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Feynman path integrals for quantum open systems

By

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Abstract

The application of infinite dimensional integration techniques of oscillatory type to the mathematical definition of Feynman path integrals is described. Recent applications to the dynamics of quantum open systems, in particular to the quantum theory of continuous measurement, are also presented.

§ 1. Introduction

Feynman integration techniques are widely used (often at an heuristic level) in many areas of theoretical physics, such as quantum field theory, nonrelativistic quantum mechanics, statistical mechanics, see e.g. [22, 25, 28, 29, 34, 36, 46, 50]. The origins of these ideas can be found the 40s [23], when R. Feynman, inspired by a suggestion by Dirac [19], proposed an alternative formulation of time evolution in quantum mechanics. Feynman's original aim was a variational Lagrangian formulation of quantum theory, reintroducing the concept of trajectory, which had been banned by the "traditional" formulation of quantum mechanics [49]. According to Feynman, the wave function ψ of a non relativistic quantum particle, namely the solution of the Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2m}\Delta\psi+V\psi \\ \psi(0,x)=\psi_0(x) \end{cases} \tag{1.1}$$

(where $\hbar$ is the reduced Planck constant, $m>0$ is the mass of the particle and $F=-\nabla V$ is an external force) should be given by a "sum over all possible histories", i.e. by a heuristic integral over the space of all possible paths γ (in the configuration space of the system) with finite energy and fixed end point of the form:

$$\psi(t,x)= \text{“}\ C\int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma\ \text{”} \tag{1.2}$$

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In formula (1.2) $S_t(\gamma)$ is the classical action of the system evaluated along the path γ:

$$S_t(\gamma) \equiv \int_0^t \left(\frac{m}{2}\|\dot\gamma(s)\|^2 - V(\gamma(s))\right) \mathrm{d}s, \tag{1.3}$$

while the symbol $\mathrm{D}\gamma$ stands for a heuristic Lebesgue "flat" measure on the space of paths and C for a normalization constant, formally $C = \int e^{\frac{i}{\hbar}\int_0^t \frac{m}{2}\|\dot\gamma(s)\|^2 \mathrm{d}s}\mathrm{D}\gamma$.

Feynman's approach to the description of quantum dynamics is particularly fascinating as it creates a connection between the classical Lagrangian description and the quantum one. Indeed it provides a quantization method, allowing, at least heuristically, to associate a quantum evolution to each classical Lagrangian. Furthermore the study of the *semiclassical limit of quantum mechanics*, i.e. the analysis of the asymptotic behavior of the solution of the Schrödinger equation (or the transition amplitudes) in the limit where the Planck constant $\hbar$ can be regarded as a small parameter converging to 0, is particularly intuitive. Indeed, according to an heuristic application of the stationary phase method to formula (1.2), when $\hbar \downarrow 0$ the paths which give the main contribution to the integral should be those that make stationary the action functional S_t (1.3). These, by Hamilton's least action principle, are exactly the classical orbits of the system. On the other hand formula (1.2), as it stands, is mathematically ill-defined: indeed neither the "infinite dimensional Lebesgue measure" $\mathrm{D}\gamma$, nor the normalization constant in front of the integral are meaningful. In fact, at the beginning of the 60's Cameron [16] proved that the heuristic *Feynman measure*, formally written as $\frac{e^{\frac{i}{\hbar}S_t(\gamma)}\mathrm{D}\gamma}{\int e^{\frac{i}{\hbar}S_t(\gamma)}\mathrm{D}\gamma}$, cannot be realized as a complex $\sigma-$additive measure with finite total variation on the space of "paths" $\gamma \in [0,t]^{\mathbb{R}^d}$, endowed with the $\sigma-$algebra generated by the *cylindrical sets* of the form

$$\{\gamma \in [0,t]^{\mathbb{R}^d} : \gamma(t_1) \in B_1, ..., \gamma(t_k) \in B_k\},$$

where $0 < t_1 < ... < t_k \leq t$ and $B_1, ..., B_k$ are Borel sets in $\mathbb{R}^d$ (see [41] for a discussion of this problem). This technical problem can be solved by changing point of view and trying to realize the "Feynman integral" in terms of a linear continuous functional on a Banach algebra of "integrable" functions, in the spirit of the Rietz-Markov theorem, that states a one to one correspondence (on suitable topological spaces X) between complex bounded measures and linear continuous functionals on $C_\infty(X)$ (the continuous functions on X vanishing at ∞). Nowadays several implementation of this program can be found in the physical and in the mathematical literature, for instance by means of analytic continuation of Wiener integrals [16, 44, 34, 47, 35, 20, 37, 43, 34], or as an infinite dimensional distribution in the framework of Hida calculus [30], via "complex Poisson measures" [38, 2], or via non standard analysis [4] or as a infinite dimensional oscillatory integral. The latter method is particularly interesting as it is the only one allowing a systematic implementation of an infinite dimensional version of the stationary phase method, as well as its application to the study of the semiclassical limit of quantum mechanics[7]. Such an approach has its roots in a couple of papers by Ito [32, 33] and was developed by S. Albeverio and R. Høegh-Krohn [7, 8], D.Elworthy and A.Truman [21], S. Albeverio and Z. Brzeźniak [1]. It is based on a generalization of the definition (and the main properties) of classical oscillatory integrals [31] to the case where the integration is performed on an infinite dimensional Hilbert space. It provides a mathematical definition of Feynman's heuristic formula (1.2) in the case where the initial datum ψ_0 and the potential V are bounded continuous function which can be written as the Fourier transform of a (complex) finite measure on $\mathbb{R}^d$ [21]. Recently more general quantum dynamical systems have been investigated by means of these functional integration techniques, see e.g. [9, 39]. Particularly interesting applications can be found in the quantum theory of open systems, a challenging area of theoretical physics which intertwines foundational issues of quantum theory (such as phenomena as *decoherence* and the process of measurement of a physical observable) and modern applications (such as, for instance, the theory of quantum computing) [22].

In the traditional formulation of quantum mechanics the continuous time evolution described by the Schrödinger equation (1.1) is valid if the quantum system is "undisturbed", i.e. if it is *isolated.* On the other hand the concept of isolated system is not realistic. We should not forget that all the informations we can have on the state of a quantum system are the result of some measurement process. According to the traditional Copenhagen formulation of quantum mechanics, when the particle interacts with the measuring apparatus, its time evolution is no longer continuous: the state of the system after the measurement is the result of a random and discontinuous change, the so-called "collapse of the wave function", which cannot be described by the Schrödinger equation. Indeed, let us consider an observables $\mathcal{A}$ represented by a self-adjoint operator A on a complex separable Hilbert space $\mathcal{H}$, whose unitary vectors represent the states of the system, and let us assume for simplicity that A is bounded and its spectrum is discrete. Let $\{a_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ and $\{\psi_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ be the corresponding eigenvalues and eigenvectors. According to the traditional mathematical formulation by von Neumann the consequences of the measurement are:

1. the *decoherence* of the state of the quantum system: because of the interaction with the measuring apparatus an initial pure state ψ becomes a mixed state, described by the density operator $\rho^{prior}(t) = \sum_i w_i P_{\psi_i}$, where P_{ψ_i} denotes the projector operator onto the eigenspace which is spanned by the vector ψ_i and $w_i = |\langle\psi_i, \psi\rangle|^2$. Considering another observable $\mathcal{B}$ (represented by a bounded self-adjoint operator B), its expectation value at time t, after the measurement of the observable $\mathcal{A}$ (but without the information of the result of the measurement of $\mathcal{A}$), is given by

$$\mathbb{E}(B)_t^{prior} = \mathrm{Tr}[\rho^{prior}(t)B]$$

The transformation mapping ψ to the so-called "prior state" $\rho^{prior}(t)$ is named "prior dynamics" or non selective dynamics.

2. The so-called "collapse of the wave function": after the reading of the result of the measurement of $\mathcal{A}$ (i.e. the real number a_i) the state of the system is the corresponding eigenstate of the measured observable:

$$\rho(t)^{post}_{a_i} = P_{\psi_i}.$$

The expectation value of another observable $\mathcal{B}$ of the system at time t (taking into account the information about the value of the measurement of A) is given by:

$$\mathbb{E}^{post}(B|A = a_i)_t = \mathrm{Tr}[\rho^{post}_{a_i}(t)B] = \langle\psi_i, B\psi_i\rangle$$

The transformation mapping the initial state ψ to one of the so-called "posterior states" $\rho^{post}_{a_i}(t)$ is called "posterior dynamics" or selective dynamics and depends on the result a_i of the measurement of $\mathcal{A}$.

As it is suggested by the collapse of the wave function, the non selective dynamics maps pure states to mixed states, while the selective one maps pure states to pure states. The relation between the posterior state and the prior state is given by:

$$\rho^{prior}(t) = \sum_i P(A = a_i)\rho^{post}_{a_i}(t)$$

where $P(A = a_i)$ is the probability that the outcome of the measurement of $\mathcal{A}$ is the eigenvalue a_i and it is given by

$$P(A = a_i) = |\langle\psi_i, \psi\rangle|^2.$$

We remark that

$$\mathbb{E}(B)^{prior}_t = \sum_i \mathbb{E}^{post}(B|A = a_i)P(A = a_i), \tag{1.4}$$

This kind of phenomena, in particular the prior and posterior dynamics, cannot be described by the traditional Schrödinger equation (1.1).

There are several efforts to include the process of measurement into the traditional quantum theory and to deduce from its laws, instead of postulating, both the process of decoherence (see point 1) and the collapse of the wave function (point 2). In particular the aim of the *quantum theory of measurement* is a description of the process of measurement taking into account the properties of the measuring apparatus, which is handled as a quantum system, and its interaction with the system submitted to the measurement [17]. Even if also this approach is not completely satisfactory (also in this case one has to postulate the collapse of the state of the compound system "measuring apparatus plus observed system" or, more generally, "system plus environment") it is able to give a better description of the process of measurement.

An example of this approach is for instance the paper by Caldeira and Legget [15], where the Lindblad equation for the evolution of the density operator ρ, describing the process of decoherence (i.e. the prior dynamics) is heuristically derived:

$$\frac{\partial}{\partial t}\rho^{prior} = \frac{1}{i\hbar}[H, \rho^{prior}] - \frac{\eta k T}{\hbar^2}[x, [x, \rho^{prior}]]. \tag{1.5}$$

The authors show how equation (1.5) is a consequence of the interaction of the system with a ensemble of oscillators that model, for instance, the normal modes of an electromagnetic field or the vibrations of the atoms in a crystal. H is the Hamiltonian of the system, k is Boltzmann constant, T is the temperature of the crystal and η is a damping constant.

Feynman formulation of quantum dynamics can provide an heuristic but really intuitive description of quantum open systems. For instance, the functional integral description of the prior dynamics of a system interacting with an external environment has been proposed by Feynman and Vernon in [26]. Let denote by ρ_S and ρ_E the initial density matrices of the system and of the environment and by S_S and S_E the action functionals of the system and of the environment respectively. Let S_I be the contribution to the total action due to the interaction between the system and the environment. Then the kernel of the reduced density operator of the system ρ^{prior} (obtained by tracing out the environmental coordinates) is heuristically given by

$$\rho^{prior}(t,x,y) = \text{``} \int_{\substack{\gamma(t)=x \\ \gamma'(t)=y}} e^{\frac{i}{\hbar}(S_S(\gamma)-S_S(\gamma'))} F(\gamma,\gamma')\rho_S(\gamma(0),\gamma'(0)) \mathrm{D}\gamma \mathrm{D}\gamma' \text{''}, \tag{1.6}$$

where F is the formal *influence functional* (shortly IF)

$$\begin{aligned} &F(\gamma,\gamma') \\ &= \text{``} \int_{\substack{\Gamma(t)=R \\ \Gamma'(t)=R}} e^{\frac{i}{\hbar}(S_E(\Gamma)-S_E(\Gamma'))} e^{\frac{i}{\hbar}(S_I(\Gamma,\gamma)-S_I(\Gamma',\gamma'))} \rho_E(\Gamma(0),\Gamma'(0)) \mathrm{D}\Gamma \mathrm{D}\Gamma' \mathrm{d}R \text{''}. \end{aligned} \tag{1.7}$$

By construction the influence functional can be regarded as a correction of the isolated dynamics of the system S which depends explicitly on the model of the environment, as well as on the form of the interaction between them. The IF formalism has been applied to the description of Markovian open quantum systems and in the study of quantum computing, where the implementation of real quantum processors is often hampered by quantum decoherence phenomena (see, e.g., [48, 12, 45], and references therein).

Another heuristic path integral formula describing the time evolution of a quantum system submitted to the continuous (unsharp) measurement of one of its observables has been proposed in [42]. From a physical point of view, the study of this problem is particularly interesting, as it involves phenomena such as the the so-called *Zeno effect*, which seems to forbid a satisfactory description of continuous measurements. Indeed if a sequence of "ideal"[1] measurements of an observable $\mathcal{A}$ (with discrete spectrum) is performed and the time interval between two measurements is sufficiently small, then the observed system does not evolve. In other words a particle whose position is continuously monitored cannot move. This result is in apparent contrast with the experience: indeed in a bubble chamber repeated measurements of the position of microscopical particles are performed without "freezing" their state. This result can be obtained by means of unsharp (fuzzy) continuous measurements. For instance, in the case where the position X of a quantum particle is continuously monitored giving a readout $[\omega] = \{\omega(s)\}_{s\in[0,t]}$, then the posterior (selective) dynamics of the system can be expressed by means or the *restricted path integral* heuristic formula[42]:

$$\psi^{post}(t,x;[\omega]) = \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)} e^{-k\int_0^t(\gamma(s)-\omega(s))^2ds}\psi_0(\gamma(0))D\gamma \tag{1.8}$$

where $k \in \mathbb{R}^+$ is a constant proportional to the accuracy of the measurement. Formally, according to formula (1.8) which contains the correction term $e^{-k\int_0^t(\gamma(s)-\omega(s))^2ds}$, the paths γ which give the main contribution to the integral are those closer to the observed trajectory $[\omega]$. We point out that the restricted path integral formalism, although just heuristic (at least at this level), is rather general as formula (1.8) does not depend on the explicit form of the environment (or the measuring apparatus) and of the interaction Hamiltonian.

An alternative mathematical description of the same physical phenomenon can be given by means of a class stochastic Schrödinger equations[14, 10, 11, 18, 27]. We consider in particular Belavkin equation, a stochastic Schrödinger equation describing the selective dynamics of a $d-$dimensional particle submitted to the measurement of one of its (possible $M-$dimensional vector) observables, described by the self-adjoint operator R on $L^2(\mathbb{R}^d)$

$$\begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt - \frac{\lambda}{2}R^2\psi(t,x)dt + \sqrt{\lambda}R\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x)\in[0,T]\times\mathbb{R}^d \end{cases} \tag{1.9}$$

where H is the quantum mechanical Hamiltonian, W is an $M-$dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $dW(t)$ is the Ito differential and $\lambda > 0$ is a coupling constant, which is proportional to the accuracy of the measurement. In the

[1] A measurement is called *ideal* if the correlation between the state of the measuring apparatus and the state of the system after the measurement is maximal

particular case of the description of the continuous measurement of position one has $R = x$, so that equation (1.9) assumes the following form:

$$
(1.10) \qquad \begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt - \frac{\lambda}{2}x^2\psi(t,x)dt + \sqrt{\lambda}x\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x) \in [0,T] \times \mathbb{R}^d, \end{cases}
$$

while in the case of momentum measurement [6], ($R = -i\hbar\nabla$) one has:

$$
(1.11) \qquad \begin{cases} d\psi(t,x) = -\frac{i}{\hbar}H\psi(t,x)dt + \frac{\lambda\hbar^2}{2}\Delta\psi(t,x)dt - i\sqrt{\lambda}\hbar\nabla\psi(t,x)dW(t) \\ \psi(0,x) = \psi_0(x) \qquad (t,x) \in [0,T] \times \mathbb{R}^d. \end{cases}
$$

The aim of the present paper is the rigorous mathematical realization of the heuristic Feynman path integral formulae (1.6) and (1.8) in terms of infinite dimensional oscillatory integration techniques. In section (2) we recall the definition and the main properties of infinite dimensional oscillatory integrals. In section 3 we apply these techniques to the mathematical definition of the Feynman-Vernon influence functional in the case of the Caldeira-Leggett model. In section 4 we construct a functional integral representation for the solution of the stochastic Schrödinger equation (1.10) of the "restricted path integral" type, i.e. similar to the heuristic Mensky formula (1.8).

§ 2. Infinite dimensional oscillatory integrals

In this section we give the definition of infinite dimensional oscillatory integral and prove some important properties which will be used in the study of the time evolution of a quantum system.

In the following we shall denote by $\mathcal{H}$ a (finite or infinite dimensional) real separable Hilbert space, whose elements will be denoted by $x, y \in \mathcal{H}$ and the scalar product by $\langle x, y\rangle$; $f : \mathcal{H} \to \mathbb{C}$ will be a Borel function on $\mathcal{H}$ and $L : \mathcal{D}(L) \subseteq \mathcal{H} \to \mathcal{H}$ an invertible, densely defined, and self-adjoint operator.

The study of oscillatory integrals on $\mathbb{R}^n$ with quadratic phase functions, called "Fresnel integrals",

$$
(2.1) \qquad \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)\mathrm{d}x, \qquad \hbar > 0,
$$

is a largely developed topic, and has strong connections with several problems in mathematics, for instance in the theory of Fourier integral operators, and physics, for instance in optics. Following Hörmander [31], the integral (2.1) can be defined even if f is not summable, by exploiting the cancellations due to the oscillatory behavior of the integrand, via a limiting procedure. More precisely, Fresnel integrals can be defined as the limit of a sequence of regularized, hence absolutely convergent, Lebesgue integrals.

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called *Fresnel integrable* if for each Schwartz test function $\phi \in \mathcal{S}(\mathbb{R}^n)$, such that $\phi(0) = 1$, the limit

$$\lim_{\epsilon \to 0} (2\pi i\hbar)^{-n/2} \int e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)\phi(\epsilon x)\mathrm{d}x$$

exists and is independent of ϕ. In this case the limit is called *Fresnel integral* of f and is denoted by

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)\mathrm{d}x\,.$$

In [21] this definition is generalized to the case where $\mathbb{R}^n$ is replaced by an infinite dimensional real separable Hilbert space $(\mathcal{H}, \langle\, , \,\rangle)$. More precisely, an *infinite dimensional Fresnel integral* can be defined as the limit of a sequence of finite dimensional approximations.

Definition 2.2. A function $f : \mathcal{H} \to \mathbb{C}$ is said *Fresnel integrable* if for any sequence $\{P_n\}_{n\in\mathbb{N}}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leqslant P_{n+1}$ and $P_n \to 1$ strongly as $n \to \infty$ (1 being the identity operator in $\mathcal{H}$), the finite dimensional approximations

$$\widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_n x, P_n x\rangle} f(P_n x)\mathrm{d}(P_n x)$$

are well defined (in the sense of Definition 2.1) and the limit

$$\lim_{n\to\infty} \widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_n x, P_n x\rangle} f(P_n x)\mathrm{d}(P_n x)$$

exists and is independent of $\{P_n\}$.
In this case the limit is called *Fresnel integral* of f and is denoted by

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)\mathrm{d}x.$$

A complete "direct description" of the largest class of Fresnel integrable functions is still missing, even in finite dimension. However, it is possible to find some interesting subsets of it.

Let us denote by $\mathcal{M}(\mathcal{H})$ the Banach space of the complex bounded variation measures μ on $\mathcal{H}$, endowed with the total variation norm

$$\|\mu\| = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\cup_i E_i = \mathcal{H}$. The space $\mathcal{M}(\mathcal{H})$ is a Banach algebra, where the product of two measures $\mu * \nu$ is by definition their convolution:

$$\mu * \nu(E) = \int_{\mathcal{H}} \mu(E - x)\mathrm{d}\nu(x), \qquad \mu, \nu \in \mathcal{M}(\mathcal{H}),$$

and the unit element is the vector δ_0.

Let $\mathcal{F}(\mathcal{H})$ be the space of complex functions on $\mathcal{H}$ which are Fourier transforms of measures belonging to $\mathcal{M}(\mathcal{H})$, namely functions $f : \mathcal{H} \to \mathbb{C}$ of the form

$$f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle}\mathrm{d}\mu_f(y) \equiv \hat{\mu}_f(x), \qquad x \in \mathcal{H},$$

for some $\mu_f \in \mathcal{M}(\mathcal{H})$. The set $\mathcal{F}(\mathcal{H})$ is a Banach algebra of functions, where the product is the pointwise one, the unit element is the function 1, i.e. $1(x) = 1\ \forall x \in \mathcal{H}$, and the norm is given by $\|f\| = \|\mu_f\|$. The following result holds.

Theorem 2.3. *Let $L : \mathcal{H} \to \mathcal{H}$ be a selfadjoint trace-class operator, such that $(I-L)$ is invertible. Let $y \in \mathcal{H}$ and let $f : \mathcal{H} \to \mathbb{C}$ be the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}$. Then the function $g : \mathcal{H} \to \mathbb{C}$ defined by*

$$g(x) = e^{-\frac{i}{2\hbar}\langle x,Lx\rangle}e^{i\langle x,y\rangle}f(x), \qquad x \in \mathcal{H}$$

is Fresnel integrable and the corresponding Fresnel integral can be explicitly computed in terms of a well defined absolutely convergent one with respect to a σ-additive measure, by means of the Parseval-type equality:

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}e^{-\frac{i}{2\hbar}\langle x,Lx\rangle}e^{i\langle x,y\rangle}f(x)\mathrm{d}x$$

$$= (\det(I-L))^{-1/2}\int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle x+y,(I-L)^{-1}(x+y)\rangle}\mathrm{d}\mu_f(x)\,, \tag{2.2}$$

where $\det(I-L) = |\det(I-L)|\,e^{-\pi i\,\mathrm{Ind}(I-L)}$ *is the Fredholm determinant of the operator $I-L$,* $|\det(I-L)|$ *its absolute value and* $\mathrm{Ind}(I-L)$ *the number of negative eigenvalues of $I-L$, counted with their multiplicity.*

Proof. The result follows directly from Theorem 2.1 in [1] (see also [21]), which states that for $g \in \mathcal{F}(\mathcal{H})$

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle}e^{-\frac{i}{2\hbar}\langle x,Lx\rangle}g(x)\mathrm{d}x = \frac{1}{\sqrt{\det(I-L)}}\int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle x,(I-L)^{-1}(x)\rangle}\mathrm{d}\mu_g(x)\,,$$

by choosing $\mu_g := \delta_y * \mu_f$. □

From expression (2.2) the following result follows easily.

Corollary 2.4. *Under the assumptions of Theorem 2.3 the functional*

$$f \in \mathcal{F}(\mathcal{H}) \mapsto \widetilde{\int} e^{\frac{i}{2\hbar}\langle x,(I-L)x\rangle} e^{i\langle x,y\rangle} f(x)\mathrm{d}x$$

is continuous in the $\mathcal{F}(\mathcal{H})$-norm.

§ 3. The Feynman-Vernon influence functional

In order to handle the description of the time evolution of the density matrix of a quantum system, in this section we introduce another type of infinite dimensional oscillatory integrals on the product space $\mathcal{H} \times \mathcal{H}$ (see [3] for more details).

Definition 3.1. A function $f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is *Fresnel integrable* if for any sequence $\{P_n\}$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leqslant P_{n+1}$ and $P_n \to 1$ strongly as $n \to \infty$ (1 being the identity operator in $\mathcal{H}$), the finite dimensional oscillatory integrals

$$\widetilde{\int}_{P_n\mathcal{H}} \widetilde{\int}_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_nx,P_nx\rangle} e^{-\frac{i}{2\hbar}\langle P_ny,P_ny\rangle} f(P_nx, P_ny)\mathrm{d}(P_nx)\mathrm{d}(P_ny)$$

are well defined and the limit

$$\lim_{n\to\infty} \widetilde{\int}_{P_n\mathcal{H}} \widetilde{\int}_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_nx,P_nx\rangle} e^{-\frac{i}{2\hbar}\langle P_ny,P_ny\rangle} f(P_nx, P_ny)\mathrm{d}(P_nx)\mathrm{d}(P_ny)$$

exists and is independent of the sequence $\{P_n\}$. In this case the limit is denoted by

$$\widetilde{\int}_{\mathcal{H}} \widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2\hbar}\langle x,x\rangle} e^{-\frac{i}{2\hbar}\langle y,y\rangle} f(x, y)\mathrm{d}x\mathrm{d}y.$$

Further, the following generalization of theorem 2.3 holds.

Theorem 3.2. *Let $L : \mathcal{H} \to \mathcal{H}$ be a trace-class operator, such that $I-L$ is invertible, and let $f : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ be the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H} \times \mathcal{H}$. Then the integral*

$$\widetilde{\int}_{\mathcal{H}} \widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2\hbar}\langle x,x\rangle} e^{-\frac{i}{2\hbar}\langle y,y\rangle} e^{-\frac{i}{2\hbar}\langle x-y,L(x+y)\rangle} f(x, y)\mathrm{d}x\mathrm{d}y$$

is well defined and is equal to

$$\frac{1}{\det(I-L)} \int_{\mathcal{H}} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle x+y,(I-L)^{-1}(x-y)\rangle} \mathrm{d}\mu_f(x, y)\,,$$

where $\det(I - L)$ *is the Fredholm determinant of* $I - L$.

Given a family $\{\mu_\alpha | \alpha \in \mathbb{R}^d\}$ in $\mathcal{M}(\mathcal{H})$, we shall denote by $\int_{\mathbb{R}^d} \mu_\alpha \mathrm{d}\alpha$ the measure defined by

$$\phi \mapsto \int_{\mathbb{R}^d} \int_{\mathcal{H}} \phi(x) \mathrm{d}\mu_\alpha(x) \mathrm{d}\alpha$$

whenever it exists. The following Fubini-type theorem on the change of order of integration between oscillatory integrals and Lebesgue integrals holds.

Theorem 3.3. *Let $L : \mathcal{H} \to \mathcal{H}$ be as in the assumptions of Theorem 3.2 and let $\mu : \mathbb{R}^d \to \mathcal{M}(\mathcal{H} \times \mathcal{H})$, $\alpha \mapsto \mu_\alpha$, be a continuous map such that*

$$\int_{\mathbb{R}^d} |\mu_\alpha| \mathrm{d}\alpha < \infty.$$

Further assume that $f_\alpha(x, y) = \hat{\mu}_\alpha(x, y)$, $(x, y) \in \mathcal{H} \times \mathcal{H}$. Then $\int_{\mathbb{R}^d} f_\alpha \mathrm{d}\alpha \in \mathcal{F}(\mathcal{H} \times \mathcal{H})$ and

$$\begin{aligned} &\int_{\mathbb{R}^d} \widetilde{\int_{\mathcal{H}}} \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} e^{-\frac{i}{2\hbar}\langle y,y\rangle} e^{-\frac{i}{2\hbar}\langle x-y, L(x+y)\rangle} f_\alpha(x,y) \mathrm{d}x\mathrm{d}y\mathrm{d}\alpha \\ &= \widetilde{\int_{\mathcal{H}}} \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} e^{-\frac{i}{2\hbar}\langle y,y\rangle} e^{-\frac{i}{2\hbar}\langle x-y, L(x+y)\rangle} \int_{\mathbb{R}^d} f_\alpha(x,y) \mathrm{d}\alpha\mathrm{d}x\mathrm{d}y \,. \end{aligned} \tag{3.1}$$

For a detailed proof of these results see [3].

Let us consider now the time evolution of a quantum system made of two linearly interacting subsystems A and B. We assume that the state space of the system A is $L^2(\mathbb{R}^d)$, while the state space of the system B is $L^2(\mathbb{R}^N)$, and consider a total Hamiltonian of the compound system of the form

$$H_{AB} = H_A + H_B + H_{INT} \,,$$

with

$$\begin{aligned} H_A &= -\frac{\Delta_{\mathbb{R}^d}}{2M} + \frac{M}{2} x\Omega_A^2 x + v_A(x) \,, \qquad x \in \mathbb{R}^d \,, \\ H_B &= -\frac{\Delta_{\mathbb{R}^N}}{2m} + \frac{m}{2} R\Omega_B^2 R + v_B(R) \,, \qquad R \in \mathbb{R}^N \,, \\ H_{INT} &= xCR \,, \end{aligned}$$

where $C : \mathbb{R}^N \to \mathbb{R}^d$ is a linear operator and Ω_A (resp. Ω_B) is a symmetric positive $d \times d$ (resp. $N \times N$) matrix, $v_A : \mathbb{R}^d \to \mathbb{R}$ and $v_B : \mathbb{R}^N \to \mathbb{R}$ continuous bounded functions. We assume that the quadratic part of the total potential, i.e. the function $(x, R) \mapsto \frac{M}{2} x\Omega_A^2 x + \frac{m}{2} R\Omega_B^2 R + xCR$ is positive definite (so that the total Hamiltonian is bounded from below) and that the density matrix of the compound system factorizes as $\rho_{AB} = \rho_A\rho_B$ and has a regular kernel $\rho_{AB}(x,y,R,Q) = \rho_A(x,y)\rho_B(R,Q)$. In the following we shall denote with Ω_{AB} the $(d+N)\times(d+N)$ matrix

$$\Omega^2_{AB} = \begin{pmatrix} \Omega^2_A & C' \\ C'^T & \Omega^2_B \end{pmatrix}, \tag{3.2}$$

(where $C' = C/\sqrt{Mm}$).

Our aim is the construction of an infinite dimensional oscillatory integral representation for the reduced density operator at time t, namely

$$\int \left(e^{-\frac{i}{\hbar}H_{AB}t}\rho_{AB}e^{\frac{i}{\hbar}H_{AB}t}\right)(x,y,R,R)\mathrm{d}R.$$

Heuristically, in the Feynman-path-integral form:

$$\text{“}\int\int_{\substack{\gamma(t)=x\\ \Gamma(t)=R}}\int_{\substack{\gamma'(t)=y\\ \Gamma'(t)=R}} e^{\frac{i}{\hbar}(S_A(\gamma)+S_B(\Gamma)+S_{INT}(\gamma,\Gamma)-S_A(\gamma')-S_B(\Gamma')-S_{INT}(\gamma',\Gamma'))}\times \tag{3.3}$$
$$\times\,\rho_A(\gamma(0),\gamma'(0))\rho_B(\Gamma(0),\Gamma'(0))\mathrm{D}\gamma\mathrm{D}\gamma'\mathrm{D}\Gamma\mathrm{D}\Gamma'\mathrm{d}R\text{”},$$

where γ and Γ represent the generic path in the configuration space of the system and of the reservoir, respectively, and

$$\begin{aligned} &S_A(\gamma)+S_B(\Gamma)+S_{INT}(\gamma,\Gamma)\\ &=\int_0^t\left(\frac{M}{2}\dot\gamma^2(s)-\frac{M}{2}\gamma(s)\Omega^2_A\gamma(s)-v_A(\gamma(s))\right)\mathrm{d}s\\ &+\int_0^t\left(\frac{m}{2}\dot\Gamma^2(s)-\frac{m}{2}\Gamma(s)\Omega^2_B\Gamma(s)-v_B(\Gamma(s))\right)\mathrm{d}s-\int_0^t\gamma(s)C\Gamma(s)\mathrm{d}s.\end{aligned}$$

Further, formula (3.3) can be written as

$$\int_{\gamma(t)=x}\int_{\gamma'(t)=y} e^{\frac{i}{\hbar}(S_A(\gamma)-S_A(\gamma'))}F(\gamma,\gamma')\rho_A(\gamma(0),\gamma'(0))\mathrm{D}\gamma\mathrm{D}\gamma',$$

where F is the *Feynman-Vernon influence functional*, formally:

$$F(\gamma,\gamma') = \int\int_{\Gamma(t)=R}\int_{\Gamma'(t)=R} e^{\frac{i}{\hbar}(S_B(\Gamma)+S_{INT}(\gamma,\Gamma)-S_B(\Gamma')-S_{INT}(\gamma',\Gamma'))}\rho_B(\Gamma(0),\Gamma'(0))\mathrm{D}\Gamma\mathrm{D}\Gamma'\mathrm{d}R$$

Let us denote by $\mathcal{H}^d_t$ the Hilbert space of absolutely continuous paths $\gamma:[0,t]\to\mathbb{R}^d$ such that $\gamma(t)=0$ and weak derivative $\dot\gamma\in L^2([0,t],\mathbb{R}^d)$, endowed with the inner product $\langle\gamma_1,\gamma_2\rangle = \int_0^t\dot\gamma_1(s)\cdot\dot\gamma_2(s)\mathrm{d}s$.

The following theorem provides a representation of the reduced density operator of the system A in terms of an infinite dimensional oscillatory integral on $\mathcal{H}^d_t$.

Theorem 3.4. *Let ρ_0^A and ρ_0^B be two density matrix operators on $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^N)$, respectively, with regular kernels $\rho_0^A(x,x')$ and $\rho_0^B(R,R')$ admitting a decomposition into pure states of the form $\rho_0(x,y) = \sum_i p_i e_i(x) e_i^*(y)$, with $p_i > 0$, $\sum_i p_i = 1$, $\langle e_i, e_j\rangle_{L^2(\mathbb{R}^d)} = \delta_{ij}$, and $e_i(x) = \hat{\mu}_i(x)$, satisfying*

$$\sum_i p_i |\mu_i|^2 < \infty. \tag{3.4}$$

Further let $\rho_0^B \in \mathcal{S}(\mathbb{R}^N \times \mathbb{R}^N)$. Let t satisfy the inequalities

$$t \neq [(n+1/2)\pi]/\omega_j^A, \qquad n \in \mathbb{Z}, \quad j = 1\dots d, \tag{3.5}$$

$$t \neq [(n+1/2)\pi]/\omega_j^B, \qquad n \in \mathbb{Z}, \quad j = 1\dots N, \tag{3.6}$$

$$t \neq [(n+1/2)\pi]/\lambda_j, \qquad n \in \mathbb{Z}, \quad j = 1\dots d+N, \tag{3.7}$$

where ω_j^A, ω_j^B, and λ_j, are the eigenvalues of the matrices Ω_A, Ω_B and Ω_{AB}, respectively (with Ω_{AB}^2 defined by (3.2)). Let us assume moreover that t is such that the determinant of the $d \times d$ left upper block of the $n \times n$ matrix $\cos(\Omega_{AB} t)$is non-vanishing.

Then the kernel $\rho_R(t,x,y)$ of the reduced density operator of the system A evaluated at time t is given by

$$\begin{aligned}
(3.8)\qquad & \rho_R(t,x,y) \\
&= e^{-\frac{it}{2\hbar} x\Omega_A^2 x} e^{\frac{it}{2\hbar} y \Omega_A^2 y} \times \\
&\times \widetilde{\int_{\mathcal{H}_t^d}} \widetilde{\int_{\mathcal{H}_t^d}} e^{\frac{i}{2\hbar}\langle \gamma, (I_d - L_A)\gamma\rangle} e^{-\frac{i}{2\hbar}\langle \gamma', (I_d - L_A)\gamma'\rangle} \times \\
&\qquad \times e^{-\frac{i}{\hbar}\int_0^t \left(x\Omega_A^2 \gamma(s) ds - y\Omega_A^2 \gamma'(s)\right) ds} e^{-\frac{i}{\hbar}\int_0^t \left(v_A'(\gamma(s)+x) - v_A'(\gamma(s)+y)\right) ds} \times \\
&\qquad \times F(\gamma,\gamma',x,y) \rho_{0,A}'(\gamma(0)+x, \gamma'(0)+y) d\gamma d\gamma',
\end{aligned}$$

where $\rho_{0,A}'(x,y) = \rho_0^A(x/\sqrt{M}, y/\sqrt{M})$, $v_a'(x) := v_A(x/\sqrt{M})$ and $F(\gamma,\gamma',x,y)$ is the influence functional

$$\begin{aligned}
(3.9)\ & F(\gamma,\gamma',x,y) \\
&= \int_{\mathbb{R}^N} e^{-\frac{it}{\hbar} xC'R} e^{+\frac{it}{\hbar} yC'R} e^{-\frac{i}{\hbar}\int_0^t (\gamma(s) - \gamma'(s)) C' R ds} \times \\
&\times \widetilde{\int_{\mathcal{H}_t^N}} \widetilde{\int_{\mathcal{H}_t^N}} e^{\frac{i}{2\hbar}\langle \Gamma, (I_N - L_B)\Gamma\rangle} e^{-\frac{i}{2\hbar}\langle \Gamma', (I_N - L_B)\Gamma'\rangle} e^{-\frac{i}{\hbar}\langle \Gamma, L^N C'^T \gamma\rangle} e^{\frac{i}{\hbar}\langle \Gamma', L^N C'^T \gamma'\rangle} \times \\
&\qquad \times e^{-\frac{i}{\hbar}\int_0^t R\Omega_B^2 (\Gamma(s) - \Gamma'(s)) ds} e^{-\frac{i}{\hbar}\int_0^t (xC'\Gamma(s) - yC'\Gamma'(s)) ds} \times \\
&\qquad \times e^{-\frac{i}{\hbar}\int_0^t \left(v_B'(\Gamma(s)+R) - v_B'(\Gamma'(s)+R)\right) ds} \times \\
&\qquad \times \rho_{0,B}'(\Gamma(0)+R, \Gamma'(0)+R) d\Gamma d\Gamma' dR,
\end{aligned}$$

with $\rho_{0,B}'(R,Q) = \rho_0^B(R/\sqrt{m}, Q/\sqrt{m})$, $v_B(R) := v_B(R/\sqrt{m})$.

These results can be applied to the study of the Caldeira-Leggett model[15] with a finite dimensional heat bath, that is to the special case in which the heat bath is described by a finite number of oscillators and thus $v_B = 0$. Further the model presume that the environment is initially in equilibrium at temperature T, i.e. that its initial density matrix $\rho_0^B(R,Q)$ is of the form $\rho_0^B(R,Q) = \prod_{j=1}^N \rho_B^{(j)}(R_j, Q_j, 0)$, where

$$\rho_B^{(j)}(R_j,Q_j,0) = \sqrt{\frac{m\omega_j}{\pi\hbar\coth(\hbar\omega_j/2kT)}}\, e^{-\left(\frac{m\omega_j}{2\hbar\sinh(\hbar\omega_j/kT)}\left((R_j^2+Q_j^2)\cosh\frac{\hbar\omega_j}{kT}-2R_jQ_j\right)\right)}$$

with ω_j, $j=1,\dots,n$, the eigenvalues of the matrix Ω_B. For notational simplicity we put again $m = M = 1$; the general case can be handled by replacing C, v_A, ρ_0^A, and ρ_0^B with C', v'_A, $\rho'_{0,A}$, and $\rho'_{0,B}$, respectively.

Inserting the terms defined above in the general formula (3.9), the influence functional becomes

$$\begin{aligned}
&F(\gamma,\gamma',x,y)\\
&= K(x,y,t)e^{-\frac{i}{2\hbar}\langle(\gamma-\gamma'),A(\gamma+\gamma')\rangle}e^{-\frac{i}{\hbar}\langle\gamma,CL^N(I_N-L_B)^{-1}v_{C,x}\rangle}\times\\
&\times e^{\frac{i}{\hbar}\langle\gamma',CL^N(I_N-L_B)^{-1}v_{C,y}\rangle}e^{\frac{i}{2\hbar}C^T(x-y)\int_0^t\frac{\sin(\Omega_Bt)\sin(\Omega_B(t-s))}{\Omega_B^2\cos(\Omega_Bt)}C^T(\gamma(s)+\gamma'(s))ds}\times\\
&\times\int_{\mathbb{R}^N}dh'_0\tilde\rho_0^B\left(h'_0-\frac{1}{2}a,h'_0+\frac{1}{2}a\right)e^{i(h_0-\frac{a}{2})\frac{1-\cos(\Omega_Bt)}{\Omega_B^2\cos(\Omega_Bt)}C^Tx}e^{-i(h_0+\frac{a}{2})\frac{1-\cos(\Omega_Bt)}{\Omega_B^2\cos(\Omega_Bt)}C^Ty}\times\\
&\qquad\times e^{i\hbar h'_0\frac{\sin\Omega_Bt}{\Omega_B\cos\Omega_Bt}a}e^{i\langle\gamma-\gamma',CL^N(I_N-L_B)^{-1}h'_0G_0\rangle},
\end{aligned}$$

where

$$K(x,y,t) = \pi^N2^Ne^{\frac{i}{2\hbar}C^T(x-y)\left(\frac{t}{\Omega_B^2}-\frac{\sin(\Omega_Bt)}{\Omega_B^3\cos(\Omega_Bt)}\right)C^T(x-y)},$$

$$e^{-\frac{i}{2\hbar}\langle(\gamma-\gamma'),A(\gamma+\gamma')\rangle} = e^{\frac{i}{2\hbar}\int_0^tC^T(\gamma-\gamma')(s)\Omega^{-1}\int_0^s\sin(\Omega_B(s-r))C^T(\gamma+\gamma')(r)drds},$$

and

$$a = -\frac{1}{\hbar}\int_0^t\cos(\Omega_Bs)C^T(\gamma(s)-\gamma'(s))ds-\frac{1}{\hbar}(\Omega_B)^{-1}\sin(\Omega_Bt)C^T(x-y).$$

By direct computation we obtain

$$\begin{aligned}
&F(\gamma,\gamma',x,y)\\
&= e^{\frac{i}{2\hbar}\int_0^tC^T(\gamma(s)+x-\gamma'(s)-y)\Omega_B^{-1}\int_0^s\sin(\Omega_B(s-r))C^T(\gamma(r)+x+\gamma'(r)+y)drds}\times\\
&\times e^{-\frac{1}{2\hbar}\int_0^tC^T(\gamma(s)+x-\gamma'(s)-y)\Omega_B^{-1}\coth\left(\frac{\hbar\Omega_B}{2kT}\right)\int_0^s\cos(\Omega_B(s-r))C^T(\gamma(r)+x-\gamma'(r)-y)drds},
\end{aligned}$$

which yields the result heuristically derived in [26].

§ 4. The stochastic Schrödinger equation

In this section we shall prove that the solution of the stochastic Schrödinger equation (1.10) can be represented by an infinite dimensional oscillatory integral of the form:

$$(4.1)\quad \psi(t,x)=\widetilde{\int} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds-\lambda\int_0^t|\gamma(s)+x|^2ds}e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds}\, e^{\int_0^t\sqrt{\lambda}(\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma.$$

The first step is the generalization of theorem 2.3 to the case of complex valued phase functions.

Theorem 4.1. *Let $\mathcal{H}$ be a real separable Hilbert space, let $y\in\mathcal{H}$ and let L_1 and L_2 be two self-adjoint, trace class commuting operators on $\mathcal{H}$ such that $I+L_1$ is invertible and L_2 is non negative. Let moreover $f\in\mathcal{F}(\mathcal{H})$, with $f\equiv\hat\mu_f$, $\mu_f\in\mathcal{M}(\mathcal{H})$.*
Then the function $g:\mathcal{H}\to\mathbb{C}$ defined as

$$g(x)=e^{\frac{i}{2\hbar}\langle x,Lx\rangle}e^{\langle y,x\rangle}f(x)$$

(L being the operator on the complexification $\mathcal{H}^{\mathbb{C}}$ of the real Hilbert space $\mathcal{H}$ given by $L=L_1+iL_2$) is Fresnel integrable (in the sense of definition 2.2) and its infinite dimensional oscillatory integral

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L)x\rangle}e^{\langle y,x\rangle}f(x)dx$$

can be explicitly computed by means of the following Parseval type equality:

$$(4.2)\quad \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,(I+L)x\rangle}e^{\langle y,x\rangle}f(x)dx=\det(I+L)^{-1/2}\int_{\mathcal{H}}e^{\frac{-i\hbar}{2}\langle k-iy,(I+L)^{-1}(k-iy)\rangle}\mu_f(dk)$$

For a detailed proof, see [5, 40].

Let us consider the stochastic Schröedinger equation (1.10), which is convenient to write in the Stratonovich equivalent form:

$$(4.3)\quad \begin{cases} d\psi=-\frac{i}{\hbar}H\psi dt-\lambda|x|^2\psi dt+\sqrt{\lambda}x\psi\circ dW(t) \\ \psi(0,x)=\psi_0(x) \qquad\qquad t\geq 0,\ x\in\mathbb{R}^d\end{cases}$$

The existence and uniqueness of a strong solution is proved in [27]. We recall that a strong solution for the stochastic equation (4.3) is a predictable process with values in $\mathcal{H}=L^2(\mathbb{R}^d)$, such that
$\psi(t)\in D(-i/\hbar H-\lambda|x|^2)$ $\mathbf{P}$-a.s.
$\mathbf{P}\Big(\int_0^T(\|\psi(t)\|^2+\|(-i/\hbar H-\lambda|x|^2)\psi\|^2)\,dt<\infty\Big)=1$

$\mathbf{P}\Big(\int_0^T \||x|\psi(t)\,dt\|^2 < \infty\Big) = 1$ and
$\mathbf{P}$ a.s. for all $t \in [0,T]$:

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \lambda|x|^2\psi dt + \sqrt{\lambda}x\cdot\psi\circ dW(t) & t\geq 0,\ x\in\mathbb{R}^d \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{4.4}$$

Let us consider the Cameron-Martin Hilbert space H_t of absolutely continuous paths $\gamma : [0,t] \to \mathbb{R}^d$ with weak derivative $\dot\gamma \in L^2([0,t],\mathbb{R}^d)$ and such that $\gamma(t) = 0$, endowed with the inner product $\langle\gamma_1,\gamma_2\rangle = \int_0^t \dot\gamma_1(s)\cdot\dot\gamma_2(s)\mathrm{d}s$. Let $H_t^{\mathbb{C}}$ be its complexification. Let $L : H_t^{\mathbb{C}} \to H_t^{\mathbb{C}}$ the operator on $H_t^{\mathbb{C}}$ defined by

$$\langle\gamma_1, L\gamma_2\rangle = -a^2\int_0^t \gamma_1(s)\cdot\gamma_2(s)\mathrm{d}s;$$

where $a^2 = -2i\lambda\hbar$. The $j-$th component of $L\gamma$, $L\gamma = (L\gamma_1,\ldots,L\gamma_d)$, is given by

$$(L\gamma)_j(s) = 2i\lambda\hbar\int_s^t \mathrm{d}s'\int_0^{s'}\gamma_j(s'')\mathrm{d}s'' \qquad j=1,\ldots,d \tag{4.5}$$

one can verify (see [21] for more details) that $iL : H \to H$ is self-adjoint with respect to the H_t-inner product, it is trace-class and its Fredholm determinant is given by:

$$\det(I+L) = \cos(at).$$

Moreover $(I+L)$ is invertible and its inverse is given by:

$$\begin{aligned}[(I+L)^{-1}\gamma]_j(s) &= \gamma_j(s) - a\int_s^t \sin[a(s'-s)]\gamma_j(s')\mathrm{d}s' \\ &\quad + \sin[a(t-s)]\int_0^t [\cos at]^{-1}a\cos(as')\gamma_j(s')\mathrm{d}s' \qquad j=1,\ldots,d.\end{aligned}$$

Let us introduce the vector $l \in H_t$ defined by

$$\langle l,\gamma\rangle = -\sqrt{\lambda}\int_0^t \omega(s)\cdot\dot\gamma(s)\mathrm{d}s = \sqrt{\lambda}\int_0^t \gamma(s)\cdot\mathrm{d}W(s), \tag{4.6}$$

which is given by $l(s) = \sqrt{\lambda}\int_s^t \omega(\tau)\mathrm{d}\tau$.

Under suitable assumptions on the potential V and the initial wave function ψ_0, it is possible to realize the heuristic expression (4.1) in terms of the following infinite dimensional oscillatory integral on the Cameron-Martin space H_t:

$$C(t,x,\omega)\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle}e^{\langle l,\gamma\rangle}e^{-2\lambda x\cdot\int_0^t\gamma(s)\mathrm{d}s}e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)\mathrm{d}s}\psi_0(\gamma(0)+x)\mathrm{d}\gamma \tag{4.7}$$

where $C(t,x,\omega) = e^{-\lambda|x|^2+\sqrt{\lambda}x\cdot\omega(t)}$ is a constant depending on t, $x\in\mathbb{R}^d$, $\omega\in\Omega$. Indeed the integrand $\exp(\frac{i}{2\hbar}\Phi)$ in (4.1), where $\Phi(\gamma) \equiv \int_0^t|\dot\gamma(s)|^2\mathrm{d}s + 2i\hbar\lambda\int_0^t|\gamma(s)+x|^2\mathrm{d}s\,-$

$2i\hbar\int_0^t \sqrt{\lambda}(\gamma(s)+x)\cdot \mathrm{d}W(s)$ can be rigorously defined as the functional on the Cameron Martin space H_t given by $\Phi(\gamma)=\langle\gamma,(I+L)\gamma\rangle-2i\hbar\langle l,\gamma\rangle-2\hbar\int_0^t a^2x\cdot\gamma(s)\mathrm{d}s-a^2|x|^2t-2i\hbar\sqrt{\lambda}x\cdot\omega(t)$, where L is the operator (4.5) and l is the vector (4.6), see [5] for details.

Theorem 4.2. *Let $V,\psi_0\in\mathcal{F}(\mathbb{R}^d)$ be Fourier transforms of complex bounded variation measures on $\mathbb{R}^d$. Then there exist a (strong) solution of the Stratonovich stochastic differential equation (4.3) and it is given by the infinite dimensional oscillatory integral with complex phase* (4.7).

Remark. The result can be extended to general initial vectors $\psi_0\in L^2(\mathbb{R}^d)$, using the fact that $\mathcal{F}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.

Proof. The proof in divided into 3 steps.
First of all we consider the case $V\equiv 0$ and approximate the trajectory $t\to\omega(t)$ of the Wiener process by a sequence of smooth curves, i.e. we consider the sequence of functions [2] $n\int_{t-\frac{1}{n}}^t\omega(s)ds\equiv\omega_n(t)$, $n\in\mathbb{N}$. We have that $\omega_n\to\omega$ uniformly on $[0,T]$, indeed

$$\sup_{s\in[0,T]}|W_n(s)-W(s)|\to 0 \quad \text{as } n\to\infty \qquad \mathbb{P}\text{ a.s.}$$

Let us consider the sequence of approximated problems:

$$\left\{\begin{array}{l} d\psi_n=-\frac{i}{\hbar}H\psi_n dt-\lambda|x|^2\psi_n dt+\sqrt{\lambda}x\cdot\psi_n dW_n(t)\\ \psi_n(0,x)=\psi_0(x)\end{array}\right. \tag{4.8}$$

where $dW_n(t)$ is an ordinary differential, i.e. $dW_n(t)=\dot{\omega}_n(t)dt$, and we can also write:

$$\left\{\begin{array}{l} \dot{\psi}_n=-\frac{i}{\hbar}H\psi_n-\lambda|x|^2\psi_n+\sqrt{\lambda}x\cdot\psi_n\dot{\omega}_n(t)\\ \psi_n(0,x)=\psi_0(x)\end{array}\right. \tag{4.9}$$

which can be recognized as a family of Schrödinger equations, with a complex potential, labeled by the random parameter $\omega\in\Omega$. By applying theorem 4.1, in the case where $\psi_0\in\mathcal{F}(\mathbb{R}^d)$, the solution of (4.9) can be computed in terms of the infinite dimensional oscillatory integral:

$$\begin{aligned}\psi_n(t,x)&=\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t|\dot{\gamma}(s)|^2\mathrm{d}s-\lambda\int_0^t|\gamma(s)+x|^2\mathrm{d}s}e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot\dot{\omega}_n(s)\mathrm{d}s}\psi_0(\gamma(0)+x)\mathrm{d}\gamma\\ &=e^{\frac{-ia^2|x|^2t}{2\hbar}+\sqrt{\lambda}x\cdot\omega_n(t)}\widetilde{\int_{H_t}}e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle}e^{\langle l_n,\gamma\rangle}\int_{\mathbb{R}^d}e^{i\alpha\cdot x}e^{i\langle b(\alpha,x),\gamma\rangle}\tilde{\psi}_0(\alpha)d\alpha d\gamma\end{aligned}$$

where $a^2=-2i\lambda\hbar$, $l_n\in\mathcal{H}_t$ is the vector defined by $l_n(s)=\sqrt{\lambda}\int_s^t\omega_n(\tau)\mathrm{d}\tau$ and $b(\alpha,x)\in H_t$, precisely:

$$b(\alpha,x)(s)=\alpha(t-s)-\frac{xa^2}{2\hbar}(t^2-s^2)$$

[2] Here we denote, as usual, the trajectory of the Wiener process $W(t)$ as $\omega(t)$.

The second step is the proof of the convergence of the sequence of approximated solutions, namely the proof that the following equation

$$(4.10)\qquad \begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \lambda|x|^2\psi dt + \sqrt{\lambda}x\cdot\psi\circ dW(t) & t>0\\ \psi(0,x)=\psi_0(x), \quad \psi_0\in S(\mathbb{R}^d) \end{cases}$$

has a unique strong solution given by the Feynman path integral

$$\psi(t,x) = \widetilde{\int} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds - \lambda\int_0^t|\gamma(s)+x|^2ds} e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma$$

rigorously realized as the infinite dimensional oscillatory integral with complex phase on H_t

$$e^{-\lambda|x|^2+\sqrt{\lambda}x\cdot\omega(t)}\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle}e^{\langle l,\gamma\rangle}e^{-2\lambda x\cdot\int_0^t\gamma(s)ds}\psi_0(\gamma(0)+x)d\gamma$$

Moreover it can be represented by the process

$$\psi(t,x)=\int_{\mathbb{R}^d}G(t,x,y)\psi_0(y)dy$$

where

$$G(t,x,y)=\frac{1}{\sqrt{2\pi i\hbar}}\sqrt{\frac{a}{\sin(at)}}e^{\sqrt{\lambda}x\cdot\omega(t)-\frac{\sqrt{\lambda}ax}{\sin(at)}\cdot\int_0^t\cos(as)\omega(s)ds}$$

$$e^{\frac{i\hbar\lambda}{2}(-a\int_0^t\omega(s)\cdot\int_s^t\omega(s')\sin[a(s'-s)]ds'ds)}$$

$$\cdot e^{\frac{i\hbar\lambda}{2}(-a\int_0^t\sin(as)\omega(s)ds\cdot\int_0^t\cos(as)\omega(s)ds - a\cot(at)|\int_0^t\cos(as)\omega(s)ds|^2)}$$

$$e^{\frac{i}{2\hbar}\left(\cot(at)(|x|^2+|y|^2)-\frac{2x\cdot y}{\sin(at)}\right)}e^{a\sqrt{\lambda}y\cdot\frac{1}{\sin(at)}(\int_0^t\cos[a(s-t)]\omega(s)ds)}$$

(see [5] for more details).

The third step is the study of the case where a potential $V\in\mathcal{F}(\mathbb{R}^d)$ is present by means of a perturbative argument. Let us consider the infinite dimensional oscillatory integrals

$$(4.11)\quad \Theta(t,0)\psi_0(x)=\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds-\lambda\int_0^t|\gamma(s)+x|^2ds}e^{-\frac{i}{\hbar}\int_0^tV(\gamma(s)+x)ds}$$
$$\cdot e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma$$

and

(4.12)

$$\Theta_0(t,0)\psi_0(x)=\widetilde{\int_{H_t}} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds-\lambda\int_0^t|\gamma(s)+x|^2ds}e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma.$$

By means of simple computations (see [5]) it is possible to show that:

$$\Theta(t,0)\psi_0(x) = \Theta_0(t,0)\psi_0(x) - i\int_0^t \Theta(t,u)(V\Theta_0(u,0)\psi_0)(x)du \tag{4.13}$$

Now the iterative solution of the latter integral equation is the Dyson series for $\Theta(t,0)$, which coincides with the corresponding power series expansion of the solution of the stochastic Schrödinger equation, which converges strongly in $L^2(\mathbb{R}^d)$. The equality holds pointwise. On the other hand, following [27], it is possible to prove that the problem (4.10) has a strong solution that verifies (4.13) in the L^2 sense, therefore $\Theta(t,0)\psi_0$ coincides with the solution $\psi(t)$. This concludes the proof of theorem 4.2. □

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