{X})$ such that for any $h\in\mathcal{H}_0$,

$$D\frac{\delta F(\gamma)}{\delta\gamma}[h]=A(\gamma)\rho h. \tag{6.1}$$

2. Linear operator $A(\gamma)$ has the integral kernel $k_\gamma(s,t)$ which is continuous in $(s,t)\in[0,T]\times[0,T]$ and we have for any $K=0,1,2,\ldots$

$$\sup_{(s,t)}\|k_\gamma(s,t)\|_{\{m,K,X_K\}}<\infty. \tag{6.2}$$

Suppose $F(\gamma)$ satisfies the above conditions and $F(\gamma^*)=0$. Let $\gamma_\theta=\theta\gamma+(1-\theta)\gamma^*$ for $0\le\theta\le1$. We define an element $\zeta(\gamma)\in\mathcal{X}$ by

$$\zeta(\gamma,t)=\int_0^1\frac{\delta F(\gamma)}{\delta\gamma(t)}\Big|_{\gamma=\gamma_\theta}d\theta. \tag{6.3}$$

Let $\tilde{W}(\gamma)$ be the multiplication operator in $\mathcal{X}$ defined by

$$\mathcal{X}\ni g(s)\to\tilde{W}(\gamma,s)g(s)\qquad(g\in\mathcal{X}), \tag{6.4}$$

where

$$\tilde{W}(\gamma,s)=\int_0^1\partial_x^2V(s,\gamma_\theta(s))\,d\theta. \tag{6.5}$$

Since $T<\delta$, $(I-\tilde{W}(\gamma)\rho\rho^*)^{-1}\in\mathcal{L}(\mathcal{X})$. We define a vector field

$$p(\gamma)=\rho^*(I-\tilde{W}(\gamma)\rho\rho^*)^{-1}\zeta(\gamma). \tag{6.6}$$

**Proposition 6.2.** *If $F(\gamma)$ satisfies our assumptions and $F(\gamma^*) = 0$, then $p(\gamma)$ is an $m$-admissible vector field and*

$$DS(\gamma)[p(\gamma)] = F(\gamma). \tag{6.7}$$

*Thus $DS(\gamma)[p(\gamma)]$ is $F$-integrable. The following equality holds:*

$$\int_{\Omega_{xy}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}[\gamma] = \int_{\Omega_{xy}} DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}[\gamma]. \tag{6.8}$$

We can apply the integration by parts theorem 5.12 and obtain

**Theorem 6.3.** *Suppose $F(\gamma)$ is an $m$-smooth functional with some $m \geq 0$ and it satisfies the additional assumption 6.1. Assume further that $F(\gamma^*) = 0$. Define $\zeta(\gamma, t)$ and $p(\gamma)$ as above. Then we have*

$$\int_{\Omega_{xy}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}[\gamma] = -(i\nu)^{-1}\int_{\Omega_{xy}} \mathrm{Div}p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}[\gamma]. \tag{6.9}$$

Apply Kumano-go's theorem of semiclassical asymptotics to (6.9), we have the following theorem.

**Theorem 6.4.** *[12]. Under the same assumption as in Theorem 6.3 the following asymptotic formula holds:*

$$\begin{aligned}&\int_{\Omega_{xy}} F(\gamma)e^{i\nu S(\gamma)}\mathcal{D}[\gamma]\\ &= \left(\frac{-i\nu}{2\pi T}\right)^{1/2} D(T,0,x,y)^{-1/2}e^{i\nu S(\gamma^*)}\big(-(i\nu)^{-1}\mathrm{Div}p(\gamma^*) + \nu^{-2}r(\nu,T,0,x,y)\big).\end{aligned}$$

*For $\forall\alpha,\beta$ there exists a constant $C_{\alpha\beta} > 0$ such that*

$$\left|\partial_x^\alpha\partial_y^\beta r(\nu,T,0,x,y)\right| \leq C_{\alpha\beta}(1 + |x| + |y|)^m. \tag{6.10}$$

Let $G_{\gamma^*}(t,s)$ be the Green function of differential equation of Jacobi field:

$$-\left(\frac{d^2}{dt^2} + \partial_x^2V(t,\gamma^*(t))\right)u(t) = f(t), \qquad u(0) = 0 = u(T). \tag{6.11}$$

Calculation shows:

**Theorem 6.5.** *Under the same assumption as in Theorem 6.4*

$$\begin{aligned}\mathrm{Div}p(\gamma^*) &= \frac{1}{2}\int_0^T\int_0^T \frac{\delta}{\delta\gamma(t)}(G_{\gamma^*}(t,s)\frac{\delta F(\gamma^*)}{\delta\gamma(s)})\,dsdt\\ &= \frac{1}{2}\int_0^T\int_0^T \frac{\delta G_{\gamma^*}(t,s)}{\delta\gamma(t)}\frac{\delta F(\gamma^*)}{\delta\gamma(s)}\,dsdt + \frac{1}{2}\int_0^T\int_0^T G_{\gamma^*}(t,s)\frac{\delta^2F(\gamma^*)}{\delta\gamma(s)\delta\gamma(t)}\,dsdt.\end{aligned}$$

**Remark 6.6 (The 2nd moment of Feynman path integral).** Let

$$F(\gamma) = \int_0^T \int_0^T (\gamma(s) - \gamma^*(s))(\gamma(t) - \gamma^*(t))a(s,t)\, dsdt.$$

Then

$$\begin{aligned}
&\int_{\Omega_{x,y}} e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma] \\
&= \Big(\frac{\nu}{2\pi i T}\Big)^{1/2} D(T,0,x,y)^{-1/2} e^{i\nu S(\gamma^*)} \\
&\quad \times \Big( - (i2\nu)^{-1} \int_0^T \int_0^T G_{\gamma^*}(s,t)a(s,t)\, dsdt + \nu^{-2} r(\nu,T,0,x,y)\Big).
\end{aligned}$$

Here $r(\nu, T, 0, x, y)$ satisfies (6.10) with $m = 2$.

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