A survey of simple formulas for generalized conditional Wiener integrals on an analogue of Wiener space

By

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Abstract

Let $C[0,T]$ denote a generalized Wiener space, the space of real-valued continuous functions on the interval $[0,T]$. In this survey paper we will introduce various simple formulas for the generalized conditional Wiener integrals of functions on $C[0,T]$ with several vector-valued conditioning functions having a drift and then investigate their relationships. As applications of the formulas we evaluate more generalized conditional Wiener integrals of functions on $C[0,T]$ including a time integral which are of interest in the Feynman integration theories and quantum mechanics.

§ 1. Introduction

A time integral is simply the Riemann integral of a function of the continuous random variable $X(x,t)=x(t)$ with respect to the parameter t for $x\in C_0[0,T]$ which is the Wiener space, the space of continuous real-valued functions x on $[0,T]$ with $x(0)=0$ [16]. This means that the time integral of $X(x,t)$ is a random variable Y on $C_0[0,T]$ satisfying

$$Y(x)=\int_0^T F(t,X(x,t))dm_L(t),$$

where $F(t,X(x,t))$ is Riemann integrable on $[0,T]$ and m_L is the Lebesgue measure on $\mathbb{R}$. The Feynman-Kac functional $F:C_0[0,t]\to\mathbb{C}$ is given by

$$F(x)=\exp\left\{-\int_0^T V(t,X(x,t))dm_L(t)\right\}$$

2010 Mathematics Subject Classification(s): 28C20, 46G12, 60G15.
Key Words: Conditional Wiener integral, Feynman integral, simple formula, time integral, Wiener integral, Wiener measure, Wiener space.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2013R1A1A2058991).
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including the time integral, where $X(x,t)$ is a standard Brownian motion process on $C_0[0,T]\times[0,T]$ and V is a complex-valued potential. Calculations involving the conditional expectations of F are important in defining and studying of a Feynman integral. To evaluate the conditional expectation of F, Yeh [18, 19] introduced an inversion formula given by a Fourier-transform which changes the conditional expectation to an ordinary (i.e., non-conditional) Wiener integral when the conditioning function is single-valued. But Yeh's inversion formula for conditional Wiener integrals is very complicated to apply when the conditioning function is vector-valued. Park and Skoug [13] introduced more simple formula to evaluate the conditional Wiener integral when the value space of the conditioning function is n-dimensional Euclidean space. Using the formula they calculated the conditional expectation of F and then generalized the Feynman-Kac formula. Furthermore they [15] extended the results of [13] to much more general conditioning functions which need not depend upon the values of the Wiener paths at only finitely many points in $[0,T]$. Furthermore, when the conditioning function is single-valued, they [3, 14] extended the results of [15] to the conditioning function which has a drift. We note that every Wiener path starts at the origin on the classical Wiener space $C_0[0,T]$.

On the other hand Ryu and Im [11, 17] introduced an analogue of Wiener measure space $(C[0,T],\mathcal{B}(C[0,T]),w_\varphi)$, where $C[0,T]$ is the space of continuous functions on the interval $[0,T]$, w_φ is a probability measure on the Borel class $\mathcal{B}(C[0,T])$ of $C[0,T]$ and φ is a probability measure on the Borel class of $\mathbb{R}$, that is, it is the initial distribution of the paths in $C[0,T]$. On this space $C[0,T]$ the author [4] derived a simple formula for a conditional expectation which corresponds to the formula in [13] and in which the conditioning function contains both the initial position $x(0)$ and present position $x(T)$ of the paths x in $C[0,T]$. Furthermore he [5] generalized the formula in [4] removing the present position $x(T)$ in the conditioning function. The author [6] also generalized the formula in [4] using the Paley-Wiener-Zygmund integral in the conditioning function and he [7] then did the formula in [6] giving a drift to the conditioning function. In the formulas in [6] and [7] the conditioning functions do not contain the initial positions of the generalized Wiener paths in $C[0,T]$. Recently the author [9] generalized the results in [4, 5, 6, 7] giving the initial position of the generalized Wiener paths to the conditioning function. We note that all the works on $C[0,T]$ reduce to those on $C_0[0,T]$ if we take $\varphi=\delta_0$, the Dirac measure concentrated at 0.

In this survey paper we will introduce various simple formulas for generalized conditional Wiener integrals of functions on $C[0,T]$ with the conditioning functions having a drift and then investigate relationships among the simple formulas. As applications of the formulas we evaluate more generalized conditional Wiener integrals of the function F on $C[0,t]$ with the drift which is of interest in the Feynman integration theories themselves and quantum mechanics.

§ 2. An analogue of Wiener space

Let $\mathbb{C}$ denote the set of complex numbers, let m_L denote the Lebesgue measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ and let $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ denote the dot product on $\mathbb{R}^m$.

For a positive real T, let $C[0,T]$ be the space of all real-valued continuous functions on the closed interval $[0,T]$ with the supremum norm. For $\vec{t}=(t_0,t_1,\cdots,t_n)$ with $0=t_0<t_1<\cdots<t_n\le T$, let $J_{\vec{t}}\colon C[0,T]\to\mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x)=(x(t_0),x(t_1),\cdots,x(t_n)).$$

For B_j $(j=0,1,\cdots,n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ let

$$m_\varphi\Big(J_{\vec{t}}^{-1}\Big(\prod_{j=0}^n B_j\Big)\Big)=\int_{B_0}\int_{\prod_{j=1}^n B_j} W_{n+1}(\vec{t};u_0,u_1,\cdots,u_n)\,dm_L^n(u_1,\cdots,u_n)\,d\varphi(u_0),$$

where

$$W_{n+1}(\vec{t};u_0,u_1,\cdots,u_n)$$
$$=\Big[\prod_{j=1}^n\frac{1}{2\pi(t_j-t_{j-1})}\Big]^{\frac12}\exp\Big\{-\frac12\sum_{j=1}^n\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\Big\}.$$

$\mathcal{B}(C[0,T])$, the Borel σ-algebra of $C[0,T]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0,T],\mathcal{B}(C[0,T]))$ such that $w_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$ [11, 17]. This measure w_φ is called an analogue of Wiener measure associated with the probability measure φ. For $t\in[0,T]$ and $x\in C_0[0,T]$ define the Wiener process W_t by $W_t(x)=x(t)$. For $t\in[0,T]$ and $x\in C[0,T]$ let $X_t(x)=x(t)$. It is well-known that the Wiener process W_t is the standard Brownian motion process. On the other hand X_t need not be a Brownian motion process since $X_0(x)=x(0)$ can take arbitrary values, that is, X_t can have arbitrary initial distribution. We note that W_t is normally distributed with mean 0 and variance t, but X_t need not be if φ is not the Dirac measure δ_0 concentrated at 0.

Let $F:C[0,T]\to\mathbb{C}$ be integrable and X_τ be a random vector on $C[0,T]$ assuming that the value space of X_τ is a normed space equipped with the Borel σ-algebra. Then we have the conditional expectation $E[F|X_\tau]$ of F given X_τ from a well known probability theory. Furthermore there exists a P_{X_τ}-integrable complex valued function ψ on the value space of X_τ such that $E[F|X_\tau](x)=(\psi\circ X_\tau)(x)$ for w_φ-a.e. $x\in C[0,T]$, where P_{X_τ} is the probability distribution of X_τ. The function ψ is called the conditional w_φ-integral of F given X_τ and it is also denoted by $E[F|X_\tau]$.

Let $\{d_k : k = 1, 2, \cdots\}$ be a complete orthonormal subset of $L_2[0,T]$ such that each d_k is of bounded variation. For v in $L_2[0,T]$ and x in $C[0,T]$ let

$$(v,x) = \lim_{n\to\infty} \sum_{k=1}^{n} \int_0^T \langle v, d_k\rangle_2 d_k(t)dx(t)$$

if the limit exists, where $\langle\cdot,\cdot\rangle_2$ denotes the inner product over $L_2[0,T]$. (v,x) is called the Paley-Wiener-Zygmund integral of v according to x [17]. Let $h \in L_2[0,T]$ be of bounded variation with $h \neq 0$ a.e. on $[0,T]$ and let $a \in C[0,T]$. Define $X_0(x) = x(0)$ for $x \in C[0,T]$ and define stochastic processes $X, Y, Z : C[0,T] \times [0,T] \to \mathbb{R}$ by

$$X(x,t) = (\chi_{[0,t]}h, x),\ Y(x,t) = (\chi_{[0,t]}h, x) + a(t)$$

and

$$Z(x,t) = (\chi_{[0,t]}h, x) + X_0(x) + a(t)$$

for $x \in C[0,T]$ and for $t \in [0,T]$. Let $b(t) = \int_0^t (h(u))^2 du$ and let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ be a partition of $[0,T]$. For $j = 1, \cdots, n, n+1$ let

$$\alpha_j^b(t) = \frac{b(t_j) - b(t)}{b(t_j) - b(t_{j-1})},\ \beta_j^b(t) = \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}$$

for $t \in [t_{j-1}, t_j]$ and

$$\sigma_j^b(s,t) = \alpha_j^b(t)\beta_j^b(s)(b(t_j) - b(t_{j-1}))$$

for $s, t \in [t_{j-1}, t_j]$. Define random vectors $Z_n : C[0,T] \to \mathbb{R}^{n+1}$ and $Z_{n+1} : C[0,T] \to \mathbb{R}^{n+2}$ by

$$Z_n(x) = (Z(x,t_0), Z(x,t_1), \cdots, Z(x,t_n))$$

and

$$Z_{n+1}(x) = (Z(x,t_0), Z(x,t_1), \cdots, Z(x,t_n), Z(x,t_{n+1}))$$

for $x \in C[0,T]$. For any function f on $[0,T]$ define the polygonal function $P_{b,n+1}(f)$ of f by

$$P_{b,n+1}(f)(t) = \sum_{j=1}^{n+1} \chi_{(t_{j-1},t_j]}(t)[\alpha_j^b(t)f(t_{j-1}) + \beta_j^b(t)f(t_j)] + \chi_{\{t_0\}}(t)f(t_0) \tag{2.1}$$

for $t \in [0,T]$, where $\chi_{(t_{j-1},t_j]}$ and $\chi_{\{t_0\}}$ denote the indicator functions. For $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$ define the polygonal function $P_{b,n+1}(\vec{\xi}_{n+1})$ of $\vec{\xi}_{n+1}$ by (2.1), where $f(t_j)$ is replaced by ξ_j for $j = 0, 1, \cdots, n, n+1$. If $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, $P_{b,n}(\vec{\xi}_n)$ is interpreted as $\chi_{[0,t_n]} P_{b,n+1}(\vec{\xi}_{n+1})$ on $[0, T]$. For $x \in C[0, T]$ and for $t \in [0, T]$ let

$$Z_{b,n+1}(x, t) = Z(x, t) - P_{b,n+1}(Z(x, \cdot))(t) \text{ and } A(t) = a(t) - P_{b,n+1}(a)(t).$$

For a function $F : C[0, t] \to \mathbb{C}$ let $F_Z(x) = F(Z(x, \cdot))$ for $x \in C[0, T]$. We have the following theorems from [9].

Theorem 2.1. *Let F be a complex valued function on $C[0, T]$ and F_Z be integrable over $C[0, T]$. Then for $P_{Z_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$E[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) = E[F(Z_{b,n+1}(x, \cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))],$$

where the expectation is taken over the variable x and $P_{Z_{n+1}}$ is the probability distribution of Z_{n+1} on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$.

Theorem 2.2. *Let F be a complex valued function on $C[0, T]$ and F_Z be integrable over $C[0, T]$. Let P_{Z_n} be the probability distribution of Z_n on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. Then for P_{Z_n}-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$*

$$E[F_Z | Z_n](\vec{\xi}_n) = \left[\frac{1}{2\pi(b(T) - b(t_n))}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(Z_{b,n+1}(x, \cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))] \times \exp\left\{-\frac{(\xi_{n+1} - a(T) - (\xi_n - a(t_n)))^2}{2(b(T) - b(t_n))}\right\} dm_L(\xi_{n+1}),$$

where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$.

Define the polygonal function $[x]$ of x on $[0, T]$ by the right hand side of (2.1) with $b(t) = t$ and with replacing $f(t_j)$ by $x(t_j)$ for $j = 0, 1, \cdots, n, n+1$. Similarly for $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define the polygonal function $[\vec{\xi}_{n+1}]$ of $\vec{\xi}_{n+1}$ on $[0, T]$ by the right hand side of (2.1) with $b(t) = t$ and with replacing $f(t_j)$ by ξ_j for $j = 0, 1, \cdots, n, n+1$.

Letting $h = 1$ and $a = 0$ on $[0, T]$ we have $Z(x, t) = x(t)$ and $b(t) = t$ for $x \in C[0, T]$ and for $t \in [0, T]$ so that we can obtain the following simple formulas in [4, 5].

Corollary 2.3. *Let $X_\tau : C[0, T] \to \mathbb{R}^{n+2}$ be given by*

$$X_\tau(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1})) \tag{2.2}$$

and let $F : C[0, T] \to \mathbb{C}$ be integrable. Then for P_{X_τ}-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ we have

$$E[F | X_\tau](\vec{\xi}_{n+1}) = E[F(x - [x] + [\vec{\xi}_{n+1}])],$$

where P_{X_τ} is the probability distribution of X_τ on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$.

Corollary 2.4. *Let $X_\kappa : C[0,T] \to \mathbb{R}^{n+1}$ be given by*

$$X_\kappa(x) = (x(t_0), x(t_1), \cdots, x(t_n))$$

and X_τ be given by (2.2). Moreover let F be defined and integrable on $C[0,T]$ and P_{X_κ} be the probability distribution of X_κ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. Then for P_{X_κ}-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, we have

$$E[F|X_\kappa](\vec{\xi}_n) = \left[\frac{1}{2\pi(T-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(x - [x] + [\vec{\xi}_{n+1}])] \exp\left\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(T-t_n)}\right\} dm_L(\xi_{n+1}),$$

where $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n)$ and $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$.

Define random vectors $Y_n : C[0,T] \to \mathbb{R}^n$ and $Y_{n+1} : C[0,T] \to \mathbb{R}^{n+1}$ by

$$Y_n(x) = (Y(x,t_1), \cdots, Y(x,t_n))$$

and

$$Y_{n+1}(x) = (Y(x,t_1), \cdots, Y(x,t_n), Y(x,t_{n+1})) \text{ for } x \in C[0,T].$$

For $\vec{\xi}_{n+1} = (\xi_1, \cdots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}$ define the polygonal function $P_{b,n+1,0}(\vec{\xi}_{n+1})$ of $\vec{\xi}_{n+1}$ by the right hand side of (2.1), where $f(t_j)$ is replaced by ξ_j for $j = 0, 1, \cdots, n, n+1$ with $\xi_0 = 0$. Let $\alpha \in \mathbb{R}$ and let φ_α be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\varphi_\alpha(B) = \varphi(B+\alpha)$ for $B \in \mathcal{B}(\mathbb{R})$.

We now have the following theorems [7].

Theorem 2.5. *Let $F : C[0,T] \to \mathbb{C}$ be a function and $F_Y(x) \equiv F(Y(x,\cdot))$ be w_{φ_α}-integrable over the variable x. Then for $P_{Y_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$ (hence for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$)*

$$E[F_Y|Y_{n+1}](\vec{\xi}_{n+1}) = E[F(Y(x,\cdot) - P_{b,n+1}(Y(x,\cdot)) + P_{b,n+1,0}(\vec{\xi}_{n+1}))],$$

where the expectation is taken over w_{φ_α} and $P_{Y_{n+1}}$ is the probability distribution of Y_{n+1} on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$.

Theorem 2.6. *Let $F : C[0,T] \to \mathbb{C}$ be a function and F_Y be w_{φ_α}-integrable over the variable x. Moreover let P_{Y_n} be the probability distribution of Y_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then for P_{Y_n}-a.e. $\vec{\xi}_n = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ (hence for a.e. $\vec{\xi}_n \in \mathbb{R}^n$)*

$$E[F_Y|Y_n](\vec{\xi}_n) = \left[\frac{1}{2\pi(b(T)-b(t_n))}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(Y(x,\cdot) - P_{b,n+1}(Y(x,\cdot)) + P_{b,n+1,0}(\vec{\xi}_{n+1}))] \exp\left\{-\frac{(\xi_{n+1} - a(T) - (\xi_n - a(t_n)))^2}{2(b(T)-b(t_n))}\right\} dm_L(\xi_{n+1}),$$

where $\vec{\xi}_{n+1} = (\xi_1, \cdots, \xi_n, \xi_{n+1})$ and the expectation is taken over w_{φ_α}.

Define a random vector $X_n : C[0,T] \to \mathbb{R}^n$ by

$$X_n(x) = (X(x,t_1), \cdots, X(x,t_n))$$

and define $X_{n+1} : C[0,T] \to \mathbb{R}^{n+1}$ by

$$X_{n+1}(x) = (X(x,t_1), \cdots, X(x,t_n), X(x,t_{n+1}))$$

for $x \in C[0,T]$.

Letting $a = 0$ we have the following corollaries [6].

Corollary 2.7. *Let $F : C[0,T] \to \mathbb{C}$ be a function and $F_X(x) \equiv F(X(x,\cdot))$ be w_{φ_α}-integrable over the variable x. Then for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$ (hence for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+1}$)*

$$E[F_X|X_{n+1}](\vec{\xi}_{n+1}) = E[F(X(x,\cdot) - P_{b,n+1}(X(x,\cdot)) + P_{b,n+1,0}(\vec{\xi}_{n+1}))],$$

where the expectation is taken over w_{φ_α} and $P_{X_{n+1}}$ is the probability distribution of X_{n+1} on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$.

Corollary 2.8. *Let $F : C[0,T] \to \mathbb{C}$ be a function and F_X be w_{φ_α}-integrable over the variable x. Moreover let P_{X_n} be the probability distribution of X_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then for P_{X_n}-a.e. $\vec{\xi}_n = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ (hence for a.e. $\vec{\xi}_n \in \mathbb{R}^n$)*

$$\begin{aligned} &E[F_X|X_n](\vec{\xi}_n) \\ &= \left[\frac{1}{2\pi(b(T)-b(t_n))}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(X(x,\cdot) - P_{b,n+1}(X(x,\cdot)) + P_{b,n+1,0}(\vec{\xi}_{n+1}))] \\ &\quad \times \exp\left\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(b(T)-b(t_n))}\right\} dm_L(\xi_{n+1}), \end{aligned}$$

where $\vec{\xi}_{n+1} = (\xi_1, \cdots, \xi_n, \xi_{n+1})$ and the expectation is taken over w_{φ_α}.

Remark. 1. The conditioning functions X_τ, X_{n+1}, Y_{n+1} and Z_{n+1} contain the present positions of the generalized Wiener paths, but X_κ, X_n, Y_n and Z_n do not. Moreover the conditioning functions X_τ, X_κ, Z_n and Z_{n+1} contain the initial positions of the generalized Wiener paths, but X_n, Y_n, X_{n+1} and Y_{n+1} do not them. These mean that we can not obtain Theorems 2.5, 2.6 and Corollaries 2.7, 2.8 from Theorems 2.1 and 2.2.

2. If $h = 1$ on $[0,T]$, then $F_X(x) = F(x - x(0))$ and $X_{n+1}(x) = (x(t_1) - x(0), \cdots, x(t_n) - x(0), x(t_{n+1}) - x(0))$ so that Corollary 2.7 generalizes Theorem 3 in [13] with $w_{\varphi_\alpha} = \delta_0$ which is the Dirac measure concentrated at 0.

3. If $\varphi = \delta_0$, then w_φ is the Wiener measure on $C_0[0,t]$ so that Corollary 2.7 is a generalization of Theorem 3 in [14].

4. Corollary 2.7 is a generalization of Theorem 2.9 in [4] only when $\varphi = \delta_0$.

5. If $a(0) = 0$, $h = \sqrt{b'}$ and $\varphi_\alpha = \delta_0$, then we can obtain the space $C_{a,b}[0,T]$ in [2] by Theorem 12 in [7]. Furthermore if Y is replaced by the generalized Brownian motion process $x(t)$ on $C_{a,b}[0,T] \times [0,T]$ and we let $\varphi_\alpha = \delta_0$, then we can also obtain Theorem 3.4 in [2] by Theorem 2.5. If we let $a = 0$ and $\varphi_\alpha = \delta_0$, then we can obtain Theorem 3 in [14] by Theorem 2.5. If we let $n = 0$ and $\varphi_\alpha = \delta_0$, then we can obtain Remark 2.2 in [3] by Theorem 2.1. Finally if we let $h = 1$, $\varphi_\alpha = \delta_0$ and $a = 0$, then we can obtain Theorem 2 in [13] by Theorem 2.5 which is among the first result expressing the conditional Wiener integrals of functions on $C_0[0,T]$ as ordinary (non-conditional) Wiener integrals.

6. All the results of this paper do not depend on a particular choice of the initial distribution φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

§ 3. Conditional expectations of functions on time integrals

In this section we evaluate generalized conditional Wiener integrals of the function $\exp\{\int_0^T Z(x,t)dm_L(t)\}$ including a time integral.

To do this let $\Sigma_m(\vec{s}) = [\sigma(s_l, s_k)]_{m\times m}$ be the matrix given by

$$\sigma(s_l, s_k) = \sigma_j^b(\min\{s_l, s_k\}, \max\{s_l, s_k\})$$

and let

$$\Psi_m(\vec{s}, \vec{u}) = \left[\frac{1}{(2\pi)^m|\Sigma_m(\vec{s})|}\right]^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\langle \Sigma_m^{-1}(\vec{s})(\vec{u} - A(\vec{s})), \vec{u} - A(\vec{s})\rangle_{\mathbb{R}}\right\}$$

for $\vec{u} \in \mathbb{R}^m$ and $\vec{s} = (s_1, \cdots, s_m) \in \mathbb{R}^m$, where $t_{j-1} < s_1 < \cdots < s_m < t_j$ and $A(\vec{s}) = (A(s_1), \cdots, A(s_m))$.

Applying Theorem 2.1 we have the following theorem [9].

Theorem 3.1. *Let $t_{j-1} < s_1 < \cdots < s_m < t_j$ and let $H_m(\vec{s}, x) = x(s_1)\cdots x(s_m)$ for $x \in C[0,T]$. Suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$. Then for $P_{Z_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$\begin{aligned}&E[(H_m(\vec{s}, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1})\\ &= \int_{\mathbb{R}^m} (u_1 + P_{b,n+1}(\vec{\xi}_{n+1})(s_1)) \cdots (u_m + P_{b,n+1}(\vec{\xi}_{n+1})(s_m))\Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}),\end{aligned}$$

where $\vec{u} = (u_1, \cdots, u_m)$.

Example 3.2. If $m = 1$, then by Theorem 3.1 we have for $P_{Z_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$E[(H_1(s_1, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1}) = A(s_1) + P_{b,n+1}(\vec{\xi}_{n+1})(s_1).$$

If $m = 2$, then

$$|\Sigma_2(\vec{s})| = \begin{vmatrix} \sigma_j^b(s_1, s_1) & \sigma_j^b(s_1, s_2) \\ \sigma_j^b(s_1, s_2) & \sigma_j^b(s_2, s_2) \end{vmatrix} = \frac{(b(t_j) - b(s_2))(b(s_2) - b(s_1))(b(s_1) - b(t_{j-1}))}{b(t_j) - b(t_{j-1})}$$

and

$$\Sigma_2^{-1}(\vec{s}) = \begin{bmatrix} \frac{b(s_2)-b(t_{j-1})}{(b(s_2)-b(s_1))(b(s_1)-b(t_{j-1}))} & -\frac{b(s_1)-b(t_{j-1})}{(b(s_2)-b(s_1))(b(s_1)-b(t_{j-1}))} \\ -\frac{b(s_1)-b(t_{j-1})}{(b(s_2)-b(s_1))(b(s_1)-b(t_{j-1}))} & \frac{b(t_j)-b(s_1)}{(b(t_j)-b(s_2))(b(s_2)-b(s_1))} \end{bmatrix}.$$

Moreover for $P_{Z_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned} E[(H_2(\vec{s}, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1}) = \sigma_j^b(s_1, s_2) &+ (A(s_1) + P_{b,n+1}(\vec{\xi}_{n+1})(s_1)) \\ &\times (A(s_2) + P_{b,n+1}(\vec{\xi}_{n+1})(s_2)) \end{aligned}$$

which is a generalization of (2) in Theorem 23 of [7].

By Theorem 2.2 we have the following theorem [9].

Theorem 3.3. *Let the assumptions and notations be as given in Theorem 3.1 and let* $j \in \{1, \cdots, n-1\}$*. Then for* P_{Z_n}*-a.e.* $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} &E[(H_m(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n) \\ &= \int_{\mathbb{R}^m} (u_1 + P_{b,n}(\vec{\xi}_n)(s_1)) \cdots (u_m + P_{b,n}(\vec{\xi}_n)(s_m)) \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}). \end{aligned}$$

Example 3.4. If $m = 1$, then by Theorem 3.3 we have for P_{Z_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$E[(H_1(s_1, \cdot))_Z | Z_n](\vec{\xi}_n) = A(s_1) + P_{b,n}(\vec{\xi}_n)(s_1).$$

If $m = 2$, then we have for P_{Z_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$E[(H_2(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n) = \sigma_j^b(s_1, s_2) + (A(s_1) + P_{b,n}(\vec{\xi}_n)(s_1))(A(s_2) + P_{b,n}(\vec{\xi}_n)(s_2))$$

which is a generalization of (2) in Theorem 24 of [7].

We also have the following theorem by Theorem 2.2 [9].

Theorem 3.5. *Let the assumptions and notations be as given in Theorem 3.1 and let* $t_n < s_1 < \cdots < s_m < t_{n+1} = T$*. For* $\xi_n \in \mathbb{R}$ *and* $\vec{u} = (u_1, \cdots, u_m) \in \mathbb{R}^m$ *let*

$$\begin{aligned} P_{\vec{s},\vec{u},\xi_n}(z) = &\left(u_1 + \xi_n + \frac{b(s_1) - b(t_n)}{b(T) - b(t_n)}(a(T) - a(t_n)) + \frac{b(s_1) - b(t_n)}{\sqrt{b(T) - b(t_n)}} z \right) \cdots \\ &\times \left(u_m + \xi_n + \frac{b(s_m) - b(t_n)}{b(T) - b(t_n)}(a(T) - a(t_n)) + \frac{b(s_m) - b(t_n)}{\sqrt{b(T) - b(t_n)}} z \right) \end{aligned}$$

$$= \sum_{l=0}^{m} a_l(\vec{s}, \vec{u}, \xi_n) z^l$$

for $z \in \mathbb{R}$, *where* $\vec{s} = (s_1, \cdots, s_m)$. *Then for* P_{Z_n}*-a.e.* $\vec{\xi_n} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$

$$E[(H_m(\vec{s}, \cdot))_Z | Z_n](\vec{\xi_n}) = \sum_{l=0}^{[\frac{m}{2}]} \frac{(2l)!}{2^l l!} \int_{\mathbb{R}^m} a_{2l}(\vec{s}, \vec{u}, \xi_n) \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}),$$

where $[\frac{m}{2}]$ *denotes the greatest integer less than or equal to* $\frac{m}{2}$.

Example 3.6. Let the assumptions and notations be as given in Theorem 3.5. If $m = 1$, then by Theorem 3.5 we have for P_{Z_n}-a.e. $\vec{\xi_n} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$

$$E[(H_1(s_1, \cdot))_Z | Z_n](\vec{\xi_n}) = a(s_1) - a(t_n) + \xi_n.$$

If $m = 2$, then we have for P_{Z_n}-a.e. $\vec{\xi_n} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} &E[(H_2(\vec{s}, \cdot))_Z | Z_n](\vec{\xi_n}) \\ &= \xi_n^2 + \xi_n(A(s_1) + A(s_2) + (a(T) - a(t_n))(\beta_{n+1}^b(s_1) + \beta_{n+1}^b(s_2))) + (A(s_1) \\ &\quad + \beta_{n+1}^b(s_1)(a(T) - a(t_n)))(A(s_2) + \beta_{n+1}^b(s_2)(a(T) - a(t_n))) + \beta_{n+1}^b(s_1) \\ &\quad \times \beta_{n+1}^b(s_2)(b(T) - b(t_n)) + \sigma_{n+1}^b(s_1, s_2) \end{aligned}$$

which is a generalization of (5) in Theorem 24 of [7].

Now we have the following two theorems [9].

Theorem 3.7. *Let the notations be as given in Theorem 3.1 and let*

$$H(x) = \exp\left\{ \int_0^T x(t) dm_L(t) \right\} \text{ for a.e. } x \in C[0, T].$$

Suppose that $\int_{\mathbb{R}} \exp\{T|u|\} d\varphi(u) < \infty$ *and* $E[\exp\{|\int_{t_{j-1}}^{t_j} X(\cdot, t) dm_L(t)|\}] < \infty$ *for* $j = 1, \cdots, n+1$. *Then for* $P_{Z_{n+1}}$*-a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ $E[\exp\{|\int_{t_{j-1}}^{t_j} Z(\cdot, t) dm_L(t)|\} | Z_{n+1}](\vec{\xi}_{n+1})$ *exists and*

$$E[H_Z | Z_{n+1}](\vec{\xi}_{n+1}) = \prod_{j=1}^{n+1} \left[1 + \sum_{m=1}^{\infty} \int_{\Delta_{m,j}} E[(H_m(\vec{s}, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1}) d(m_L)^m(\vec{s}) \right],$$

where $\Delta_{m,j} = \{(s_1, \cdots, s_m) : t_{j-1} < s_1 < \cdots < s_m < t_j\}$ *and* $E[(H_m(\vec{s}, \cdot))_Z | Z_{n+1}]$ *is as given in Theorem 3.1. Moreover if* $E[\exp\{|\int_{t_{j-1}}^{t_j} Z(\cdot, t) dm_L(t)|\} | Z_{n+1}](\vec{0})$ *exists for* $j = 1, \cdots, n+1$, *then* $E[H_Z | Z_{n+1}](\vec{\xi}_{n+1})$ *can be represented by*

$$E[H_Z | Z_{n+1}](\vec{\xi}_{n+1}) = \exp\left\{ \int_0^T P_{b,n+1}(\vec{\xi}_{n+1})(t) dm_L(t) \right\} \prod_{j=1}^{n+1} \left[1 + \sum_{m=1}^{\infty} \int_{\Delta_{m,j}} \int_{\mathbb{R}^m} u_1 \cdots u_m \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}) d(m_L)^m(\vec{s}) \right].$$

Theorem 3.8. *Let the notations and the first part of the assumptions in Theorem 3.7 be given. Then for P_{Z_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ $E[H_Z|Z_n](\vec{\xi}_n)$ is given by the right hand side of the first equality in Theorem 3.7 with replacing $E[(H_m(\vec{s},\cdot))_Z|Z_{n+1}](\vec{\xi}_{n+1})$ by $E[(H_m(\vec{s},\cdot))_Z|Z_n](\vec{\xi}_n)$, where the $E[(H_m(\vec{s},\cdot))_Z|Z_n](\vec{\xi}_n)$s are as given in Theorems 3.3 and 3.5. Moreover if the second part of the assumptions in Theorem 3.7 holds, then $E[H_Z|Z_n](\vec{\xi}_n)$ is given by the right hand side of the second equality in Theorem 3.7 with replacing*

$$\exp\left\{\int_0^T P_{b,n+1}(\vec{\xi}_{n+1})(t)dm_L(t)\right\}$$

by

$$\exp\left\{\int_0^{t_n} P_{b,n}(\vec{\xi}_n)(t)dm_L(t) + \frac{1}{2}(b(T)-b(t_n))(B(T)-B(t_n))^2 + (a(T)-a(t_n))(B(T)-B(t_n)) + \xi_n(T-t_n)\right\},$$

where $\frac{d}{dt}B(t) = \beta^b_{n+1}(t) = \frac{b(t)-b(t_n)}{b(T)-b(t_n)}$ and $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n)$.

Example 3.9. 1. Let $h = 1$. Then $X(x,t) = x(t)-x(0)$ so that for $j = 1,\cdots,n+1$

$$E\left[\exp\left\{\left|\int_{t_{j-1}}^{t_j} X(\cdot,t)dm_L(t)\right|\right\}\right] \le E\left[\exp\left\{T \sup_{0\le t\le T}|x(t)-x(0)|\right\}\right]$$

which is finite by Theorem 1.4 of [12]. Hence $E[H_Z|Z_{n+1}]$ and $E[H_Z|Z_n]$ are given by Theorems 3.7 and 3.8, respectively, with $b(t) = t$ and $\frac{d}{dt}B(t) = \frac{t-t_n}{T-t_n}$ for $t \in [0,T]$.

2. Let $h(u) = T-u$ for $u \in [0,T]$ and suppose that $\int_{\mathbb{R}} \exp\{M|u|\}d\varphi(u) < \infty$, where $M = \max\{T, 2T^2\}$. Then for $j = 1,\cdots,n+1$ and $x \in C[0,T]$ we have

$$\left|\int_{t_{j-1}}^{t_j} X(x,t)m_L(t)\right| \le 2T^2 \sup_{0\le t\le T}|x(t)-x(0)| + T^2|x(0)|$$

so that

$$\int_{C[0,T]} \exp\left\{\left|\int_{t_{j-1}}^{t_j} X(x,t)m_L(t)\right|\right\}dw_\varphi(x) \le \left[\int_{\mathbb{R}} \exp\{2T^2|u|\}d\varphi(u)\right]^{\frac{1}{2}} \left[\int_{C[0,T]} \exp\left\{4T^2 \sup_{0\le t\le T}|x(t)-x(0)|\right\}dw_\varphi(x)\right]^{\frac{1}{2}}$$

which is finite by Theorem 1.4 of [12]. Now $E[H_Z|Z_{n+1}]$ and $E[H_Z|Z_n]$ are given by Theorems 3.7 and 3.8, respectively, with $b(t) = \frac{1}{3}[T^3 - (T-t)^3]$ and $\frac{d}{dt}B(t) = 1 - (\frac{T-t}{T-t_n})^3$ for $t \in [0,T]$.

§ 4. Evaluation formulas for other functions

In this section, using the simple formulas in the previous section, we derive evaluation formulas for generalized conditional Wiener integrals of various functions which are interested in Feynman integration theories themselves and quantum mechanics.

We have the following theorems from Theorem 3.2 of [8] and Theorems 21, 22, 23, 24, 25 and 26 of [7].

Theorem 4.1. *Let $m \in \mathbb{N}$ and $F_m(x) = \int_0^T (x(t))^m dm_L(t)$ for $x \in C[0,T]$. Suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$. Then for $P_{Z_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$E[(F_m)_Z|Z_{n+1}](\vec{\xi}_{n+1}) = \sum_{j=1}^{n+1}\sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l l!(m-2l)!}\int_{t_{j-1}}^{t_j} (A(t) + P_{b,n+1}(\vec{\xi}_{n+1})(t))^{m-2l}(\sigma_j^b(t,t))^l dm_L(t),$$

where $[\frac{m}{2}]$ denotes the greatest integer less than or equal to $\frac{m}{2}$.

Theorem 4.2. *Let the assumptions be as given in Theorem 4.1 and for $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$ let*

$$\Xi(\vec{\xi}_n) = \sum_{j=1}^{n}\sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l l!(m-2l)!}\int_{t_{j-1}}^{t_j} (A(t) + P_{b,n}(\vec{\xi}_n)(t))^{m-2l}(\sigma_j^b(t,t))^l dm_L(t).$$

Then for P_{Z_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} E[(F_m)_Z|Z_n](\vec{\xi}_n) &= \Xi(\vec{\xi}_n) + m!\sum_{l=0}^{[\frac{m}{2}]}\sum_{k=0}^{m-2l}\sum_{p=0}^{k}\sum_{q=0}^{[\frac{p}{2}]} \frac{\xi_n^{k-p}(a(T)-a(t_n))^{p-2q}(b(T)-b(t_n))^q}{2^{l+q}l!q!(k-p)!(p-2q)!(m-2l-k)!} \\ &\times \int_{t_n}^{T} (\sigma_{n+1}^b(t,t))^l(\beta_{n+1}^b(t))^p(A(t))^{m-2l-k} dm_L(t). \end{aligned}$$

Theorem 4.3. *Let $0 < s_1 < s_2 \le T$ and let $s_1 \in [t_{l-1} - t_l]$, $s_2 \in [t_{j-1} - t_j]$. For $x \in C[0,T]$ let $G(x) = x(s_1)x(s_2)$ and suppose that $\int_{\mathbb{R}} u^2 d\varphi(u) < \infty$.*

1. *If $l \neq j$, then for $P_{Z_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$E[G_Z|Z_{n+1}](\vec{\xi}_{n+1}) = (A(s_1) + P_{b,n+1}(\vec{\xi}_{n+1})(s_1))(A(s_2) + P_{b,n+1}(\vec{\xi}_{n+1})(s_2)).$$

2. *If $l \le n$, $j \le n$ and $l \neq j$, then for P_{Z_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$*

$$E[G_Z|Z_n](\vec{\xi}_n) = (A(s_1) + P_{b,n}(\vec{\xi}_n)(s_1))(A(s_2) + P_{b,n}(\vec{\xi}_n)(s_2)).$$

3. *If $l \le n$ and $j = n+1$, then for P_{Z_n}-a.e. $\vec{\xi_n} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$*

$$E[G_Z|Z_n](\vec{\xi_n}) = (A(s_2) + \xi_n + \beta^b_{n+1}(s_2)(a(T) - a(t_n)))(A(s_1) + P_{b,n}(\vec{\xi_n})(s_1)).$$

For $j = 1, \cdots, n+1$, let $g_j = \frac{1}{\sqrt{b(t_j) - b(t_{j-1})}} \chi_{(t_{j-1}, t_j]} h$, let V be the subspace of $L_2[0,T]$ generated by $\{g_1, \cdots, g_{n+1}\}$, let $V^\perp$ be the orthogonal complement of V and $\mathcal{P}^\perp : L_2[0,T] \to V^\perp$ be the orthogonal projection. Let $\mathcal{M}(L_2[0,T])$ be the class of all complex valued Borel measures of bounded variation over $L_2[0,T]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F which for $\sigma \in \mathcal{M}(L_2[0,T])$ have the form

$$F(x) = \int_{L_2[0,T]} \exp\{i(v,x)\} d\sigma(v) \tag{4.1}$$

for w_φ-a.e. $x \in C[0,T]$. We note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra [11].

Theorem 4.4. *Let a be absolutely continuous on $[0,T]$. Let $F \in S_{w_\varphi}$ and $\sigma \in \mathcal{M}(L_2[0,T])$ be related by (4.1). Then for $P_{Z_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $E[F_Z|Z_{n+1}](\vec{\xi}_{n+1})$ is given by*

$$E[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = \int_{L_2[0,T]} \exp\{i(v, A(t) + P_{b,n+1}(\vec{\xi}_{n+1}))\} \exp\Big\{-\frac{1}{2}\|\mathcal{P}^\perp(vh)\|_2^2\Big\} d\sigma(v).$$

Theorem 4.5. *Let the assumptions be as given in Theorem 4.4 and for $\vec{\xi_n} \in \mathbb{R}^{n+1}$ let*

$$D(v, \vec{\xi_n}) = \exp\Big\{i(v, A(t) + P_{b,n}(\vec{\xi_n})) - \frac{1}{2}\|\mathcal{P}^\perp(vh)\|_2^2\Big\}.$$

Then for P_{Z_n}-a.e. $\vec{\xi_n} \in \mathbb{R}^{n+1}$, $E[F_Z|Z_n](\vec{\xi_n})$ is given by

$$E[F_Z|Z_n](\vec{\xi_n}) = \int_{L_2[0,T]} D(v, \vec{\xi_n}) \exp\Big\{i\frac{\langle vh, g_{n+1}\rangle_2 (a(T) - a(t_n))}{\sqrt{b(T) - b(t_n)}} - \frac{1}{2}\langle vh, g_{n+1}\rangle_2^2\Big\} d\sigma(v).$$

Remark. 1. $\xi_0 = 0$ in [1, 2, 3, 6, 7, 13, 14, 15] because the conditioning functions in the references have no initial distribution, that is, they have the Dirac measure $\varphi = \delta_0$ as a initial distribution. On the other hand ξ_0 in [4, 5, 9, 11] can have arbitrary real number according to the initial distribution φ.

2. Using the methods as used in [10, 20, 21] with the conditioning functions Z_n and Z_{n+1} we can obtain the conditional w_φ-integrals of the cylinder functions and the product of the cylinder functions and F given by (4.1).

3. Suppose that $\int_{\mathbb{R}} u^2 d\varphi(u) < \infty$, if necessary, in each theorem, corollary and example of this paper. Then by Lemma 2.5 of [8] both $E[X_0]$ and $Var[X_0]$ exist. Let $m_Z(t) = a(t) + E[X_0]$ and $b_Z(t) = b(t) + Var[X_0]$ for $t \in [0,T]$. Since for $t_1, t_2 \in [0,T]$

$$m_Z(t_2) - m_Z(t_1) = a(t_2) - a(t_1) \text{ and } b_Z(t_2) - b_Z(t_1) = b(t_2) - b(t_1),$$

a and b can be replaced by m_Z and b_Z, respectively, in each result of this paper.

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