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White noise approach to path integrals: From Lagrangian to Hamiltonian

By

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Abstract

We discuss the white noise approach to Feynman path integrals. First we recall the Lagrangian path integral and see that the method can be applied to the Hamiltonian path integrals by using the same idea.

PART I

§ 1. Introduction

Our original idea is to give a reasonable interpretation to the formulation of a propagator in quantum mechanics by using the white noise analysis.

By the well-known theory, the classical trajectories fluctuate, so that there are many possible trajectories around the classical one which is uniquely determined by the variational calculus applied to the action functional.

Now one may ask what does a possible trajectories mean. We have proposed

Here is a history.

(1) We proposed the idea of taking a Brownian bridge to express the fluctuation. 1981 Berlin Conference, L. Streit, and T.H.

Then, some information on this from L. Streit;

Scientists: Inomata, DeWitt-Morette, M. Grothaus, J.Klauder have contributed much.

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W. Bock, Hamiltonian path integrals in white noise analysis. Univ. Kaiserslautern. 2013.

Papers: T. Kuna, L. Streit and W. Westerkamp, Feynman integrals for a class of exponentially growing potentials. J. Math. Physics 39 (1998), 4476-4491.

M. de Faria, M. J. Oliveira and L. Streit, Feynman integrals for non-smooth and rapidly growing potentials.

L. Streit, Feynman integrals as generalized functions on path space: Things done and open problems.Dec. 2007.

Conference: Bielefeld Conf. 2013 in honour of Prof. Ludwig Streit.

Literatures of historical interest:

Daubechies and J.R. Klauder, Quantum mechanical path integrals with Wiener measure for all polynomial Hamiltonians. II. J. Math. Phys. 26 (1985), 2239-2256.

(2) Information from Statistical Mechanics,

A typical example is due to Tomohiro Sasamoto. He is working to get exact solution of the KPZ (Kardar-Parisi-Zhang) equation (1938) of the form

$$\frac{\partial}{\partial t}h = \frac{1}{2}\lambda(\frac{\partial}{\partial x}h)^2 + \nu\frac{\partial^2}{\partial x^2}h + \sqrt{D}\eta,$$

where η is the space time noise parameterized by $x \in R^d$ and $t \in R^1$.

T. Sasamoto has obtained the exact solution of the equation by establishing the calculus of the functionals of the space-time noise. It is noted that he obtained necessary formulas of generalized white noise functionals including Feynman path integral, Donsker's delta function (for space-time noise), exponentials of regular functionals on noise, and so forth. We feel that some of our results (obtained in purely theoretical way) have been concretized. Here are some literatures related to this direction.

M. Kardar, G. Parisi and Y-C Zhang, Dynamic scaling of growing interfaces. Physical Review Letters. 56 no.9 (1938). 889-892,

T.Sasamoto and H. Spohn, One-dimensional Kardar-Parisi-Zhang equation: An exact solution and its universality. Physical Review Letters. 104, 230602 (2010), 230602 1-4.

T.Sasamoto and H. Spohn, Exact height distributions for the KPZ equation with narrow wedge initial condition. Nucl. Physics. B 834 (2010), 523-545.

§ 2. Brownian bridge and a setup of the propagator

First we have to explain why the Brownian bridge is involved in the class of quantum mechanical possible trajectories.

In [2] §32, Action principle, there is a statement that $B(t,s) = \int_t^s L(u)du$ satisfies a chain rule, by which we may imagine the formula for the transition probabilities of a **Markov process**.

To fix the idea, we consider the case where the time interval is taken to be $[0,T]$. Now the term z that expresses the quantity of fluctuation can be a Markov process $X(t), 0 \le t \le T$. Further assumptions on $X(t)$ can be made as follows.

1) $X(t)$ is a Gaussian process, since it is a sort of noise.

2) As a usual requirement, the Gaussian process satisfies $E(X(t)) = 0$ and has the canonical representation by Brownian motion, namely

$$X(t) = \int_0^t F(t,u)\dot{B}(u)du.$$

and $X(0) = X(T) = 0$ (bridged).

3) $X(t)$ is a Gaussian 1-ple Markov process.

4) The normalized process $Y(t)$ enjoys the projective invariance under time-change.

Theorem 2.1. *The Brownian bridge $X(t)$ over the time interval $[0,T]$ is characterized by the above conditions 1) - 4).*

This theorem we have proved before and the proof is omitted here. We only note that the canonical representation of $X(t)$ is given by

$$X(t) = (T-t)\int_0^t \frac{1}{T-u}\dot{B}(u)du,$$

and the covariance $\Gamma(t,s)$ is

$$\Gamma(t,s) = \sqrt{\frac{s(T-t)}{t(T-s)}}, s \le t.$$

Namely,

$$\Gamma(t,s) = \sqrt{(0,T;s,t)}, s \le t,$$

where $(\cdot,\cdot;\cdot,\cdot)$ is the anharmonic ratio.

[Remark] Heuristically speaking, it was 1981 when we proposed a white noise approach to path integrals to have quantum mechanical propagators (Hida-Streit paper presented 1981 Berlin Conference on Math-Phys. Later Streit-Hida [17]). We had, at that time, some idea in mind for the use of a Brownian bridge, and we had practically many good examples of integrand with various kinds of potentials, and satisfactory results have been obtained.

With this background we are ready to propose how to form quantum mechanical propagators.

The possible quantum mechanical trajectories $x(t), t \in [0,T]$ are expressed in the form

$$x(t) = y(t) + \sqrt{\frac{\hbar}{m}} X(t),$$

where $X(t)$ is a Brownian bridge over the time interval $[0,T]$. The fluctuation z in the earlier expression is now taken to be a Brownian bridge.

Remind that the classical trajectory $y(t), t \in [0,T]$, is uniquely determined by the variational principle for the action

$$A[x] = \int_0^T L(x, \dot{x}) dt,$$

where the Lagrangian $L(x, \dot{x})$ in question is assumed to be of the form

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x).$$

The potential $V(x)$ is usually assumed to be regular, but later we can extend the theory to the case where V has certain singularity, even time-dependent Mainly by the Streit school).

The actual expression and computations of the propagator are given successively as follows:

We follow the Lagrangian dynamics. The possible trajectories are sample paths $y(s), s \in [0,t]$, expressed in the form

$$y(s) = x(s) + \sqrt{\frac{\hbar}{m}} B(s), \tag{2.1}$$

where the $B(t)$ is an ordinary Brownian motion. Hence the action S is expressed in the form in terms of quantum trajectory y:

$$A = \int_0^t L(y(s), \dot{y}(s)) ds.$$

Note that the effect of forming a bridge is given by putting the delta-function $\delta_0(y(t)-y_2)$ as a factor of the integrand, where $y_2 = x(t)$.

Now we set

$$S(t_0, t_1) = \int_{t_0}^{t_1} L(t)dt. \tag{2.2}$$

and set

$$\exp\left[\frac{i}{\hbar}\int_{t_0}^{t_1} L(t)dt\right] = \exp\left[\frac{i}{\hbar}S(t_0, t_1)\right] = B(t_0, t_1).$$

Then, we have (see Dirac [2]), for $0 < t_1 < t_2 < \cdots < t_n < t$,

$$B(0,t) = B(0,t_1) \cdot B(t_1, t_2) \cdots B(t_n, t).$$

See [2] Section 32.

Theorem 2.2. *The quantum mechanical propagator $G(0,t;y_1,y_2)$ is given by the following average*

$$G(0,t;y_1,y_2) = \left\langle N e^{\frac{i}{\hbar}\int_0^t L(y,\dot{y})ds + \frac{1}{2}\int_0^t \dot{B}(s)^2 ds} \delta_o(y(t)-y_2) \right\rangle, \tag{2.3}$$

where N is the amount of multiplicative renormalization. The average $\langle\ \rangle$ is understood to be the integral with respect to the white noise measure μ.

§ 3. Generalized white noise functionals revisited

It is well-known that there are two classes of generalized white noise functionals; $(L^2)^-$ and $(S)^*$. We use them without discrimination except it is necessary to choose one of them specifically.

It seems better to explain the concept of "renormalization" which makes formal but important functionals of the $\dot{B}(t)$'s to be acceptable as generalized white noise functionals. To save time we refer the interpretation to the literatures [8] and [9].

We should note that there are generalized white noise functionals involved in the expectation in Theorem 2. For instance, there is involved the delta function, in fact the Donsker's delta function $\delta_o(y(t)-y_2)$, which is a generalized white noise functional.

There is used a Gauss kernel of the form $\exp[c\int_0^t \dot{B}(s)^2 ds]$, the ideal case is $c = -\frac{1}{2}$. In general, if $c \neq \frac{1}{2}$, then it can be a generalized functional after having the multiplicative renormalization. Now we have the exceptional case, but it can be accepted by combining with other factor of an exponential; this is just the case. In reality, we combine it with the term that comes from the kinetic energy.

The factor $\exp[\frac{1}{2}\int_0^t \dot{B}(s)^2 ds]$ serves as the **flattening effect** of the white noise measure. One may ask why the functional does so. An intuitive answer to this question is as follows: If we write a Lebesgue measure (exists only virtually) on E^* by dL, the white noise measure μ may be expressed in the form $\exp[-\frac{1}{2}\int_0^t \dot{B}(s)^2 ds]dL$. Hence, the the factor in question is put to make the measure μ to be a flat measure dL. In fact, this makes sense eventually.

Returning to the average (3) (in Theorem 2), which is an integral with respect to the white noise measure μ, it is important to note that the integrand (i.e. the inside of the angular bracket) is integrable, in other words, it is a bilinear form of a generalized functional and a test functional.

There have to follow short notes to be reminded. They are rather crucial. The formula (3) involves a product of functionals of the form like $\varphi(x)\cdot\delta(\langle x, f\rangle - a), f \in L^2(R), a \in \mathbf{C}$. To give a correct interpretation to the expectation of (3) with this choice, it should be checked that it can be regarded as a bilinear form of a pair of a test functional and a generalized functional. The following assertion answers to this question.

Theorem 3.1. *(Streit et al [10]) Let $\varphi(x)$ be a generalized white noise functional. Assume that the $\mathcal{T}$-transform $(\mathcal{T}\varphi)(\xi), \xi \in E$, of φ is extended to a functional of f in $L^2(R)$, in particular a function of $\xi + \lambda f$, and that $(\mathcal{T}\varphi)(\xi - \lambda f)$ is an integrable function of λ for any fixed ξ. If the transform of $(\mathcal{T}\varphi)(\xi - \lambda f)$ is a U-functional, then the pointwise product $\varphi(x)\cdot\delta(\langle x, f\rangle - a)$ is defined and is a generalized white noise functional.*

Proof. First a formula for the δ-function is provided.

$$\delta_a(t) = \delta(t-a) = \frac{1}{2\pi}\int e^{ia\lambda}e^{-i\lambda x}d\lambda \text{ (in distribution sense).}$$

Hence, for $\varphi \in (S)^*$ and $f \in L^2(R)$ we have

$$\begin{aligned} \mathcal{T}(\varphi(x)\delta(\langle x, f\rangle - a))\xi) &= \frac{1}{2\pi}\int e^{ia\lambda}e^{-i\lambda\langle x,f\rangle}e^{i\langle x,\xi\rangle}\varphi(x)d\mu(x)d\Lambda \\ &= \frac{1}{2\pi}\int e^{ia\lambda}(\mathcal{T}\varphi)(\xi^\lambda f)d\lambda. \end{aligned} \tag{3.1}$$

By assumption this determines a U-functional, which means the product $\varphi(x)\cdot\delta(\langle x,f\rangle - a)$ makes sense and it is a generalized white noise functional.

□

Example 3.2. The above theorem can be applied to a Gauss kernel $\varphi_c(x) = N\exp[c\int x(t)^2dt]$, with $c \neq \frac{1}{2}$.

i) The case where c is real and $c < 0$.
We have

$$
\begin{aligned}
(\mathcal{T}\varphi)(\xi - \lambda f) &= \exp[\frac{c}{1-2c}\int(\xi(t) - \lambda f(t))^2dt] \\
&= \exp[\frac{c}{1-2c}(\|\xi\|2 - 2\lambda(\xi, f) + \lambda^2\|f\|^2]).
\end{aligned}
$$

This is an integrable function of real λ. Hence, by the above Theorem 10.3, we have a generalized white noise functional.

ii) The case where $c = \frac{1}{2} + ia, a \in R - \{0\}$.
The same expression as in i) is given.

Example 3.3. In the following case, exact values of the propagators can be obtained and, of course, they agree with the known results.
i) Free particle
ii) Harmonic oscillator.
iii) Potentials which are Fourier transforms of measures (the the Albeverio-Hohkron potential).
iv) Others.

§ 4. Some of further developments and related topics

[I] In addition to Example 2, we have some more interesting potentials,including those are much singular and time depending. There are satisfactory results in the recent developments.

Example 4.1. Streit et al have obtained explicit formulae in the following cases:

1) a time depending Lagrangian of the form

$$
L(x(t), \dot{x}(t), t) = \frac{1}{2}m(t)\dot{x}(t)^2 - k(t)^2x(t)^2 - \dot{f}(t)x(t),
$$

where $m(t), k(t)$ and $f(t)$ are smooth functions.

2) A singular potential $V(x)$ of the form

$$V(x) = \sum_n c^{-n^2} \delta_n(x), \quad c > 0,$$

and others.

[II] The Hopf equation.

There are many approaches to the Navier-Stokes equation.

$$u_{\alpha,t} + u_\beta u_{\alpha,\beta} = -p{\cdot}\alpha + \mu u_{\alpha,\beta\beta},$$

where $\alpha, \beta = 1, 2, 3$ and where the following notations are used:

$$f_{\alpha,t} = \frac{\partial f_\alpha}{\partial t},$$

$$f_{\alpha,\beta} = \frac{\partial f_\alpha}{\partial x_\beta}$$

and

$$f_{\alpha,\beta\gamma} = \frac{\partial^2 f_\alpha}{\partial x_\beta \partial x_\gamma}.$$

There is an approach to this equation by using the characteristic functional Φ of the measure $P^t(du)$ defined on the phase space $\{u = (u_1, u_2, u_3)\}$:

$$\Phi(\xi, t) = \int e^{i<\xi,u>} P^t(du).$$

E. Hopf shows that the characteristic functional $\Phi(\xi, , t)$ satisfies the following functional differential equation, called **Hopf equation**:

$$\frac{\partial \Phi}{\partial t} = \int_R \xi_\alpha(x)[i\frac{\partial}{\partial x_\beta}\frac{\partial^2 \Phi}{\partial \xi_\beta(x)dx\partial \xi_\alpha(x)dx} + \mu\Delta_x \frac{\partial \Phi}{\partial \xi_\alpha(x)dx} - \frac{\partial \Pi}{\partial x_\alpha}]dx.$$

Studying this approach, we may think of the two matters. One is a similarity to the Feynman integral in the sense that both cases deal with functional obtained in the form

$$E(\exp[f(u)]),$$

where $f(u)$ is a function of a path (trajectory) u.The expectation ia taken with respect to the probability measure introduced on the path space.

As the second point, one may think of equations $\Phi_n, n \geq 0$ that come from the Hopf equation and the Fock space expansion of generalized white noise functionals. In this case we expect that the calculus can be done in a similar manner to the white noise calculus.

We may remind an interesting approach to the Navier-Stokes equation by A.Inoue.

[III] Towards noncommutative white noise calculus. This comes from many reasons: amomg others

i) noncommutative geometry,

ii) Hamiltonian dynamics using both variables, p, q.

§ 5. Two remarks

(1) There appears a particular quadratic form in the white noise analysis, i.e.

$$\int : \dot{B}(t)^2 : dt.$$

There are somewhat general quadratic form

$$\int f(t) : \dot{B}(t)^2 : dt + \int\int F(u,v) : \dot{B}(u)\dot{B}(v) : dudv$$

which is called *normal functional*, the first term is called the singular part and the second term is the regular part. The two terms can be characterized from our viewpoint and play significant roles, respectively. Remind the role of singular part in the path integral.

(2) Our method of path integrals enables us to deal with the case of very irregular potentials to have the propagator, by L. Streit and others.

PART II

Hamiltonian dynamics

1) Background

We should like to mention some historical stories.

I. Daubechies and J.R. Klauder, Quantum mechanical path integrals with Wiener measure for all polynomial Hamiltonians. II. J. Math. Phys. 26 (1985), 2239-2256.

While there is recent topics.

W. Bock, Hamilotonian path integrals in white noise analysis. Kaiserslautern Dissetation 2013.

M.A. de Gosson, Symplectic methods in harmonic analysis and in mathematical physics. Birkhäuser, 2011.

Definition 5.1. $R^{2n} = \{z = (x,p); x = (x_1, x_2, \cdots, x_n), p = (p_1, p_2, \cdots, p_n)\}$ is the phase space. There is a time-dependent Hamiltonian given by the function satisfying $H \in C^\infty(R^{2n+1})$ (Hamiltonian equation).

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}(x,p,t) \tag{5.1}$$

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j}(x,p,t). \tag{5.2}$$

Assuming that this equation is given on some subdomain of $z \leq 1$ with, we can prove that there exists the unique solution under the assumption $t \in [-T, T]$ and $z(0) = z_0$.

Example. The case where the equation does not depend on t. The hamiltonian is expressed in the following form;

$$H(x,p) = \sum_1^n \frac{p_j^2}{2m_j} + U(x).$$

The potential U is now assumed to be $U \in C^\infty(R^n)$.

Proposition 5.2. *Further if U satisfy $U(x) \geq a$ for some a, then there exists the unique solution of the Hamiltonian equation*

$$\frac{dx_j}{dt} = \frac{p_j}{m_j} \tag{5.3}$$

$$\frac{dp_j}{dt} = -\frac{\partial U}{\partial x_j}(x) \tag{5.4}$$

Proof. To fix the idea, we set $a = 0, m = 1, n = 1$. Then, we have

$$\frac{dx}{dt} = p \tag{5.5}$$

$$\frac{dp}{dt} = -\frac{\partial U}{\partial x}(x) \tag{5.6}$$

This guarantees the existence of the unique solution under the suitable initial condition.

2) Hamiltonian fields.

Now we introduce some notations to make formulas simpler.

$\frac{\partial}{\partial x}$ is simply written as ∂_x, the gradient is ∂_x, and $\partial_z = \{\partial_x, \partial_p\}$ and so forth in a similar manner.

The matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is denoted by J. Then, the Hamiltonian equation is simply written as

$$\dot{z} = J\partial_z H(z).$$

Definition 5.3.

$$X_H = J\partial_z H = (\partial_x H, -\partial_p H)$$

is called the **Hamiltonian vector field** and $J\partial_z$ is called the **symplectic gradient**.

We continue discussion on Hamiltonian path integral.

There is an additional remark. Unlike the case on Lagrangian dynamics where we understand $p = m\frac{dq}{dt}$, we now discriminate the position x and momentum p (momentum), indeed they are independent variables.

In fact, the relation ship between x and q is expressed in the form $dx \wedge dp$, so that we see a noncommutative realization.

We are now in a position to have a quick overview of our method towards the Hamiltonian path integral with some additional notes. For this purpose, we follow the line due to Klauder-Grothaus-Bock.

Hamiltonian $H(x,p,t)$ is given by

$$H(x,p,t) = \frac{1}{2m}p^2 + V(x,p,t).$$

The Hamiltonian action $S(x,p,t)$ is expressed in the form

$$S(x,p,t) = \int_0^t p(\tau)\dot{x}(\tau) - H(x(\tau), p(\tau), \tau)d\tau.$$

First take the path integral over the configuration (coordinate space) path integral, then take that on the momentum space. Their relationship can be seen with the help of the Fourier transform. The main tool is, of course, the white noise analysis on generalized functionals.

1. The path integral on configuration space.

A trajectory of a Brownian motion starting from x_0:

$$x(\tau) = x_0 + \sqrt{\hbar/m}B(\tau), 0 \le \tau \le t. \tag{5.7}$$

The constant $\sqrt{\hbar/m}$ is determined by the dimension calculus. The momentum p is obtained by another Brownian motion ω, which is independent of $B(t)$ above. Thus,

$$p(\tau) = \sqrt{\hbar m}\omega(\tau), 0 \le \tau \le t,$$

Thus, the Feynman integrand I_c is given by:

$$I_c = N\exp[\frac{i}{\hbar}\int_0^t p(\tau)\dot{x}(\tau) - \frac{p(\tau)^2}{2m}d\tau + \frac{1}{2}\int_0^t \dot{x}(\tau)^2 + p(\tau)^2 d\tau)$$
$$\cdot \exp[-\frac{i}{\hbar}\exp[-\frac{i}{\hbar}\int_0^t V(x(\tau), p(\tau), \tau)d\tau]\delta(x(t) - y),$$

where N is a (multiplicative) renormalizing constant, the delta function is used for the pinning effect.

[Remark 1] In the above equation, it seems to take a Brownian bridge rather than Donsker's delta function, but either way gives the same result. It is a matter of taste.

[Remark 2] The multiplicative renormalizing constant can be derived from the formulas for exponential of quadratic form, the exact form comes from that of Brownian functional.

There one can see the exact formula, in particular the constant sitting in front.

With those remarks given above we can carry on the integration with respect to the white noise measure.

2. Hamiltonian path integral on momentum space.

The variable $p(\tau)$ involves only fluctuation by a Brownian motion:

$$p(\tau) = p_0 + \frac{\sqrt{\hbar m}}{t}B(\tau), \quad 0 \le \tau \le t.$$

The space variable $x(\tau)$ consists only of noise.

$$x(\tau) = \sqrt{\hbar/m}t\omega(\tau), \quad 0 \le \tau \le t.$$

Note that the two Brownian motions $B(\tau)$ and $\omega(\tau)$ are independent.

Then, Feynman integrand I_m is given by the following formula:

$$I_m = N\exp[\frac{i}{\hbar}\int_0^t (-x(\tau)\dot{p}(\tau) - \frac{p(\tau)^2}{2m})d\tau + \frac{1}{2}\int_0^t (\omega(\tau)^2 + B(\tau)^2)d\tau]$$
$$\cdot \exp[-\frac{i}{\hbar}\int_0^t V(x(\tau), p(\tau), \tau)d\tau]\delta(p(t) - p').$$

Integrate this formula with respect to direct product measure of two white noise measures to get the quantum mechanical propagator.

Non commutativity follows naturally.

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