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Change of Scale Formulas for Wiener Integrals

By

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Abstract

We survey various change of scale formulas for Wiener integrals that have been established since Cameron and Storvick first discovered in 1987. In particular, we introduce several classes of functions, for which the change of scale formula hold, of interest in Feynman integration theory and quantum mechanics.

Contents

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§ 1. Introduction and Preliminary

It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [3] and under translations [2], that is, unlike the Riemann integral it is known that

$$\int_{C_0[0,T]} F(\rho x)\, dm(x) \neq \frac{1}{\rho} \int_{C_0[0,T]} F(x)\, dm(x).$$

Cameron and Storvick [8] expressed the analytic Feynman integral on classical Wiener space as a limit of Wiener integrals. In doing so, they discovered nice change of scale formulas for Wiener integrals on classical Wiener space $(C_0[0,1], m)$ [7]. In [34, 35], Yoo and Skoug extended these results to an abstract Wiener space (B, H, ν). Moreover Yoo, Song, Kim and Chang [36, 37] established a change of scale formula for Wiener integrals of some unbounded functionals on (a product) abstract Wiener space. Recently Yoo, Kim and Kim [33] obtained a change of scale formula for a function space integral on a generalized Wiener space $C_{a,b}[0,T]$.

In this paper we survey various change of scale formulas for Wiener integrals that have been established since Cameron and Storvick. In particular, we introduce several classes of functions, for which the change of scale formula hold, of interest in Feynman integration theory and quantum mechanics.

Let $C_0[0,T]$ denote the Wiener space, that is, the space of real valued continuous functions x on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. Then $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a function F by

$$\int_{C_0[0,T]} F(x)\, dm(x).$$

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable [18] provided ρE is measurable for each $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.).

Let $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the sets of complex numbers with positive real part and the complex numbers with nonnegative real part, respectively. Let F be a complex valued measurable functional on $C_0[0,T]$ such that the Wiener integral

$$J_F(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2} x)\, dm(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, then $J_F^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0,T]}^{\text{anw}_\lambda} F(x)\, dm(x) = J_F^*(\lambda). \tag{1.1}$$

If the following limit exists for nonzero real q, then we call it the analytic Feynman integral of F over $C_0[0,T]$ with parameter q and we write

$$\int_{C_0[0,T]}^{\text{anf}_q} F(x)\, dm(x) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x)\, dm(x) \tag{1.2}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

§ 2. Change of scale formulas for Wiener integrals of functionals in $\mathcal{S}$

In this section we introduce the Cameron and Storvick's change of scale formulas for Wiener integrals. Let us begin with this section by introducing the class of functionals that we work on in this section.

Let $\mathcal{S} = \mathcal{S}(L_2[a,b])$ be the space of functionals expressible in the form

$$F(x) = \int_{L_2[a,b]} \exp\Big\{ i \int_a^b v(t)\, dx(t) \Big\}\, d\mu(v) \tag{2.1}$$

for s-almost all $x \in C_0[a,b]$, where $\mu \in \mathcal{M}(L_2[a,b])$, the class of complex measures of finite variation defined on $\mathcal{B}(L_2[a,b])$ [5].

It has been shown by Johnson [16] that the space $\mathcal{S}$ is isometrically isomorphic to the Fresnel space $\mathcal{F}(H)$ of Albeverio and Høugh-Krohn [1]. Moreover the Banach algebra $\mathcal{S}$ is a very rich class of functionals. For example, functionals of the form

$$F(x) = \exp\Big\{ \int_0^T \int_0^T f(s,t,x(s),x(t))\, ds\, dt \Big\} \tag{2.2}$$

were discussed in the book by Feynman and Hibbs [13] on path integrals, and in Feynman's original paper [12]. Chang, Johnson and Skoug showed in [9] that for appropriate $f : [0,T]^2 \times \mathbb{R}^2 \to \mathbb{C}$, functionals of the form (2.2) are known to belong to $\mathcal{S}$.

Cameron and Storvick [6] proved that functionals in $\mathcal{S}$ is analytic Wiener and analytic Feynman integrable as follows.

Theorem 2.1. *Let $F \in \mathcal{S}$ be given by* (2.1)*. Then F is analytic Wiener integrable and*

$$\int_{C_0[a,b]}^{\text{anw}_\lambda} F(x)\, dm(x) = \int_{L_2[a,b]} \exp\Big\{ -\frac{1}{2\lambda} \int_a^b (v(t))^2\, dx(t) \Big\}\, d\mu(v). \tag{2.3}$$

Moreover F is analytic Feynman integrable and

$$\int_{C_0[a,b]}^{\mathrm{anf}_q} F(x)\, dm(x) = \int_{L_2[a,b]} \exp\Big\{-\frac{i}{2q}\int_a^b (v(t))^2\, dx(t)\Big\}\, d\mu(v) \tag{2.4}$$

for every nonzero real q.

In [8], Cameron and Storvick gave relationships between Wiener integral and analytic Feynman integral for functionals in $\mathcal{S}$, that is, they expressed Feynman integral in terms of Wiener integrals.

In Theorem 2.2 below the Wiener integrals are associated with a sequence of subdivisions of the time interval $[a,b]$, while in Theorem 2.3, the Wiener integrals are associated with a complete orthonormal set of functions.

Theorem 2.2. *Let $\langle\sigma_n\rangle$ be a sequence of subdivisions of $[a,b]$, let σ_n has m_n intervals and let $\|\sigma_n\| \to 0$ as $n \to \infty$. Let $\langle\lambda_n\rangle$ be a sequence of complex numbers with* $\mathrm{Re}(\lambda_n) > 0$ *for all n such that $\lambda_n \to -iq$ as $n \to \infty$. Let $x \in C_0[a,b]$ and let x_{σ_n} be the polygonal function that equals x at the division points of σ_n and is linear and continuous between them. Then if $F \in \mathcal{S}$,*

(2.5)

$$\int_{C_0[a,b]}^{\mathrm{anf}_q} F(x)\, dm(x) = \lim_{n\to\infty} \lambda_n^{m_n/2} \int_{C_0[a,b]} \exp\Big\{\frac{1-\lambda_n}{2}\int_a^b \Big\|\frac{dx_{\sigma_n}(s)}{ds}\Big\|^2 ds\Big\} F(x)\, dm(x)$$

for each nonzero real number q.

Theorem 2.3. *Let $\langle\phi_n\rangle$ be a complete orthonormal sequence of functions on $[a,b]$. Let $F \in \mathcal{S}$. Let $\langle\lambda_n\rangle$ be a sequence of complex numbers with* $\mathrm{Re}(\lambda_n) > 0$ *for all n such that $\lambda_n \to -iq$ as $n \to \infty$. Then the analytic Feynman integral of F exists and*

(2.6)

$$\int_{C_0[a,b]}^{\mathrm{anf}_q} F(x)\, dm(x) = \lim_{n\to\infty} \lambda_n^{n/2} \int_{C_0[a,b]} \exp\Big\{\frac{1-\lambda_n}{2}\sum_{k=1}^{n}\Big(\int_a^b \phi_k(t)\, dx(t)\Big)^2\Big\} F(x)\, dm(x)$$

for each nonzero real number q.

If ρ is positive real number and we set $\lambda_n = \rho^{-2}$ for all n in Theorems 2.2 and 2.3, then we obtain the following change of scale formulas, respectively.

Theorem 2.4 (Cameron and Storvick [7]). *Let $\langle\sigma_n\rangle$ be a sequence of subdivisions of $[a,b]$ such that $\|\sigma_n\| \to 0$ as $n \to \infty$, and let m_n be the number of subintervals in σ_n. Then if $F \in \mathcal{S}$,*

(2.7)

$$\int_{C_0[a,b]} F(\rho x)\, dm(x) = \lim_{n\to\infty} \rho^{-m_n} \int_{C_0[a,b]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\int_a^b \Big\|\frac{dx_{\sigma_n}(s)}{ds}\Big\|^2 ds\Big\} F(x)\, dm(x)$$

for each $\rho > 0$.

Theorem 2.5 (Cameron and Storvick [7]). *Let $\langle\phi_n\rangle$ be a complete orthonormal sequence of functions on $[a,b]$. Then if $F \in \mathcal{S}$,*
(2.8)
$$\int_{C_0[a,b]} F(\rho x)\,dm(x) = \lim_{n\to\infty} \rho^{-n} \int_{C_0[a,b]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\Big(\int_a^b \phi_k(t)\,dx(t)\Big)^2\Big\}F(x)\,dm(x)$$

for each $\rho > 0$.

We are interested in the change of scale formula of the form (2.8) in the rest of this paper.

The space $\mathcal{S}$ is a Banach algebra and hence it is a complete linear normed space. However Johnson and Skoug have shown in [19] that it is not closed with respect to pointwise or even uniform convergence. We shall denote the closure of $\mathcal{S}$ under uniform convergence s-almost everywhere by $\mathrm{Cl}_u\mathcal{S}$. It can be seen that $\mathrm{Cl}_u\mathcal{S}$ is a Banach algebra with norm given by

$$\|F\| = \inf_B\{B : |F(x)| \le B \text{ for } s\text{-almost all } x \in C_0[a,b]\}.$$

The change of scale formulas (2.7) and (2.8) for functions in $\mathcal{S}$ can be extended to for functions in $\mathrm{Cl}_u\mathcal{S}$, indeed. For details, see [7].

The following example was given in [7], and we compute Wiener integrals of a functional under a change of scale transformation explicitly.

Example 2.6. *Let $[a,b] = [0,\pi]$ and define $\phi_j(t) = \sqrt{2/\pi}\sin jt$ for $j = 1,2,\ldots$. Then $\langle\phi_j\rangle$ is a complete orthonormal sequence on $[0,\pi]$. Define*

$$F(x) = \exp\Big\{\alpha\int_0^\pi x(t)\cos t\,dt\Big\}$$

for $x \in C_0[0,\pi]$ and α is a real or complex number. We evaluate the Wiener integrals on each side of the change of scale formula (2.8) above. The left hand side is

$$L = \int_{C_0[0,\pi]} \exp\Big\{\alpha\rho\int_0^\pi x(t)\cos t\,dt\Big\}\,dm(x).$$

Using integration by parts and Paley-Wiener-Zygmund theorem [31], we have

$$L = (2\pi)^{-1/2}\int_{\mathbb{R}} \exp\Big\{-\alpha\rho\Big(\frac{\pi}{2}\Big)^{1/2} - \frac{u^2}{2}\Big\}\,du = \exp\Big\{\frac{\alpha^2\rho^2\pi}{4}\Big\}.$$

On the other hand, consider

$$R \equiv \int_{C_0[a,b]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\Big(\int_0^\pi \phi_k(t)\,dx(t)\Big)^2 - \alpha\Big(\frac{\pi}{2}\Big)^{1/2}\int_0^\pi \phi_1(t)\,dx(t)\Big\}\,dm(x).$$

We evaluate the Wiener integral above using Paley-Wiener-Zygmund theorem to obtain

$$R = \rho^n \exp\Big\{\frac{\alpha^2\rho^2\pi}{4}\Big\}.$$

Thus we have established that the change of scale formula (2.8) *is valid for all complex number* α.

If α is pure imaginary in Example 2.6, $F \in \mathcal{S}$, so F is an example of a functional to which the change of scale formula applies. On the other hand, if $\mathrm{Re}(\alpha) \neq 0$, F is unbounded so $F \notin \mathcal{S}$ and also $F \notin \mathrm{Cl}_u\mathcal{S}$. Thus this example shows that the class of functionals for which the change of scale formula holds is more extensive than $\mathrm{Cl}_u\mathcal{S}$.

Recently, Kim, Song and Yoo [25] established the following change of scale formula for a generalized Wiener integrals for functionals in $\mathcal{S}$, that is,

$$\begin{aligned} &\int_{C_0[0,T]} F(\rho Z_h(x,\cdot))\, dm(x) \\ &= \lim_{n\to\infty}\int_{C_0[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}(\phi_k, Z_h(x,\cdot))^2\Big\}F(Z_h(x,\cdot))\, dm(x) \end{aligned} \tag{2.9}$$

for s-a.e. $y \in C_0[0,T]$, where Z_h is a Gaussian process $Z_h(x,t) = \int_0^t h(s)\,dx(s)$. They also showed that (2.9) holds for functionals of the form

$$F(x) = G(x)\Psi((\alpha_1,x),\dots,(\alpha_r,x)), \tag{2.10}$$

where $G \in \mathcal{S}$, $\Psi = \psi + \phi$ where $\psi \in L_p(\mathbb{R}^r)$ for $1 \le p < \infty$, $\alpha_k = \gamma_k/h$ with $\{\gamma_1,\dots,\gamma_r\}$ a orthonormal set in $L_2[0,T]$ and ϕ is the Fourier transform of a complex Borel measure of bounded variation on $\mathbb{R}^r$. Note that $F(x)$ need not be bounded or continuous.

Moreover Kim, Song and Yoo [26] extended (2.9) as follows. For functionals of the form (2.1) or (2.10),

$$\begin{aligned} &\int_{C_0[0,T]} F(\rho Z_h(x,\cdot)+y)\, dm(x) \\ &= \lim_{n\to\infty}\int_{C_0[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}(\phi_k, Z_h(x,\cdot))^2\Big\}F(Z_h(x,\cdot)+y)\, dm(x) \end{aligned} \tag{2.11}$$

for s-a.e. $y \in C_0[0,T]$.

§ 3. Other classes of functionals

In this section we introduce some other classes of functionals for which the change of scale formula similar to (2.8) hold. These classes are of interest in Feynman integration theory and quantum mechanics.

§ 3.1. Cylinder function

Let (B, H, ν) be the abstract Wiener space [28]. In [27] Kim established a change of scale formula for Wiener integrals of cylinder functions on B. That is, for $F(x) = f((h_1,x)^\sim, \dots, (h_n,x)^\sim)$, he proved that

$$\int_B F(\rho x)\,d\nu(x) = \rho^{-n}\int_B \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}[(h_k,x)^\sim]^2\Big\}F(x)\,d\nu(x) \tag{3.1}$$

where $\{h_1, \dots, h_n\}$ is an orthonormal set in H and $\rho > 0$.

Note that in the change of scale formula (2.8) by Cameron and Storvick, $\langle\phi_n\rangle$ may be any complete orthonormal set of functions in $L_2[0,T]$ and it requires the limiting procedure. While in the change of scale formula (3.1), although it does not require the limiting procedure but $\{h_1, \dots, h_n\}$ in the exponential of the integrand must be the same as the elements used to define the cylinder function F.

Recently Kim [23] expressed the analytic Feynman integral of cylinder function of single variable on $C_0[0,T]$ as a limit of Wiener integrals. And he obtained the original version of a change of scale formula for Wiener integral of cylinder function. Of course the change of scale formula by in [27] can be obtained as a corollary of the result in [23].

Let α be a nonzero function with $\|\alpha\| = 1$ in $L_2[0,T]$. For $1 \le p < \infty$ let $\mathcal{A}^{(p)}$ be the space of all functionals F on $C_0[0,T]$ of the form

$$F(x) = f(\langle\alpha, x\rangle)$$

for s-a.e. x in $C_0[0,T]$, where $f : \mathbb{R} \to \mathbb{R}$ is in $L_p(\mathbb{R})$ and $\langle\alpha, x\rangle$ denote the Paley-Wiener-Zygmund stochastic integral $\int_0^T \alpha(t)\,dx(t)$. Let $\mathcal{A}^{(\infty)}$ be the space of all functionals of the form $F(x) = f(\langle\alpha, x\rangle)$ with $f \in C_0(\mathbb{R})$, the space of bounded continuous functions on $\mathbb{R}$ that vanish at infinity.

Then we have the following change of scale formula for Wiener integral.

Theorem 3.1. *Let $1 \le p \le \infty$ and let $F \in \mathcal{A}^{(p)}$ be given, where $\|\alpha\| = 1$. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_2[0,T]$. Then we have*

$$\int_{C_0[0,T]} F(\rho x)\,dm(x) = \lim_{n\to\infty}\rho^{-n}\int_{C_0[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\langle\phi_k, x\rangle^2\Big\}F(x)\,dm(x) \tag{3.2}$$

for all $\rho > 0$.

If $\{\phi_1, \dots, \phi_n, \alpha\}$ is linearly dependent for some $n = 1, 2, \dots$, then we have the following corollary. In fact, Kim [27] considered in the case when $\phi_1 = \alpha$.

Corollary 3.2. *Let $1 \le p \le \infty$ and let $F \in \mathcal{A}^{(p)}$ be given, where $\|\alpha\| = 1$. Let n be a positive integer and let $\{\phi_1, \dots, \phi_n\}$ be an orthonormal set of functions in $L_2[0,T]$*

such that $\{\phi_1, \dots, \phi_n, \alpha\}$ *is linearly dependent. Then we have*

$$\int_{C_0[0,T]} F(\rho x)\, dm(x) = \rho^{-n} \int_{C_0[0,T]} \exp\Big\{\frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^{n} \langle \phi_k, x\rangle^2\Big\} F(x)\, dm(x) \tag{3.3}$$

for all $\rho > 0$.

§ 3.2. Fresnel class $\mathcal{F}(B)$ on abstract Wiener space

Let $\{e_j\}$ be a complete orthonormal system in H such that the e_j's are in B^*. For each $h \in H$ and $x \in B$, define a stochastic inner product $(h, x)^\sim$ as follows:

$$(h, x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \langle h, e_k\rangle (x, e_k), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that for every $h \in H$, $(h, x)^\sim$ exists for ν-a.e. $x \in B$ and is a Borel measurable function having a Gaussian distribution with mean zero and variance $|h|^2$. Furthermore, $(h, x)^\sim = (x, h)$ for ν-a.e $x \in B$ if $h \in B^*$.

Let $M(H)$ denote the space of complex-valued countably additive Borel measures on H. Under the total variation norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity.

The Fresnel class $\mathcal{F}(B)$ of functionals on B is defined as the space of all s-equivalence classes of functions F on B of the form

$$F(x) = \int_H \exp\{i(h, x)^\sim\}\, d\mu(h) \tag{3.4}$$

for some $\mu \in M(H)$. It is known that $\mathcal{F}(B)$ is a Banach algebra with the norm $\|F\| = \|\sigma\|$ and the mapping $\mu \to F$ is a Banach algebra isomorphism. Moreover, Kallianpur and Bromley [20] showed that every functionals in $\mathcal{F}(B)$ is analytic Wiener and analytic Feynman integrable.

Yoo and Skoug [34] showed that a change of scale formula for Wiener integrals holds for functionals in $\mathcal{F}(B)$.

Theorem 3.3. *Let* $\{e_j\}$ *be a complete orthonormal set of functions in* H. *Then for* $F \in \mathcal{F}(B)$ *we have*

$$\int_B F(\rho x)\, d\nu(x) = \lim_{n\to\infty} \rho^{-n} \int_B \exp\Big\{\frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^{n} [(e_k, x)^\sim]^2\Big\} F(x)\, d\nu(x) \tag{3.5}$$

for all $\rho > 0$.

We close this subsection by introducing two extensions of Theorem 3.3. The Banach algebra $\mathcal{F}(B)$ is not closed with respect to pointwise or even uniform convergence, and thus its closure $\mathrm{Cl}_u\mathcal{F}(B)$ with respect to uniform convergence s-a.e. is a larger space than $\mathcal{F}(B)$. We can extend the change of scale formula (3.5) for functionals in $\mathrm{Cl}_u\mathcal{F}(B)$. For details, see [34].

All functions in $\mathcal{F}(B)$ are bounded. Yoo, Song, Kim and Chang [37] established change of scale formula for Wiener integrals of functions of the form

$$F(x) = G(x)\Psi((e_1, x)^\sim, \dots, (e_n, x)^\sim), \tag{3.6}$$

where $G \in \mathcal{F}(B)$ and $\Psi = \psi + \phi$ where $\psi \in L_p(\mathbb{R}^n)$ and ϕ is the Fourier transform of a complex Borel measure of bounded variation on $\mathbb{R}^n$. Note that $F(x)$ need not be bounded or continuous.

§ 3.3. Banach algebra $\mathcal{S}(L_2(Q))$ on Yeh-Wiener space

Let $C_2(Q)$ denotes the Yeh-Wiener space, that is, the space of continuous functions x on $Q = [a,b] \times [c,d]$ such that $x(a,t) = x(s,c) = 0$ for all $(s,t) \in Q$. Let $M(L_2(Q))$ be the class of complex measures of finite variation defined on $\mathcal{B}(L_2(Q))$, the Borel class of $L_2(Q)$.

The Banach algebra $\mathcal{S}(L_2(Q))$ consists of all functionals F on $C_2(Q)$ expressible in the form

$$F(x) = \int_{L_2(Q)} \exp\Big\{ i \int_Q v(s,t)\, dx(s,t) \Big\}\, d\mu(v) \tag{3.7}$$

for s-a.e. $x \in C_2(Q)$ and for some $\mu \in M(L_2(Q))$.

Yoo and Yoon [38] established the following change of scale formula for Yeh-Wiener integral.

Theorem 3.4. *Let $\{\phi_n\}$ be a complete orthonormal sequence of functions on Q. Then for $F \in \mathcal{S}(L_2(Q))$ we have*

(3.8)

$$\int_{C_2(Q)} F(\rho x)\, dx = \lim_{n\to\infty} \rho^{-n} \int_{C_2(Q)} \exp\Big\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^{n} \Big[\int_Q \phi_k(s,t)\, dx(s,t) \Big]^2 \Big\} F(x)\, dx$$

for all $\rho > 0$.

The Banach algebra $\mathcal{S}(L_2(Q))$ of analytic Yeh-Feynman integrable functionals is not closed under the uniform convergence [38]. Hence the change of scale formula for $\mathcal{S}(L_2(Q))$ can be extended to the closure $\mathrm{Cl}_u\mathcal{S}(L_2(Q))$ of $\mathcal{S}(L_2(Q))$.

§ 3.4. Generalized Fresnel class $\mathcal{F}_{A_1,A_2}$

Let A_1 and A_2 be bounded and non negative self-adjoint operators on H. Let $\mathcal{F}_{A_1,A_2}$ be the space of all s-equivalence classes of functions on $B \times B$ which have the form

$$F(x_1,x_2) = \int_H \exp\{i[(A_1^{1/2}h,x_1)^\sim + (A_1^{1/2}h,x_1)^\sim]\}\, d\mu(h) \tag{3.9}$$

for some finite complex Borel measure μ on H. Let $M(H)$ denote the space of finite complex Borel measures μ on H. Then $M(H)$ is a Banach algebra over the complex numbers under convolution, with the norm $\|\mu\|$ equal to the total variation of μ. The map $\mu \to [F]$ sets up an algebra isomorphism between $M(H)$ and $\mathcal{F}_{A_1,A_2}$ if the range of $A_1 + A_2$ is dense in H. In this case, $\mathcal{F}_{A_1,A_2}$ becomes a Banach algebra under the norm $\|F\| = \|\mu\|$.

Let A be a bounded self-adjoint operators on H. Then $A = A^+ - A^-$, where A^+ and A^- are each bounded and non negative self-adjoint. Take $A_1 = A^+$ and $A_2 = A^-$. If A^+ is the identity and A^- is the zero operator, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(H)$ and $\mathcal{F}(B)$.

Yoo and Skoug [34] established the following change of scale formula for Wiener integrals on a product abstract Wiener space.

Theorem 3.5. *Let $\{e_n\}$ be a complete orthonormal sequence in H. Then for $F \in \mathcal{F}_{A_1,A_2}$ we have*

(3.10)

$$\begin{aligned}&\int_{B\times B} F(\rho_1 x_1, \rho_2 x_2)\, d(m\times m)(x_1,x_2)\\ &= \lim_{n\to\infty} (\rho_1\rho_2)^{-n} \int_{B\times B} \exp\Big\{\sum_{j=1}^{2}\Big(\frac{\rho_j^2-1}{2\rho_j^2}\sum_{k=1}^{n}[(e_k,x_j)^\sim]^2\Big)\Big\} F(x_1,x_2)\, d(m\times m)(x_1,x_2)\end{aligned}$$

for all $\rho_1 > 0$ and $\rho_2 > 0$.

The Banach algebra $\mathcal{F}_{A_1,A_2}$ is not closed with respect to pointwise or even uniform convergence, and thus its uniform closure $\mathrm{Cl}_u\mathcal{F}_{A_1,A_2}$ with respect to uniform convergence s-a.e. is a larger space than $\mathcal{F}_{A_1,A_2}$. Change of scale formula (3.10) for $\mathcal{F}_{A_1,A_2}$ can be extended to the closure $\mathrm{Cl}_u\mathcal{F}_{A_1,A_2}$ [34].

Yoo, Song and Kim [36] extended Theorem 3.5 for functionals of the form

$$F(x_1,x_2) = G(x_1,x_2)\Psi(X_{n_1,n_2}(x_1,x_2)), \tag{3.11}$$

where $G \in \mathcal{F}_{A_1,A_2}$, $\Psi = \psi + \phi$ where $\psi \in L_p(\mathbb{R}^{n_1+n_2})$ for $1 \le p < \infty$ and ϕ is a Fourier transform of a complex Borel measure of bounded variation on $\mathbb{R}^{n_1+n_2}$, and

$$X_{n_1,n_2}(x_1,x_2) = (X_{1,n_1}(x_1), X_{2,n_2}(x_2))$$

with $X_{j,n_j}(x_j) = ((e_{j,1}, x_j)^\sim, \dots, (e_{j,n_j}, x_j)^\sim)$ and $\{e_{j,1}, \dots, e_{j,n_j}\}$ is an orthonormal set in H for $j = 1, 2$.

§ 3.5. $\mathcal{S}''_{n,B}$ over paths in abstract Wiener space

Let $C_0(B) = C_0([0,T], B)$ denote the set of abstract Wiener space valued continuous functions on $[0,T]$ which vanish at origin. The Brownian motion in B induces a probability measure m_B on $(C_0(B), \mathcal{B}(C_0(B)))$ which is non-zero Gaussian.

Let $\Delta_n = \{(s_1, \dots, s_n) \in [0,T]^n : 0 = s_0 < s_1 < \cdots < s_n \le T\}$. Let $\mathcal{M}''_n = \mathcal{M}''_n(\Delta_n \times H^n)$ be the class of complex Borel measures on $\Delta_n \times H^n$ and let $\|\mu\| = \mathrm{var}\mu$, the total variation of $\mu \in \mathcal{M}''_n$. Let $\mathcal{S}''_{n,B} = \mathcal{S}''_{n,B}(\Delta_n \times H^n)$ be the space of functionals of the form

$$F(x) = \int_{\Delta_n \times H^n} \exp\Big\{ i \sum_{k=1}^{n} (h_k, x(s_k))^\sim \Big\}\, d\mu(\vec{s}, \vec{h}) \tag{3.12}$$

for s-a.e. x in $C_0(B)$ where $\mu \in \mathcal{M}''_n$.

Kim and Kim [24] established a relationship between Wiener integral and analytic Feynman integral for functionals in $\mathcal{S}''_{n,B}$. They also expressed analytic Wiener integral as a limit of a sequence of Wiener integrals over $C_0(B)$, and obtained a change of scale formula for Wiener integral over $C_0(B)$ of these functionals.

Theorem 3.6. *Let $F \in \mathcal{S}''_{n,B}$. Let $\{e_n\}$ be a complete orthonormal sequence in H. Then we have*

$$\int_{C_0(B)} F(\rho x)\, dm_B(x) = \lim_{m \to \infty} \rho^{-mn} \int_{C_0(B)} F_{\rho^{-2}}(x)\, dm_B(x), \tag{3.13}$$

where

$$\begin{aligned} F_{\rho^{-2}}(x) = \int_{\Delta_n \times H^n} \exp\Big\{ & \frac{\rho^2 - 1}{2\rho^2} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{[(e_j, x(s_k) - x(s_{k-1})^\sim]^2)}{s_k - s_{k-1}} \\ & + i \sum_{k=1}^{n} (h_k, x(s_k))^\sim \Big\}\, d\mu(\vec{s}, \vec{h}). \end{aligned} \tag{3.14}$$

for all $\rho > 0$.

Kim and Kim [24] also extended the result in Theorem 3.6 for functionals

$$F(x) = G(x)\psi(x(T)), \tag{3.15}$$

where G belongs to $\mathcal{S}''_{n,B}$ and ψ belong to the Fresnel class $\mathcal{F}(B)$.

§ 3.6. Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ on a function space

Let $a(t)$ be an absolutely continuous real valued function on $[0,T]$ with $a(0) = 0$ and $a'(t) \in L^2[0,T]$. Let $b(t)$ be a strictly increasing continuous differentiable real valued function with $b(0) = 0$. It is known that the probability measure μ induced by a generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$, taking a separable version, is supported by $C_{a,b}[0,T]$ [31].

Let $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ be the function space(generalized Wiener space) induced by Y. The Wiener process W on $C_0[0,T] \times [0,T]$ is free of drift and is stationary in time, while the stochastic process Y on $C_{a,b}[0,T] \times [0,T]$ is subject to a drift $a(t)$ and nonstationary in time.

Let $L^2_{a,b}[0,T]$ be the Hilbert space of continuous functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$. The Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ is the space of all s-equivalence classes of functionals F on $C_{a,b}[0,T]$ which have the form

$$F(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x\rangle\}\, df(v) \tag{3.16}$$

where the associated measure f is a complex valued countably additive Borel measures on $L^2_{a,b}[0,T]$ and $\langle v, x\rangle$ denotes the Paley-Wiener-Zygmund integral.

Chang and Skoug [10] introduced a function space integral and a generalized Feynman integral on $C_{a,b}[0,T]$. They showed that every functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ is generalized analytic Feynman integrable under some conditions on the associated measure f.

If $a(t) = 0$ and $b(t) = t$ on $[0,T]$, then the function space integral and the generalized analytic Feynman integral reduce to the Wiener integral and the analytic Feynman integral, respectively.

Yoo, Kim and Kim [33] established a relationship between the function space integral and the generalized analytic Feynman integral on $C_{a,b}[0,T]$) for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$. Moreover, they obtained a change of scale formula for a function space integral on $C_{a,b}[0,T]$ of these functionals.

Theorem 3.7. *Let $|a(t)| = cb(t)$ on $[0,T]$ for some constant $c \geq 0$. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L^2_{a,b}[0,T]$. Let $F \in \mathcal{S}(L^2_{a,b}[0,T])$. Then*

$$\begin{aligned}\int_{C_{a,b}[0,T]} F(\rho x)\, d\mu(x) = \lim_{n\to\infty} \rho^{-n} \int_{C_{a,b}[0,T]} \exp\Big\{ &\frac{\rho^2-1}{2\rho^2} \sum_{k=1}^{n} \frac{\langle \phi_k, x\rangle^2}{B_k} \\ &+ (\rho^{-1}-1) \sum_{k=1}^{n} \frac{A_k\langle \phi_k, x\rangle}{B_k} \Big\} F(x)\, d\mu(x)\end{aligned} \tag{3.17}$$

for all $\rho > 0$.

In Theorem 3.7 we require the condition that $|a(t)| = cb(t)$. Example 3.4 in [33] shows that the relationship (3.17) is not necessarily valid if $|a(t)| \neq cb(t)$. Moreover taking $a(t) = 0$ and $b(t) = t$ in Theorem 3.7, we have the change of scale formula for Wiener integrals on classical Wiener space.

§ 4. Change of scale formula for conditional Wiener integrals

Let $F : C_0[0,T] \to \mathbb{C}$ be integrable and let X be a random vector on $C_0[0,T]$. Then we have a conditional expectation $E[F|X]$ given X from a well-known probability theory. Further, there exists a P_X-integrable function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for m-a.e. $x \in C_0[0,T]$, where P_X is the probability distribution of X. The function ψ is called the conditional Wiener integral of F given X and it is denoted by $E[F|X]$.

In [29] Park and Skoug introduced a simple formula for conditional Wiener integrals which evaluate the conditional Wiener integral of a function given X_τ as a Wiener integral of the function and in [11], using this formula, they expressed the analytic Feynman integral of the functions in $\mathcal{S}$ as an integral of the conditional analytic Feynman integral of the functions.

Let $\{v_1, \ldots, v_r\}$ be an orthonormal subset of $L_2[0,T]$. For $1 \leq p \leq \infty$, let $\mathcal{A}_r^{(p)}$ be the space of all cylinder functions F_r on $C_0[0,T]$ of the form

$$F_r(x) = f(\langle v_1, x\rangle, \ldots, \langle v_r, x\rangle) \tag{4.1}$$

for s-a.e. x in $C_0[0,T]$, where $F : \mathbb{R}^r \to \mathbb{R}$ is in $L_p(\mathbb{R}^r)$. Let $\mathcal{A}_r^{(\infty)}$ be the space of all functions of the form (4.1) with $f \in L_\infty(\mathbb{R}^r)$, the space of essentially bounded functions on $\mathbb{R}^r$.

In [32] Yoo, Chang, Cho, Kim and Song established a relationship between Wiener integral and conditional analytic Feynman integral on Wiener space.

Theorem 4.1. *Let q be a nonzero real number and let $\{\lambda_n\}$ be a sequence in $\mathbb{C}_+$ with $\lambda_n \to -iq$ as $n \to \infty$. Let $G_r(x) = F(x)[F_r(x) + K_r(x)]$, where F belongs to $\mathcal{S}$, F_r belongs to $\mathcal{A}_r^{(1)}$, and $K_r(x) = \phi(\langle v_1, x\rangle, \ldots, \langle v_r, x\rangle)$ for s-a.e. $x \in C_0[0,T]$. Define $X_k : C_0[0,T] \to \mathbb{R}^k$ by $X_k(x) = ((\alpha_1, x), \ldots, (\alpha_k, x))$ for $x \in C_0[0,T]$ where $\{\alpha_1, \ldots, \alpha_k\}$ is an orthonormal subset of $L_2[0,T]$. Then we have*

$$\begin{aligned} &E^{\mathrm{anf}_q}[G_r|X_k](\vec{\xi}) \\ &= \lim_{n\to\infty} \lambda_n^{n/2} \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda_n}{2}\sum_{j=1}^{n}(e_j, x)^2\Big\} G_r(x - x_k + \vec{\xi}_k)\, dm(x) \end{aligned} \tag{4.2}$$

for a.e. $\vec{\xi} \in \mathbb{R}^k$.

The following change of scale formula for conditional Wiener integrals of possibly unbounded functions on Wiener space now follows immediately from Theorem 4.1.

Theorem 4.2. *Let F_r belong to $\mathcal{A}_r^{(p)}$ for $1 \leq p \leq \infty$ and let F belong to $\mathcal{S}$. Let $K_r(x) = \phi(\langle v_1, x\rangle, \ldots, \langle v_r, x\rangle)$ for s-a.e. $x \in C_0[0,T]$ and let $G_r(x) = F(x)[F_r(x) + K_r(x)]$. Define $X_k : C_0[0,T] \to \mathbb{R}^k$ by $X_k(x) = ((\alpha_1, x), \ldots, (\alpha_k, x))$ for $x \in C_0[0,T]$ where $\{\alpha_1, \ldots, \alpha_k\}$ is an orthonormal subset of $L_2[0,T]$. Then for a.e. $\vec{\xi} \in \mathbb{R}^k$, we have*

$$
\begin{aligned}
&E[G_r(\gamma\cdot)|X_k(\gamma\cdot)](\vec{\xi})\\
&= \lim_{n\to\infty} \gamma^{-n} \int_{C_0[0,T]} \exp\Big\{\frac{\gamma^2-1}{2\gamma^2}\sum_{j=1}^{n}(e_j,x)^2\Big\} G_r(x - x_k + \vec{\xi}_k)\, dm(x)
\end{aligned}
\tag{4.3}
$$

for all $\gamma > 0$.

§ 5. Change of scale formula for Wiener integrals related with Fourier-Feynman transform and convolution

In 1976, Cameron and Storvick [4] introduced as L_2 analytic Fourier-Feynman transform. In 1979, Johnson and Skoug [17] developed an L_p analytic Fourier-Feynman transform for $1 \leq p \leq 2$ that extended the results by Cameron and Storvick. In 1995, Huffman, Park and Skoug [14] defined a convolution product for functionals on Wiener space and showed that the Fourier-Feynman transform of the convolution product was a product of Fourier-Feynman transforms. For a detailed survey of the previous work on the Fourier-Feynman transform and related topics, see [30].

In this section, we express the Fourier-Feynman transform and convolution product of functionals in $\mathcal{S}$ as limits of Wiener integrals on Wiener space. Moreover we introduce change of scale formulas for Wiener integrals related to Fourier-Feynman transform and convolution product of these functionals.

Let F be a functional on $C_0[0,T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let

$$
T_\lambda(F)(y) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x+y)\, dm(x).
\tag{5.1}
$$

We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by ($\lambda \in \mathbb{C}_+$)

$$
T_q^{(1)}(F)(y) = \lim_{\lambda\to -iq} T_\lambda(F)(y),
\tag{5.2}
$$

for s-a.e. $y \in C_0[0,T]$, whenever this limit exist. For $1 < p < \infty$, we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F on $C_0[0,T]$ by the formula ($\lambda \in \mathbb{C}_+$)

$$
T_q^{(p)}(F)(y) = \operatorname*{l.i.m.}_{\lambda\to -iq} T_\lambda(F)(y),
\tag{5.3}
$$

whenever this limit exists; that is, for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_0[0,T]} |T_\lambda(F)(\rho x) - T_q^{(p)}(F)(\rho x)|^{p'} \, dm(x) = 0$$

where $1/p + 1/p' = 1$.

Huffman, Park and Skoug [15] established the existence of the Fourier-Feynman transform on $C_0[0,T]$ for functionals in $\mathcal{S}$.

Kim, Kim and Yoo [21] gave a relationship between analytic Fourier-Feynman transform and the Wiener integral on $C_0[0,T]$ for functionals in $\mathcal{S}$. They expressed Fourier-Feynman transform of functionals in $\mathcal{S}$ as a limit of Wiener integrals as follows.

Theorem 5.1. *Let $F \in \mathcal{S}$. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_2[0,T]$. Let q be a nonzero real number and let $\{\lambda_n\}$ be a sequence of complex numbers in $\mathbb{C}_+$ such that $\lambda_n \to -iq$. Then we have*

$$T_q^{(p)}(F)(y) = \lim_{n \to \infty} \lambda_n^{n/2} \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda_n}{2} \sum_{k=1}^{n} \langle \phi_k, x\rangle^2\Big\} F(x+y) \, dm(x) \tag{5.4}$$

for s-a.e. $y \in C_0[0,T]$.

The following change of scale formula for Wiener integral related to Fourier-Feynman transform of functionals in $\mathcal{S}$ follows from Theorem 5.1 above.

Theorem 5.2. *Let $F \in \mathcal{S}$. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_2[0,T]$. Then for each $\rho > 0$*

$$\int_{C_0[0,T]} F(\rho x + y) \, dm(x) = \lim_{n \to \infty} \rho^{-n} \int_{C_0[0,T]} \exp\Big\{\frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^{n} \langle \phi_k, x\rangle^2\Big\} F(x+y) \, dm(x) \tag{5.5}$$

for s-a.e. $y \in C_0[0,T]$.

Letting $y = 0$ is (5.5), we have the change of scale formula (2.8) for Wiener integrals on classical Wiener space.

In the following example, and we compute a Wiener integral of a functional under a change of scale transformation explicitly.

Example 5.3. *Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_2[0,T]$. Define $F(x) = \exp\{\alpha\langle\phi_1, x\rangle\}$ for $x \in C_0[0,T]$ and α is a real or complex number. We evaluate the Wiener integrals on each side of* (5.5). *The left hand side of* (5.5) *can be evaluated as follows.*

$$L \equiv \int_{C_0[0,T]} F(\rho x + y) \, dm(x) = \int_{C_0[0,T]} \exp\{\alpha\rho\langle\phi_1, x\rangle + \alpha\langle\phi_1, y\rangle\} \, dm(x).$$

By the Paley-Wiener-Zygmund theorem, we have

$$L = \exp\Big\{\frac{\alpha^2\rho^2}{2} + \alpha\langle\phi_1, y\rangle\Big\}.$$

Next we evaluate the Wiener integral on the right hand side of (5.5).

$$R \equiv \int_{C_0[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\langle\phi_k, x\rangle^2\Big\}F(x+y)\,dm(x).$$

By the Paley-Wiener-Zygmund theorem again, we have

$$R = \rho^n \exp\Big\{\frac{\alpha^2\rho^2}{2} + \alpha\langle\phi_1, y\rangle\Big\}.$$

Thus we have established that equation (5.5) *is valid for* $F(x) = \exp\{\alpha\langle\phi_1, x\rangle\}$.

Note that in Example 5.3, α is a real or complex number. If α is pure imaginary, $F \in \mathcal{S}$ and F is an example of a functional to which Theorem 5.2 applies. On the other hand, if the real part of α is not equal to 0, then F can be unbounded. Thus this example shows that the class of functionals for which (5.5) holds is more extensive than $\mathcal{S}$.

Let F and G be functionals on $C_0[0,T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, we define their convolution product $(F * G)_\lambda$ by

$$(F * G)_\lambda(y) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\,dm(x) \tag{5.6}$$

if it exists. Moreover for nonzero real number q, the convolution product $(F * G)_q$ is defined by

$$(F * G)_q(y) = \int_{C_0[0,T]}^{\mathrm{anf}_q} F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\,dm(x) \tag{5.7}$$

if it exists.

Huffman, Park and Skoug [15] established the existence of convolution product of functionals in $\mathcal{S}$.

Kim, Kim and Yoo [21] gave a relationship between convolution product and the Wiener integral on $C_0[0,T]$ for functionals in $\mathcal{S}$. They expressed convolution product of functionals in $\mathcal{S}$ as a limit of Wiener integrals as follows.

Theorem 5.4. *Let F and G be elements of $\mathcal{S}$ with associated complex Borel measures f and g, respectively. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_2[0,T]$. Let q be a nonzero real number and let $\{\lambda_n\}$ be a sequence of complex numbers* *in $\mathbb{C}_+$ such that $\lambda_n \to -iq$. Then we have*

(5.8)

$$(F*G)_q(y) = \lim_{n\to\infty} \lambda_n^{n/2} \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda_n}{2}\sum_{k=1}^{n}\langle\phi_k,x\rangle^2\Big\} F\Big(\frac{y+x}{\sqrt{2}}\Big) G\Big(\frac{y-x}{\sqrt{2}}\Big)\, dm(x)$$

for s-a.e. $y \in C_0[0,T]$.

The following change of scale formula for Wiener integral related to convolution product of functionals in $\mathcal{S}$ follows from Theorem 5.4 above.

Theorem 5.5. *Let F and G be elements of $\mathcal{S}$ with associated complex Borel measures f and g, respectively. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_2[0,T]$. Then for each $\rho > 0$*

$$\begin{aligned}(5.9)\qquad &\int_{C_0[0,T]} F\Big(\frac{y+\rho x}{\sqrt{2}}\Big) G\Big(\frac{y-\rho x}{\sqrt{2}}\Big)\, dm(x)\\ &= \lim_{n\to\infty} \rho^{-n} \int_{C_0[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\langle\phi_k,x\rangle^2\Big\} F\Big(\frac{y+x}{\sqrt{2}}\Big) G\Big(\frac{y-x}{\sqrt{2}}\Big)\, dm(x)\end{aligned}$$

for s-a.e. $y \in C_0[0,T]$.

Similar example as in Example 5.3 shows that the class of functionals for which the change of scale formula related to convolution product holds is more extensive than $\mathcal{S}$.

Recently Kim, Kim and Yoo [22] published that similar results in this section holds for functionals in a Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ on a generalized function space $C_{a,b}[0,T]$. That is, they obtained change of scale formulas for function space integrals related with generalized Fourier-Feynman transform and convolution product of these functionals.

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