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ON THE EXISTENCE OF p-ELASTIC CLOSED CURVES AND FLAT-CORE SOLUTIONS IN $\mathbf{S}^2(G)$

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1. Introduction

This paper considers a problem concerning a generalization of elastic curves in $\mathbf{S}^2(G)$, where $\mathbf{S}^2(G)$ is a 2-dimesional sphere with Gaussian curvature G. More concretely, we consider a functional including the p-th power of the absolute value of the curvature of $c \in \mathbf{S}^2(G)$. The motivation for this problem comes from [15]. In [15], we consider a similar problem in $\mathbf{R}^2$ and show the existence of rather curious solutions called "flat-core" solutions for the case $p > 2$. Here we note the concept of flat-core solution itself was introduced by Guedda-Veron [6] and recently developed by Takeuchi [13] in another context (not elastica). One of the purpose of this article is to seek whether the flat-core type solutions exist or not in $\mathbf{S}^2(G)$. Moreover, we proceed to construct various solutions other than flat-core type solutions. For the construction of these non-flat-core type solutions, precise estimate for so called "time map" is necessary. For the estimation of time-map, its differentiability (for some variable) is evidently needed. However, unfortunately we could not find relevant articles ensuring this differentiability. So, in Appendix, we would like to attempt to prove this property in detail. In the next section, we precisely describe the problem to deal with.

2. Formulation of the problem

Let $\mathcal{S}$ denote the space of C^2 closed curves in $\mathbf{S}^2(G)$:

$$\mathcal{S} = \{c \in C^2([0,1], \mathbf{S}^2(G)) \mid c_t(t) \neq 0\,(t \in [0,1]),$$
$$c(0) = c(1), c_t(0) = c_t(1), c_{tt}(0) = c_{tt}(1)\}.$$

Let $\epsilon > 0$. We say a mapping $c(w,t) : (-\epsilon, \epsilon) \times [0,1] \to \mathbf{S}^2(G)$ is a "variation" of $c \in \mathcal{S}$ if it satisfies $c(0,t) = c(t)$ and for each $w \in (-\epsilon, \epsilon)$, $c(w, \cdot)$ is an element of $\mathcal{S}$. We call

$\partial c(w,\cdot)/\partial w$ a "variational vector field" of c. Here we put some natural assumptions for a variation $c(w,\cdot)$.

Assumption 1. *Let $c(w,\cdot)$ be a variation of $c \in \mathcal{S}$. We assume for each $t \in [0,1]$, $c(\cdot,t)$ is smooth with respect to $w \in (-\epsilon,\epsilon)$ and for each $w \in (-\epsilon,\epsilon)$, $c(w,\cdot)$ is twice continuously differentiable with respect to $t \in [0,1]$.*

To state the problem to be considered, we further define the following.

Definition 1. *Let $c(t)$ ($t \in [0,1]$) be a curve in $\mathcal{S}$ and s represents its arclength parameter. We introduce the p-elastic energy E of c as:*

$$E(c) = \int_0^L |\nabla_{c_s} c_s|^p ds = \int_0^L |\kappa(s)|^p ds = \int_0^1 |\kappa(t)|^p \langle c_t, c_t \rangle_{\mathbf{R}^3}^{\frac{1}{2}} dt,$$

where κ is the curvature of c defined by Frenet-Serret formula, L is a total length of the curve c and $\langle \cdot,\cdot \rangle_{\mathbf{R}^3}$ is the standard inner product of $\mathbf{R}^3$. We consider the relaxed functional J_λ of E which coincides with the one treated in Langer-Singer [9] when $p=2$:

$$J_\lambda(c) = \int_0^L \left(|\kappa(s)|^p + \lambda\right) ds = \int_0^1 \left(|\kappa(t)|^p + \lambda\right) \langle c_t, c_t \rangle_{\mathbf{R}^3}^{\frac{1}{2}} dt \tag{1}$$

Let $c(w,t)$ ($(w,t) \in (-\epsilon,\epsilon) \times [0,1]$) be a variation of c. We say c is a "stationary curve" in $\mathcal{S}$ if the first variation of J_λ vanish at c, i.e.

$$\left.\frac{dJ_\lambda(c(w,\cdot))}{dw}\right|_{w=0} = 0$$

for all variations of c.

The purpose of this article is obtaining stationary curves as many as possible and analyzes their properties.

Remark 1. *From Theorem 1 and Lemma 3 below, we see that there exist C^2 but not C^4 stationary curves. Arroyo-Garay-Mencia [2] (see; also [7]) seek stationary curves of J_0 whose regularity is of class C^4. One of their results (Proposition 8) is that when $p > 2$, only stationary curves of J_0 are geodesics. However, by relaxing the regularity of solutions to C^2, we can find stationary curves of J_0 other than geodesics; see Proposition 3. Moreover, if $p > 2$ and $\lambda > 0$ we can find novel curious solutions, which we call flat-core solutions.*

3. Lemmas

To obtain the first variation formula of J_λ, we represent the curvature of a curve on $\mathbf{S}^2(G)$ with local coordinate. We represent a point (x,y,z) on $\mathbf{S}^2(G)$ with polar coordinate as

$$(x,y,z) = (r\sin v\cos u, r\sin v\sin u, -r\cos v), \quad (0 \le u < 2\pi, 0 \le v \le \pi), \tag{2}$$

and assume a Riemannian metric tensor which is induced from the embedding: $\mathbf{S}^2(G) \to \mathbf{R}^3$;

$$g_{uu} = r^2 \sin^2 v, \; g_{uv} = 0, \; g_{vv} = r^2. \tag{3}$$

We note

$$G = \frac{1}{r^2}$$

will be used throughout this article. Let

$$c(s) = (r \sin v(s) \cos u(s), r \sin v(s) \sin u(s), -r \cos v(s)) \tag{4}$$

be a curve on $\mathbf{S}^2(G)$ belonging to $\mathcal{S}$ and s represents an arc-length parameter, i.e. it satisfies

$$r^2 \big(\sin v(s)\big)^2 u_s(s)^2 + r^2 v_s(s)^2 = 1. \tag{5}$$

Let $\mathbf{e}_1(s) = c_s(s)$ and $\mathbf{e}_2(s)$ be its $\pi/2$ (rad) anti-clockwise rotation. Then, by Frenet-Serret formulas we have

$$\mathbf{e}_1(s) = c_s(s), \quad \nabla_{\mathbf{e}_1(s)} \mathbf{e}_1(s) = \kappa(s) \mathbf{e}_2(s). \tag{6}$$

Here $\nabla_{\mathbf{e}_1(s)}$ is the covariant derivative for the direction $\mathbf{e}_1(s)$, and the concrete expressions of $\nabla_{\mathbf{e}_1(s)} \mathbf{e}_1(s)$ and $\mathbf{e}_2(s)$ are

$$\begin{aligned} \nabla_{\mathbf{e}_1(s)} \mathbf{e}_1(s) &= c_{ss}(s) - \frac{\big\langle c_{ss}(s), c(s) \big\rangle_{\mathbf{R}^3}}{r^2} c(s) \\ &= r \begin{pmatrix} -u_s^2 \cos u \big(\cos v\big)^2 \sin v - 2u_s v_s \cos v \sin u - u_{ss} \sin u \sin v + v_{ss} \cos u \cos v \\ -u_s^2 \sin u \big(\cos v\big)^2 \sin v + 2u_s v_s \cos u \cos v + u_{ss} \cos u \sin v + v_{ss} \sin u \cos v \\ -u_s^2 \cos v \big(\sin v\big)^2 + v_{ss} \sin v \end{pmatrix}^T, \end{aligned} \tag{7}$$

$$\mathbf{e}_2(s) = \frac{c(s) \times c_s(s)}{\langle c(s) \times c_s(s), c(s) \times c_s(s) \rangle_{\mathbf{R}^3}^{\frac{1}{2}}} = r \begin{pmatrix} u_s \cos u \cos v \sin v + v_s \sin u \\ u_s \sin u \cos v \sin v - v_s \cos u \\ u_s \big(\sin v\big)^2 \end{pmatrix}^T,$$

where $\times$ is the outer product. From these, we obtain the expression of the curvature as follows:

Lemma 1. *Assume s represents an arclength parameter of $c \in \mathcal{S}$. Then the curvature κ of c is expressed as*

$$\begin{aligned} \kappa(s) &= \big\langle \nabla_{\mathbf{e}_1(s)} \mathbf{e}_1(s), \mathbf{e}_2(s) \big\rangle_{\mathbf{R}^3} \\ &= r^2 \Big(-u_{ss}(s) v_s(s) \sin v(s) + u_s(s) v_{ss}(s) \sin v(s) \\ &\qquad - 2u_s(s) v_s(s)^2 \cos v(s) - u_s(s)^3 \big(\sin v(s)\big)^2 \cos v(s) \Big). \end{aligned} \tag{8}$$

We would like to see the expression of κ of c when t does not represent its arclength. In this case, by changing the variable

$$\frac{ds}{dt} = \sqrt{r^2\big(\sin v(t)\big)^2 u_t(t)^2 + r^2 v_t(t)^2}, \tag{9}$$

the curvature of c is expressed as follows.

$$\kappa(t) = \frac{r^2\Big(-u_{tt}v_t \sin v(t) + u_t v_{tt} \sin v(t) - 2u_t v_t^2 \cos v(t) - u_t^3 \big(\sin v(t)\big)^2 \cos v(t)\Big)}{\Big\{r^2\big(\sin v(t)\big)^2 u_t^2 + r^2 v_t^2\Big\}^{\frac{3}{2}}}. \tag{10}$$

4. Main Results

Under the assumption that $|\kappa|^{p-2}\kappa$ is of class C^2 we have the following theorem.

Theorem 1. *Let $p > 1$ and $\lambda \in \mathbf{R}$. Let $(u, v) \in \mathcal{S}$ be a curve such that $0 < \inf_{t\in[0,1]} v(t)$, $\sup_{t\in[0,1]} v(t) < \pi$, and $|\kappa|^{p-2}\kappa$ is of class C^2, where κ is the curvature of (u, v). Then for each variation $c(w, t) = (u(w, t), v(w, t)) : (-\varepsilon, \varepsilon) \times [0, 1] \to \mathbf{S}^2(G)$ of (u, v) with $\varepsilon > 0$, there holds*

$$\left.\frac{dJ_\lambda(c(w,\cdot))}{dw}\right|_{w=0} = \int_0^L X(s)\sin v(s)\big(-v_s(s)u_w(0,s) + u_s(s)v_w(0,s)\big)\,ds$$

in the coordinate (2), where s is an arclength parameter of $(u(0,\cdot), v(0,\cdot))$, L is the total length of $(u(0,\cdot), v(0,\cdot))$ and

$$X(s) = p\big(|\kappa(s)|^{p-2}\kappa(s)\big)_{ss} + Gp|\kappa(s)|^{p-2}\kappa(s) + (p-1)|\kappa(s)|^p\kappa(s) - \lambda\kappa(s).$$

For ensuring the assumption that $|\kappa|^{p-2}\kappa$ is of class C^2, we introduce the auxiliary equation:

$$\begin{cases} p^2\omega_s(s)^2 = d - F(\omega(s)), & s \in \mathbf{R}, \\ \omega_s(0) = 0, \quad F(\omega(0)) = d, & \end{cases} \tag{11}$$

where

$$F(\omega) = (p-1)^2|\omega|^{\frac{2p}{p-1}} + Gp^2\omega^2 - 2\lambda(p-1)|\omega|^{\frac{p}{p-1}} \quad \text{for } \omega \in \mathbf{R}. \tag{12}$$

We show in [11], for certain triple of (p, λ, d), (11) has C^2 periodic solution. We denote the period of the solution of (11) by $T_{p;\lambda,d} = \inf\{s > 0, | \, \omega(s) = \omega(0),\, \omega_s(s) = \omega_s(0)\}$.

Lemma 2. *Assume $p > 1$, $\lambda \in \mathbf{R}$ and a triple (p, λ, d) admits C^2 periodic solution ω ($\omega \not\equiv 0$) of (11). For such ω, we define*

$$v(s) = \arccos\Big(-\frac{\sqrt{G}p}{\sqrt{d+\lambda^2}}\omega(s)\Big), \tag{13}$$

$$u_s(s) = \frac{\sqrt{G}\big(\lambda - (p-1)|\omega(s)|^{\frac{p}{p-1}}\big)}{\sqrt{d+\lambda^2}\big(1 - \frac{Gp^2}{d+\lambda^2}\omega(s)^2\big)}. \tag{14}$$

Then s represents the arclength parameter i.e. it satisfies (5).

Using Lemma 2, we obtain the following result.

Theorem 2. *Let $p>1$, $\lambda \in \mathbf{R}$ and a triple (p,λ,d) admits C^2 periodic solution ω of* (11). *Further, let ω be a solution of* (11). *If $T_{p;\lambda,d}=0$, then*

$$c(t) = (2\pi t, v_0) \quad (t \in [0,1]) \tag{15}$$

is a stationary curve of J_λ in S, where

$$v_0 = \operatorname{arccot}(-r|\omega(0)|^{\frac{2-p}{p-1}}\omega(0)), \tag{16}$$

and if $T_{p;\lambda,d}>0$, (u,v) is defined through (13) *and* (14), *and there is $m \in \mathbb{N}$ such that*

$$u(0)=u(mT_{p;\lambda,d}), \quad u_s(0)=u_s(mT_{p;\lambda,d}), \quad u_{ss}(0)=u_{ss}(mT_{p;\lambda,d}),$$
$$v(0)=v(mT_{p;\lambda,d}), \quad v_s(0)=v_s(mT_{p;\lambda,d}), \quad v_{ss}(0)=v_{ss}(mT_{p;\lambda,d}),$$

then

$$c(t) = (u(mT_{p;\lambda,d}\,t), v(mT_{p;\lambda,d}\,t)) \quad (t \in [0,1]) \tag{17}$$

is a stationary curve of J_λ in S and it does not pass either the north pole or the south pole in the coordinate (2).

Here we put

$$H(\omega) = (p-1)|\omega|^{\frac{p}{p-1}} + Gp|\omega|^{\frac{p-2}{p-1}} - \lambda. \tag{18}$$

Suppose $p>1$ and $\lambda \in \mathbf{R}$ are given constant. We see from (18) that $H(\omega)=0$ has at most two positive real roots. Assume $\omega_{1;\lambda}$ and $\omega_{2;\lambda}$ ($\omega_{2;\lambda}<\omega_{1;\lambda}$) be two positive real roots of $H(\omega)=0$ (in the case only single positive root exists, we put this $\omega_{1;\lambda}$). We enumerate the behavior of F.

Lemma 3. *For various (p,λ,d), behavior of F is classified as follows.*

(a) *The case $p>2, \lambda \le 0$ or $p=2, \lambda \le 2G$ or $1<p<2, \lambda \le 2G^{\frac{p}{2}}\left(\frac{2-p}{p-1}\right)^{\frac{p-2}{2}}$,*

ω		0	
F_ω	$-$		$+$
F		0	

TABLE 1. Behavior of F for the case (a).

(b) *The case $p>2, \lambda>0$ or $p=2, \lambda>2G$,*
(c) $1<p<2$, $2G^{\frac{p}{2}}\left(\frac{2-p}{p-1}\right)^{\frac{p-2}{2}} < \lambda \le (Gp)^{\frac{p}{2}}(p-1)^{1-p}(2-p)^{\frac{p-2}{2}}$,
(d) $1<p<2$, $\lambda > (Gp)^{\frac{p}{2}}(p-1)^{1-p}(2-p)^{\frac{p-2}{2}}$,

ω		$-\omega_{1;\lambda}$		0		$\omega_{1;\lambda}$	
F_ω	$-$		$+$		$-$		$+$
F		$-$		0		$-$	

TABLE 2. Behavior of F for the case (b).

ω		$-\omega_{1;\lambda}$		$-\omega_{2;\lambda}$		0		$\omega_{2;\lambda}$		$\omega_{1;\lambda}$	
F_ω	$-$		$+$		$-$		$+$		$-$		$+$
F		$+$ *or* 0		$+$		0		$+$		$+$ *or* 0	

TABLE 3. Behavior of F for the case (c).

ω		$-\omega_{1;\lambda}$		$-\omega_{2;\lambda}$		0		$\omega_{2;\lambda}$		$\omega_{1;\lambda}$	
F_ω	$-$		$+$		$-$		$+$		$-$		$+$
F		$-$		$+$		0		$+$		$-$	

TABLE 4. Behavior of F for the case (d).

Suppose $d \in \mathbf{R}$ is given, then from Lemma 3, we see that the equation $d - F(\omega) = 0$ has at most three positive real roots. We put these roots $\omega_{1;\lambda,d}$, $\omega_{2;\lambda,d}$ and $\omega_{3;\lambda,d}$ and assume $\omega_{1;\lambda,d} > \omega_{2;\lambda,d} > \omega_{3;\lambda,d}$ (in the case, only single root exist, we put this $\omega_{1;\lambda,d}$). We classify the solutions of (11) to five types:

(I) Constant solution $\omega \equiv \omega_0$. (ω_0 is one of 0, $\pm\omega_{1;\lambda}$ and $\pm\omega_{2;\lambda}$.)

(II) Positive periodic solution which oscillates between $\omega_{1;\lambda,d}$ and $\omega_{2;\lambda,d}$. (For the sake of completeness, if ω oscillates between $\omega_{1;\lambda,0}$ and 0, we do not call it type (II) solution, and we call it type (V) solution; see (V). Recall that if $\omega_{2;\lambda,d}$ is defined, it is positive.)

(III) Negative periodic solution which oscillates between $-\omega_{1;\lambda,d}$ and $-\omega_{2;\lambda,d}$.

(IV) Sign changing periodic solution which oscillates between $-\omega_{1;\lambda,d}$ and $\omega_{1;\lambda,d}$. (d must be positive.)

(V) Solution constructed along the following rule.
- It consists of the following solutions (Vi), (Vii) and (Viii) of (11).
- It includes at least one of (Vii) or (Viii).
- It is glued in arbitrary order.

(Vi) Constant solution $\omega \equiv 0$.

(Vii) One period of positive periodic solution ω which oscillates between $\omega_{1;\lambda,0}$ and 0. Here, one period means that the function $\omega(s)$ defined on $0 \le s \le T_{p;\lambda,d}$ and it satisfies $\omega(0) = \omega(T_{p;\lambda,d}) = 0$ and $\omega(T_{p;\lambda,d}/2) = \omega_{1;\lambda,0}$.

(Viii) One period of negative periodic solution which oscillates between $-\omega_{1;\lambda,0}$ and 0. (The meaning of one period is similar as above.)

We sometimes call type (V) solution "flat-core solution". Glued solutions such as (Vii), (Vi)-(Vii)-(Viii)-(Vi), (Vii)-(Viii), (Vii)-(Vi)-(Viii) are examples of flat-core solutions.

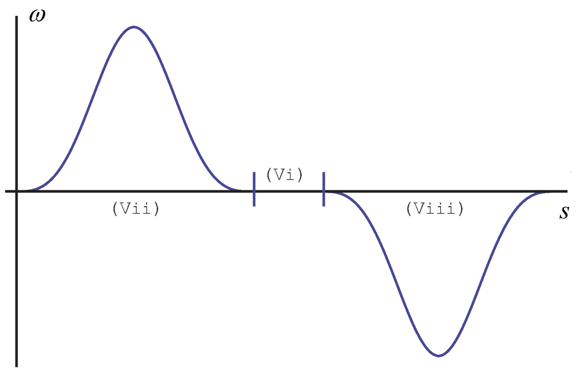

FIGURE 1. A flat-core solution of the equation (11) composed by type (Vii),(Vi),(Viii) solution.

5. Closedness of curves

Let c be a stationary curve and κ its curvature. From here we call stationary curves constructed in Theorem 2, type (I), (II), (III), (IV) and flat-core solution if $\omega = |\kappa|^{p-2}\kappa$ is a type (I), (II), (III), (IV) and (V) solution of (11).

For ensuring the existence of stationary curves, we have to show the closedness of the curve $c = (u, v)$ satisfying (13) and (14). From the expression (13), we see that v and u_s are C^2 periodic functions. Hence for the closedness, we have only to show that $u(0) = u(L)$. To show this property, we explicitly write down the period $T_{p;\lambda,d}$ of ω and angular change $\Lambda_{p;\lambda,d}$. We write $T^{(\mathrm{II})}_{p;\lambda,d}$ instead of $T_{p;\lambda,d}$ if ω is type (II), $\Lambda^{(\mathrm{IV})}_{p;\lambda,d}$ instead of $\Lambda_{p;\lambda,d}$ if ω is type (IV) and so on if any distinction is necessary. More precisely, we define

$$T^{(\mathrm{II})}_{p;\lambda,d}(=T^{(\mathrm{III})}_{p;\lambda,d}) = \int_{\omega_{2;\lambda,d}}^{\omega_{1;\lambda,d}} \frac{2p}{\sqrt{d-F(\omega)}}d\omega, \qquad T^{(\mathrm{IV})}_{p;\lambda,d} = \int_0^{\omega_{1;\lambda,d}} \frac{4p}{\sqrt{d-F(\omega)}}d\omega,$$

and

(19)
$$\begin{aligned}\Lambda^{(\mathrm{II})}_{p;\lambda,d}(=\Lambda^{(\mathrm{III})}_{p;\lambda,d}) &= \frac{2p\sqrt{G}}{\sqrt{d+\lambda^2}} \int_{\omega_{2;\lambda,d}}^{\omega_{1;\lambda,d}} \frac{\lambda-(p-1)\omega^{\frac{p}{p-1}}}{\left(1-\frac{Gp^2}{d+\lambda^2}\omega^2\right)\sqrt{d-F(\omega)}}\,d\omega,\\ \Lambda^{(\mathrm{IV})}_{p;\lambda,d} &= \frac{4p\sqrt{G}}{\sqrt{d+\lambda^2}} \int_0^{\omega_{1;\lambda,d}} \frac{\lambda-(p-1)\omega^{\frac{p}{p-1}}}{\left(1-\frac{Gp^2}{d+\lambda^2}\omega^2\right)\sqrt{d-F(\omega)}}\,d\omega.\end{aligned}$$

Here, we define $\gamma_{n,m}$ the set of stationary curves satisfying

(20)
$$\Lambda_{p;\lambda,d} = \frac{2n\pi}{m}.$$

Concretely, for $n \in \mathbf{Z}$, $m \in \mathbf{N}$ with $\gcd(n,m)=1$, we define

$$\gamma_{n,m} := \Big\{ (u,v) \mid (u,v) \text{ satisfies (13) and (14)}, \Lambda_{p;\lambda,d} = \frac{2n\pi}{m} \Big\}.$$

Suppose a triple (p,λ,d) is given and for such (p,λ,d), $c=(u,v)\in\gamma_{n,m}$. Put $L = mT_{p;\lambda,d}$, then we obtain

$$\int_0^L u_s(s)ds = \int_0^{mT_{p;\lambda,d}} u_s(s)ds = m\int_0^{T_{p;\lambda,d}} u_s(s)ds = 2n\pi. \tag{21}$$

This means the curve which belongs to $\gamma_{n,m}$ close up after m period of its curvature (given by (8)) and n trips around the small circle (or the equator).

Here for $\lambda \geq 0$, we put

$$d_\lambda = F\Big(\Big(\frac{\lambda}{p-1}\Big)^{\frac{p-1}{p}}\Big) = Gp^2\Big(\frac{\lambda}{p-1}\Big)^{\frac{2(p-1)}{p}} - \lambda^2.$$

We note that both $\sqrt{(d+\lambda^2)/(Gp^2)}$ and $(\lambda/(p-1))^{\frac{p-1}{p}}$ are roots of equations $1 - Gp^2\omega^2/(d+\lambda^2)=0$ and $(p-1)\omega^{\frac{p}{p-1}} - \lambda = 0$ respectively, which appear in the expression of $\Lambda_{p;\lambda,d}$. Next lemma asserts the order relations between $\omega_{1;\lambda,d}$, $\sqrt{(d+\lambda^2)/(Gp^2)}$ and $(\lambda/(p-1))^{\frac{p-1}{p}}$.

Lemma 4. *Let $p>1$, $\lambda\in\mathbf{R}$ and $d \geq \min_{\omega\in\mathbf{R}} F(\omega)$. Then there hold $d+\lambda^2\geq 0$,*

$$F\left(\sqrt{\frac{d+\lambda^2}{Gp^2}}\right) \geq d, \tag{22}$$

and in the case $\lambda \geq 0$,

$$F\left(\sqrt{\frac{d+\lambda^2}{Gp^2}}\right) = d \Leftrightarrow d = d_\lambda. \tag{23}$$

Moreover, if $d > \min_{\omega\in\mathbf{R}} F(\omega)$ or there exists $\omega\neq 0$ such that $F(\omega)\leq 0$, then in the case $\lambda\leq 0$, there holds

$$\omega_{1;\lambda,d} < \sqrt{\frac{d+\lambda^2}{Gp^2}}, \tag{24}$$

and in the case $\lambda>0$, there holds

$$\begin{cases} \omega_{1;\lambda,d} < \sqrt{\dfrac{d+\lambda^2}{Gp^2}} < \Big(\dfrac{\lambda}{p-1}\Big)^{\frac{p-1}{p}} & \text{in the case } d<d_\lambda, \\ \omega_{1;\lambda,d} = \sqrt{\dfrac{d+\lambda^2}{Gp^2}} = \Big(\dfrac{\lambda}{p-1}\Big)^{\frac{p-1}{p}} & \text{in the case } d=d_\lambda, \\ \Big(\dfrac{\lambda}{p-1}\Big)^{\frac{p-1}{p}} < \omega_{1;\lambda,d} < \sqrt{\dfrac{d+\lambda^2}{Gp^2}} & \text{in the case } d>d_\lambda. \end{cases} \tag{25}$$

Here we define

$$\mathcal{W} = \Big\{(\lambda, d) \in \mathbf{R}^2 \,|\, \lambda \leq 0,\, d \geq 0\Big\} \cup \Big\{(\lambda, d) \in \mathbf{R}^2 \,|\, \lambda > 0,\, \min_{\omega \in \mathbf{R}} F(\omega) \leq d < d_\lambda\Big\},$$
$$\mathcal{O} = \Big\{(\lambda, d) \in \mathbf{R}^2 \,|\, \lambda > 0,\, d > d_\lambda\Big\}$$

(Figure 2 shows the sets $\mathcal{W}$ and $\mathcal{O}$ for the case $p > 2$). We also define for fixed $\lambda \in \mathbf{R}$,

$$\mathcal{W}_\lambda = \Big\{d \in \mathbf{R} \,|\, (\lambda, d) \in \mathcal{W}\Big\} \text{ and } \mathcal{O}_\lambda = \Big\{d \in \mathbf{R} \,|\, (\lambda, d) \in \mathcal{O}\Big\}$$

(note that when $\lambda \leq 0$, $\mathcal{O}_\lambda = \emptyset$). We can easily see that $\Lambda_{p;\lambda,d}$ is continuous with respect to d when $d \in \mathcal{W}_\lambda$ or $d \in \mathcal{O}_\lambda$. However, it may have discontinuity on d_λ. Indeed, using

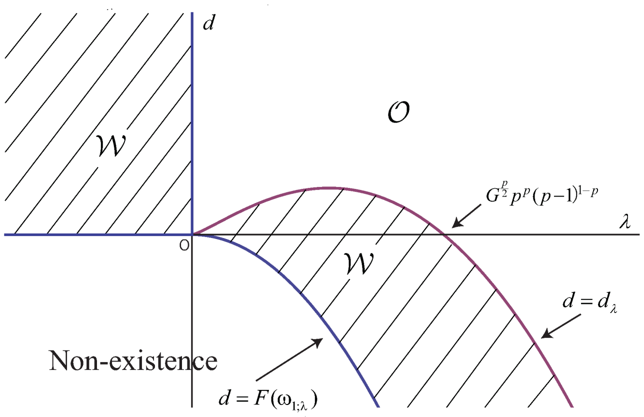

FIGURE 2. Domains of $\mathcal{W}$ and $\mathcal{O}$.

elliptic integrals, we can show this discontinuity in the case of $p = 2$. We note in this case it holds that

$$d_\lambda = Gp^2 \Big(\frac{\lambda}{p-1}\Big)^{\frac{2(p-1)}{p}} - \lambda^2 = 4G\lambda - \lambda^2.$$

Proposition 1. *Assume $p = 2$. Then, in the case $\lambda > 4G$, it holds*

$$\lim_{d \to d_\lambda + 0} \Lambda^{(II)}_{2;\lambda,d} = 2K\left(\sqrt{\frac{4G}{\lambda}}\right) - \pi \tag{26}$$

$$\lim_{d \to d_\lambda - 0} \Lambda^{(II)}_{2;\lambda,d} = 2K\left(\sqrt{\frac{4G}{\lambda}}\right) + \pi. \tag{27}$$

In the case $0 < \lambda < 4G$, it holds

$$\lim\nolimits_{d \to d_\lambda + 0} \Lambda^{(IV)}_{2;\lambda,d} = 2\sqrt{\tfrac{\lambda}{G}} K\left(\sqrt{\tfrac{\lambda}{4G}}\right) - 2\pi \tag{28}$$
$$\lim\nolimits_{d \to d_\lambda - 0} \Lambda^{(IV)}_{2;\lambda,d} = 2\sqrt{\tfrac{\lambda}{G}} K\left(\sqrt{\tfrac{\lambda}{4G}}\right) + 2\pi.$$

Remark 2. *If $(\lambda, d) \in \mathcal{W}$, from* (14) *and* (25), *we see that $u(s)$, $s \in (0, T_{p;\lambda,d})$ is monotone increasing, while if $(\lambda, d) \in \mathcal{O}$, $u_s(s)$ changes its sign on some $s \in (0, T_{p;\lambda,d})$. Thus shapes of stationary curves change drastically between $\mathcal{W}$ and $\mathcal{O}$.*

Definition 2. *Assume a stationary curve (u, v) generated by* (11), (13), (14) *has (λ, d) belonging to $\mathcal{W}$ (resp. $\mathcal{O}$). Then we say (u, v) is "wavelike" (resp. "orbitlike").*

5.1. **Existence of the flat-core solutions.** We can show the existence of flat-core solutions.

Proposition 2. *Assume $p > 2$. Then if $0 < \lambda < G^{\frac{p}{2}} p^p (p-1)^{1-p}$, there exist wavelike flat-core solutions and if $G^{\frac{p}{2}} p^p (p-1)^{1-p} < \lambda$, there exist orbitlike flat-core solutions.*

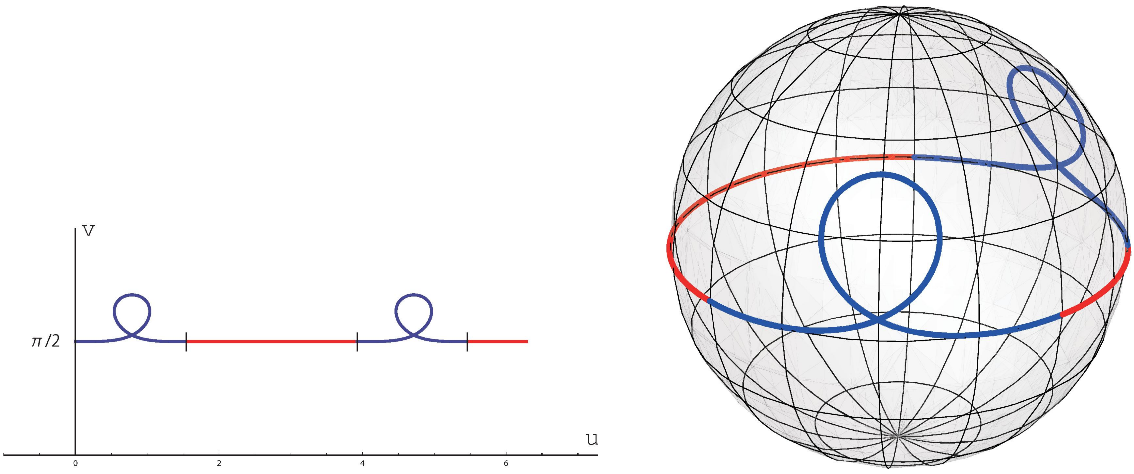

FIGURE 3. Orbitlike flat-core solution of for $p = 3, \lambda = 100, G = 1$.

Figures 3-5 show examples of flat-core solutions.

5.2. **Existence of type (IV) solutions.** Let $\lambda \le 0$ be fixed. Then if

$$\frac{2n\pi}{m} \in \left\{ \Lambda^{(IV)}_{p;\lambda,d} \,\middle|\, 0 < d < \infty \right\} \tag{29}$$

holds, clearly type (IV) wavelike stationary curves belonging to $\gamma_{n,m}$ exist. We show that $n \in \mathbf{Z}$ and $m \in \mathbf{N}$ satisfying (29) actually exists.

Proposition 3. *Suppose $n \in \mathbf{Z}$ and $m \in \mathbf{N}$ satisfy*

$$\begin{cases} -2\pi < \dfrac{2n\pi}{m} < 0, & 1 < p < 2, \lambda < 0 \\ -2\pi\sqrt{\dfrac{2G}{2G-\lambda}} < \dfrac{2n\pi}{m} < 0, & p = 2, \lambda < 0 \\ 0 < -\dfrac{2n\pi}{m} \ll 1, & 1 < p \le 2 \text{ and } \lambda = 0 \text{ or } p > 2, \lambda \le 0. \end{cases} \tag{30}$$

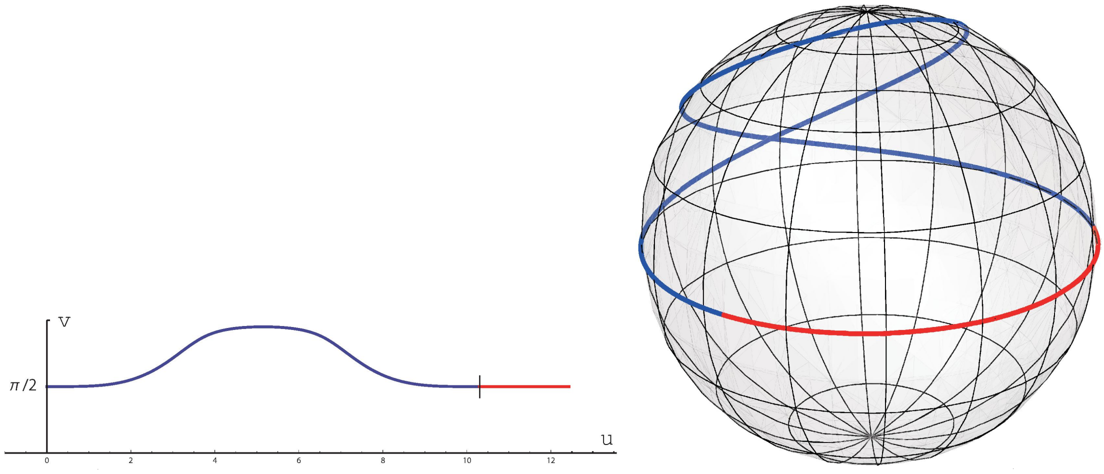

FIGURE 4. Wavelike flat-core solution of for $p = 3, \lambda = 5, G = 1$.

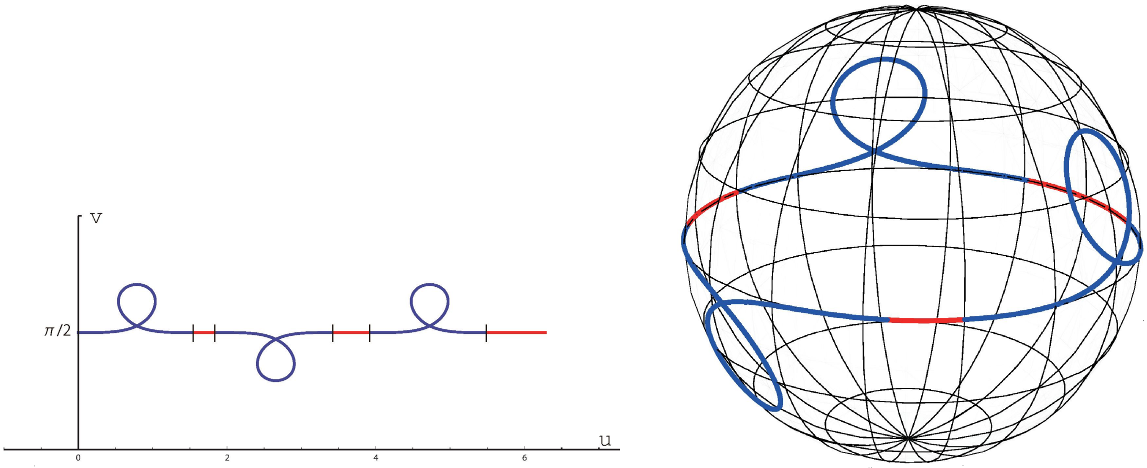

FIGURE 5. Orbitlike flat-core solution of for $p = 3, \lambda = 100, G = 1$.

Then (29) *is satisfied. Especially if* $p > 2$, $\lambda < 0$ *and* $0 < -2n\pi/m \ll 1$, *there exists at least two distinct stationary curves belonging to* $\gamma_{n,m}$.

Although Proposition 3 asserts the existence of type (IV) solutions belonging to $\gamma_{n,m}$ where $0 < -2n\pi/m \ll 1$, numerical result indicates the existence of type (IV) solutions for small m. Figure 6 shows an example of such type (IV) wavelike solution which belongs to $\gamma_{1,2}$.

We show the result concerning the existence of type (IV) orbitlike solutions. First we define for p satisfying $1 < p < 2$,

$$\lambda_p = 2G^{\frac{p}{2}} \left(\frac{2-p}{p-1} \right)^{\frac{p-2}{2}}.$$

We have the following lemma.

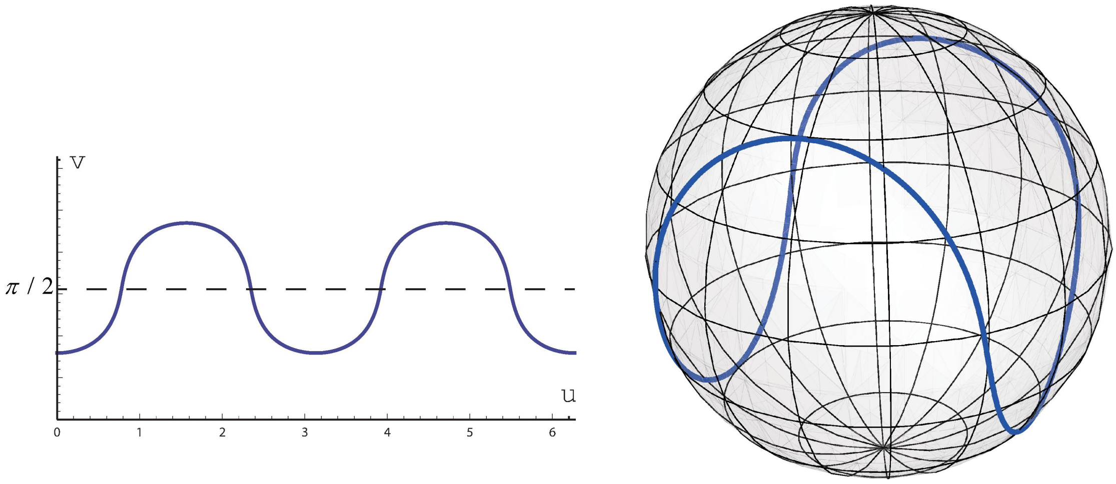

FIGURE 6. Wavelike type (III) stationary curve belonging to $\gamma_{1,2}$; $p = 3, \lambda = -1$ and $d \sim 30.75, G = 1$.

Lemma 5. *Let $1 < p < 2$. Then, on the interval (λ_p, ∞), the graphs $\lambda \mapsto F(\omega_{2;\lambda})$ and $\lambda \mapsto d_\lambda$ intersect only once at $\lambda = \lambda_*$. Moreover, λ_* satisfies*

$$\lambda_p < \lambda_* < G^{\frac{p}{2}} p^p (p-1)^{1-p}. \tag{31}$$

Further we define $\bar{\lambda} \in (2G, 4G)$ the unique solution of

$$\sqrt{\frac{\lambda}{G}} K\left(\sqrt{\frac{\lambda}{4G}}\right) = \pi.$$

The following proposition shows the existence of type (IV) orbitlike solutions.

Proposition 4. *There hold the following.*

(1) *In the case $1 < p < 2$ and $\lambda > \lambda_p$,*
   (a) *if $\lambda_p < \lambda < \lambda_*$, there exists an orbitlike type* (IV) *solution,*
   (b) *if $\lambda \geq \lambda_*$, for each $m, n \in \mathbf{N}$, there exists an orbitlike type* (IV) *solution belonging to $\gamma_{n,m}$,*

(2) *In the case $p = 2$ and $\lambda > 2G$,*
   (a) *if $2G < \lambda < 4G$ and $\lambda \neq \bar{\lambda}$, for each $m, n \in \mathbf{N}$ such that*

$$\frac{2n\pi}{m} \in \begin{cases} \left(2\sqrt{\frac{\lambda}{G}} K\left(\sqrt{\frac{\lambda}{4G}}\right) - 2\pi, 0\right) & \text{if } \lambda \in (2G, \bar{\lambda}), \\ \left(0, 2\sqrt{\frac{\lambda}{G}} K\left(\sqrt{\frac{\lambda}{4G}}\right) - 2\pi\right) & \text{if } \lambda \in (\bar{\lambda}, 4G), \end{cases}$$

   there exists an orbitlike type (IV) *solution belonging to $\gamma_{n,m}$,*
   (b) *if $2G < \lambda < 4G$ and $\lambda = \bar{\lambda}$, there exists an orbitlike type* (IV) *solution,*

(c) *if* $\lambda \geq 4G$, *for each* $m, n \in \mathbf{N}$, *there exists an orbitlike type* (IV) *solution belonging to* $\gamma_{n,m}$.

(3) *In the case* $p > 2$ *and* $\lambda > 0$, *there exists an orbitlike type* (IV) *solution.*

Figure 7 shows a numerical example of type (IV) orbitlike solution for the case $p = 3$.

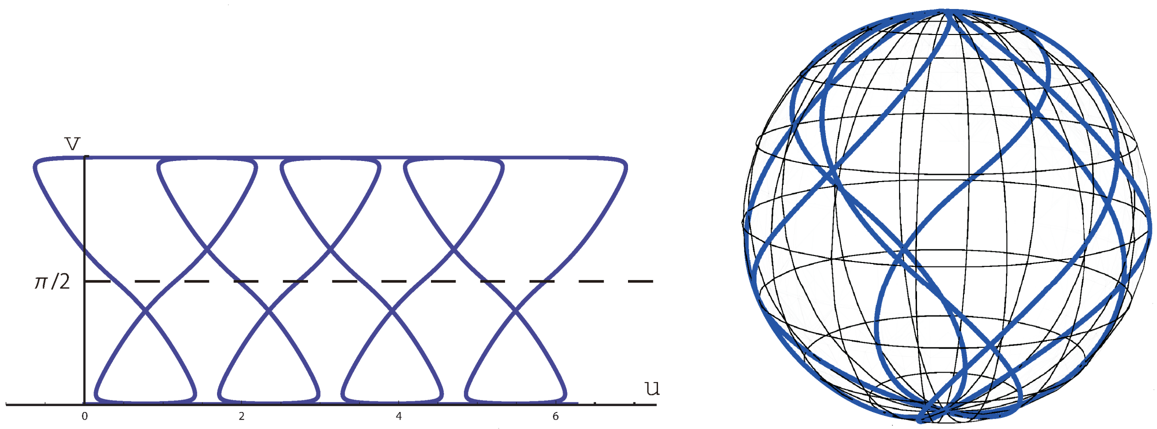

FIGURE 7. Orbitlike type (IV) stationary curve belonging to $\gamma_{1,4}$; $p = 3, \lambda = 3, d \sim 6.762, G = 1$.

We can show the existence of type (I) solutions and type (II) or (III) solutions. These results are seen in [11]. Figure 8 and 9 are examples of type (II) wavelike solutions. Both Figure 8 and 9 take $p = 1.5, \lambda = 2.4$.

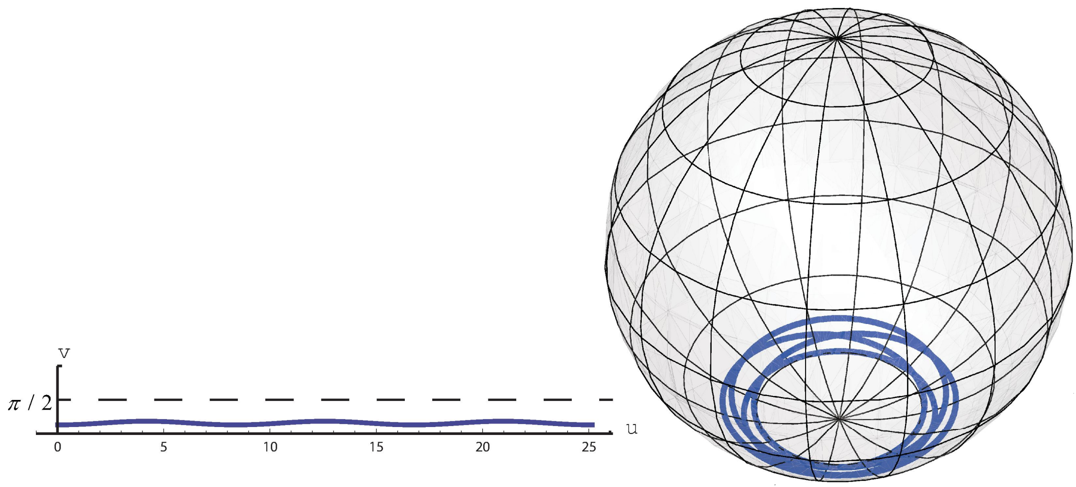

FIGURE 8. Wavelike type (II) stationary curve belonging to $\gamma_{4,3}$; $p = 1.5, \lambda = 2.4$, $d \sim -0.273, G = 1$.

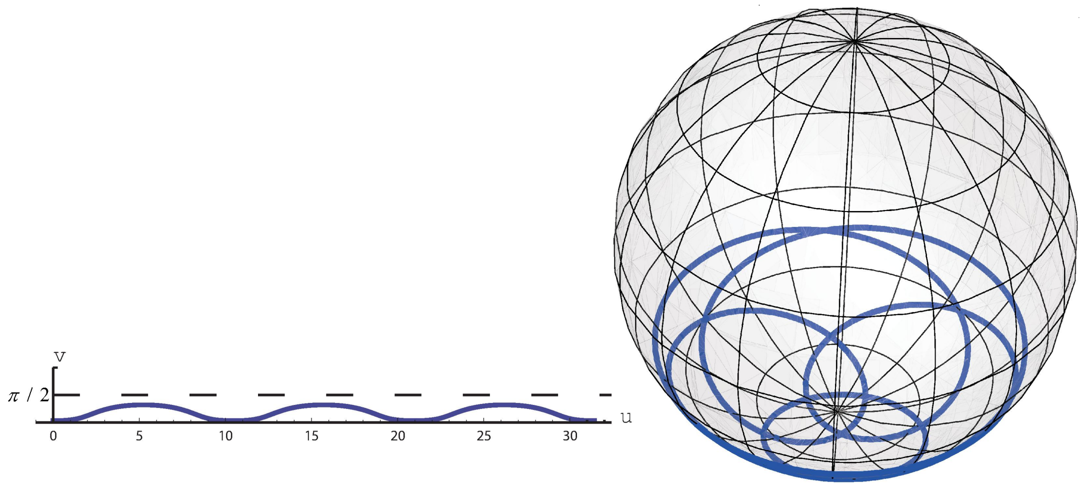

FIGURE 9. Wavelike type (II) stationary curve belonging to $\gamma_{5,3}$; $p = 1.5, \lambda = 2.4,\ d \sim 0.236, G = 1$.

6. APPENDIX

In this section, as mentioned in the introduction, we will show the differentiability of time map $T^{(IV)}_{p,\lambda,d}$. In [11], we show this property with implicit function theorem. Here we give an alternative proof of its differentiability which may be useful for the case that the implicit function theorem is not applicable. Let $\epsilon_0 > 0$ be sufficiently small and $\beta \in (\omega_{1;\lambda} - \epsilon_0, \omega_{1;\lambda} + \epsilon_0)$. For such β, clearly there uniquely exists $\alpha(\beta) > 0$ such that $\alpha(\beta)$ is continuous with respect to β, $F(\alpha(\beta)) = F(\beta)$, $\alpha(\beta) \neq \beta$ if $\beta \neq \omega_{1;\lambda}$ and $\alpha(\omega_{1;\lambda}) = \omega_{1;\lambda}$. Putting $d = F(\beta)$, the time map $T^{(IV)}_{p;\lambda,d}$ can be regarded as a function of β. In some circumstances, we need the differentiability of the time map with respect to β.

Lemma 6. *It holds that $\alpha(\beta)$ is a smooth function of β, moreover*

(32)
$$\alpha(\omega_{1;\lambda}) = \omega_{1;\lambda},\ \alpha'(\omega_{1;\lambda}) = -1,\ \alpha''(\omega_{1;\lambda}) = -\frac{2F'''(\omega_{1;\lambda})}{3F''(\omega_{1;\lambda})},\ \alpha'''(\omega_{1;\lambda}) = -\frac{2F'''(\omega_{1;\lambda})^2}{3F''(\omega_{1;\lambda})^2}.$$

Proof. Since the smoothness of α at $\beta = \omega_{1;\lambda}$ seems not so clear, we show its smoothness in the neighborhood of $\beta = \omega_{1;\lambda}$. Now put

$$G(\alpha, \beta) = F(\alpha) - F(\beta).$$

We note G is C^∞ in the neighborhood of $(\alpha, \beta) = (\omega_{1;\lambda}, \omega_{1;\lambda})$ and

$$G_\alpha(\omega_{1;\lambda}, \omega_{1;\lambda}) = G_\beta(\omega_{1;\lambda}, \omega_{1;\lambda}) = F_\omega(\omega_{1;\lambda}) = 0$$
$$G_{\alpha\alpha}(\omega_{1;\lambda}, \omega_{1;\lambda}) = F_{\omega\omega}(\omega_{1;\lambda}) > 0,\ G_{\beta\beta}(\omega_{1;\lambda}, \omega_{1;\lambda}) = -F_{\omega\omega}(\omega_{1;\lambda}) < 0,$$

so $(\omega_{1;\lambda}, \omega_{1;\lambda})$ is a non-degenerate critical point of G. Hence by Morse's lemma, there exists a coordinate chart $U; (x, y)$ and C^∞-diffeomorphism $\Phi : (x, y) \mapsto (\alpha(x, y), \beta(x, y)) \in \mathbf{R}^2$

such that $(0,0) \in U$ and

$$\Phi(0,0) = (\omega_{1;\lambda}, \omega_{1;\lambda}),\ G(\alpha(x,y), \beta(x,y)) = G(\omega_{1;\lambda}, \omega_{1;\lambda}) + x^2 - y^2 = x^2 - y^2.$$

We note it holds

$$\{(x,y) \in U \mid G(\alpha(x,y), \beta(x,y)) = 0\} = \{(x,x) \in U\} \cup \{(x,-x) \in U\}.$$

We put

$$\Phi((x,x)) = (\alpha(x,x), \beta(x,x)) = (\alpha_1(x), \beta_1(x))$$
$$\Phi((x,-x)) = (\alpha(x,-x), \beta(x,-x)) = (\alpha_2(x), \beta_2(x)).$$

From $G(\alpha_1(x), \beta_1(x)) = 0$ and $G(\alpha_2(x), \beta_2(x)) = 0$, clearly $\alpha_1(x) = \beta_1(x)$ or $\alpha_2(x) = \beta_2(x)$ holds. Assume $\alpha_2(x) = \beta_2(x)$ holds (we only consider this case, since the case $\alpha_1(x) = \beta_1(x)$ can be treated similarly). Since Φ gives the C^∞-diffeomorphism from U to $\Phi(U)$, it holds that

$$\left(\frac{d\alpha_1}{dx}(0), \frac{d\beta_1}{dx}(0)\right) \neq (0,0).$$

In the case $d\beta_1(0)/dx \neq 0$, $x(\beta_1)$ is C^∞ for $\beta_1 \in (\omega_{1;\lambda} - \epsilon_0, \omega_{1;\lambda} + \epsilon_0)$, where $\epsilon_0 > 0$ is sufficiently small. Thus $\alpha_1(x(\beta_1))$ is C^∞ with respect to β_1. Setting $\beta = \beta_1$, $\alpha = \alpha_1$, we obtain that α is a C^∞ function of β. In the case $d\alpha_1(0)/dx \neq 0$, similarly we obtain β_1 is C^∞ for $\alpha_1 \in (\omega_{1;\lambda} - \epsilon_0, \omega_{1;\lambda} + \epsilon_0)$. Setting $\beta = \alpha_1$, $\alpha = \beta_1$, we obtain that α is a C^∞ function of β.

Formula (32) can be obtained by derivating

$$F(\beta) = F(\alpha(\beta))$$

with β several times. □

Lemma 7. *Let $\epsilon_0 > 0$ be sufficiently small. Then $T^{(IV)}_{p;\lambda,d(\beta)}$ is $C^2(\omega_{1;\lambda} - \epsilon_0, \omega_{1;\lambda} + \epsilon_0)$ with respect to β.*

Proof. We show that $T^{(IV)}_{p;\lambda,d(\beta)}$ is C^1 in detail, for the reason that the proof for this case is not so complex. Since the proof for C^2 differentiability is quite similar, we would like to note only essential points.

Now, putting $\omega = \beta - (\beta - \alpha(\beta))z$, we have

$$T^{(IV)}_{p;\lambda,d} = 2p \int_\alpha^\beta \frac{d\omega}{\sqrt{F(\beta) - F(\omega)}} = 2p \int_0^1 \frac{\beta - \alpha(\beta)}{\sqrt{F(\beta) - F(\beta - (\beta - \alpha(\beta))z)}} dz.$$

We have

$$\frac{d}{d\beta}\left(\frac{\beta-\alpha(\beta)}{\sqrt{F(\beta)-F(\beta-(\beta-\alpha(\beta))z)}}\right)$$
$$=\frac{2(F(\beta)-F(\omega))(1-\alpha'(\beta))-(\beta-\alpha(\beta))\left(F'(\beta)-F'(\omega)(1-(1-\alpha'(\beta))z)\right)}{2\left(F(\beta)-F(\omega)\right)^{\frac{3}{2}}}$$
$$=\frac{2(F(\beta)-F(\omega))(1-\alpha'(\beta))+(\omega-\alpha(\beta))F'(\omega)-(\beta-\alpha(\beta))F'(\beta)+(\beta-\omega)\alpha'(\beta)F'(\omega)}{2\left(F(\beta)-F(\omega)\right)^{\frac{3}{2}}}.$$

We put

$$\begin{aligned}\varphi_1(\omega)=&2(F(\beta)-F(\omega))(1-\alpha'(\beta))+(\omega-\alpha(\beta))F'(\omega)-(\beta-\alpha(\beta))F'(\beta)\\&+(\beta-\omega)\alpha'(\beta)F'(\omega)\end{aligned}$$

and show for ω satisfying $\omega\in[\omega_{1;\lambda},\beta]$, there exists a constant $C_1>0$ such that

$$|\varphi_1(\omega)|\le C_1\epsilon^2(\beta-\omega) \tag{33}$$

holds. We can easily see that

$$\varphi_1(\beta)=0. \tag{34}$$

Next we estimate $\varphi_1'(\omega)$. We note it holds

$$\varphi_1'(\omega)=F'(\omega)(-1+\alpha'(\beta))+\left(\omega-\alpha(\beta)+(-\omega+\beta)\alpha'(\beta)\right)F''(\omega).$$

Substituting $\beta=\omega_{1;\lambda}+\epsilon$, we obtain from Taylor series expansion for ω and ϵ that

$$\begin{aligned}\varphi_1'(\omega)=&\varphi_1'(\omega_{1;\lambda})+(\omega-\omega_{1;\lambda})\varphi_1''(\omega_{1;\lambda})+O((\omega-\omega_{1;\lambda})^2)\\=&\left(\omega_{1;\lambda}-\alpha(\beta)+(-\omega_{1;\lambda}+\beta)\alpha'(\beta)\right)F''(\omega_{1;\lambda})\\&+(\omega-\omega_{1;\lambda})\left(\omega_{1;\lambda}-\alpha(\beta)+(-\omega_{1;\lambda}+\beta)\alpha'(\beta)\right)F'''(\omega_{1;\lambda})\\&+O((\omega-\omega_{1;\lambda})^2)\\=&\left(-\frac{1}{3}F'''(\omega_{1;\lambda})\epsilon^2+O(\epsilon^3)\right)+(\omega-\omega_{1;\lambda})\left(-\frac{F'''(\omega_{1;\lambda})^2}{3F''(\omega_{1;\lambda})}\epsilon^2+O(\epsilon^3)\right)\\&+O\left((\omega-\omega_{1;\lambda})^2\right).\end{aligned} \tag{35}$$

Since by Lemma 6, $|\omega-\omega_{1;\lambda}|\le 2\epsilon$ for $\omega\in[\alpha(\beta),\beta]$, (35) means that there exists a constant $C_1>0$ such that

$$|\varphi_1'(\omega)|\le C_1\epsilon^2 \quad \text{for } \omega\in[\alpha(\beta),\beta]. \tag{36}$$

From (34) and above inequality, we have

$$|\varphi_1(\omega)|\le\int_\omega^\beta|\varphi_1'(u)|\,du\le C_1\epsilon^2(\beta-\omega),$$

which shows (33) on $\omega\in[\omega_{1;\lambda},\beta]$.

Next, noting $F(\alpha(\beta)) = F(\beta)$, we see that

$$\varphi_1(\alpha(\beta)) = 0. \tag{37}$$

Hence from (36), (37), we obtain as in the case $\omega \in [\omega_{1;\lambda}, \beta]$,

$$|\varphi_1(\omega)| \le C_1\epsilon^2(\omega - \alpha(\beta)), \qquad \omega \in [\alpha(\beta), \omega_{1;\lambda}]. \tag{38}$$

Here, let $z_{1;\lambda} \in (0,1)$ be the point satisfying $\omega_{1;\lambda} = \beta - (\beta - \alpha(\beta))z_{1;\lambda}$. Then we obtain

$$\min_{0\le z\le z_{1;\lambda}} \frac{F(\beta) - F(\beta - (\beta - \alpha(\beta)z))}{z} = \min_{\omega_{1;\lambda}\le\omega\le\beta} \frac{F(\beta) - F(\omega)}{\beta - \omega}(\beta - \alpha(\beta)) \tag{39}$$
$$= \frac{F(\beta) - F(\omega_{1;\lambda})}{\beta - \omega_{1;\lambda}}(\beta - \alpha(\beta)),$$

$$\min_{z_{1;\lambda}\le z\le 1} \frac{F(\beta) - F(\beta - (\beta - \alpha(\beta)z))}{1 - z} = \min_{\alpha\le\omega\le\omega_{1;\lambda}} \frac{F(\alpha(\beta)) - F(\omega)}{\omega - \alpha(\beta)}(\beta - \alpha(\beta)) \tag{40}$$
$$= \frac{F(\alpha(\beta)) - F(\omega_{1;\lambda})}{\omega_{1;\lambda} - \alpha(\beta)}(\beta - \alpha(\beta)).$$

Using (33), (38), (39), (40) and noting the relation, $\beta - \omega_{1;\lambda} = \epsilon$, we obtain

$$\left|\frac{d}{d\beta}\left(\frac{\beta - \alpha(\beta)}{\sqrt{F(\beta) - F(\beta - (\beta - \alpha(\beta))z)}}\right)\right| = \frac{|\varphi_1(\omega)|}{2\,(F(\beta) - F(\omega))^{\frac{3}{2}}} \tag{41}$$
$$= \frac{|\varphi_1(\beta - (\beta - \alpha(\beta))z)|}{2\,(F(\beta) - F(\beta - (\beta - \alpha(\beta))z))^{\frac{3}{2}}}$$
$$\le \frac{C_1(\beta - \alpha(\beta))\epsilon^2 z}{2z^{\frac{3}{2}}\left(\frac{F(\beta)-F(\omega_{1;\lambda})}{\beta-\omega_{1;\lambda}}\right)^{\frac{3}{2}}(\beta - \alpha(\beta))^{\frac{3}{2}}} + \frac{C_1(\beta - \alpha(\beta))\epsilon^2(1 - z)}{2(1-z)^{\frac{3}{2}}\left(\frac{F(\alpha)-F(\omega_{1;\lambda})}{\omega_{1;\lambda}-\alpha(\beta)}\right)^{\frac{3}{2}}(\beta - \alpha(\beta))^{\frac{3}{2}}}$$
$$= \frac{C_1}{2}\sqrt{\frac{\epsilon}{\beta - \alpha(\beta)}}\left(\frac{\epsilon}{\beta - \omega_{1;\lambda}}\right)^{\frac{3}{2}}\left(\frac{F(\beta) - F(\omega_{1;\lambda})}{(\beta - \omega_{1;\lambda})^2}\right)^{-\frac{3}{2}}\frac{1}{z^{\frac{1}{2}}}$$
$$+ \frac{C_1}{2}\sqrt{\frac{\epsilon}{\beta - \alpha(\beta)}}\left(\frac{\epsilon}{\omega_{1;\lambda} - \alpha(\beta)}\right)^{\frac{3}{2}}\left(\frac{F(\alpha(\beta)) - F(\omega_{1;\lambda})}{(\omega_{1;\lambda} - \alpha(\beta))^2}\right)^{-\frac{3}{2}}\frac{1}{(1-z)^{\frac{1}{2}}}$$
$$= \frac{C_1}{2}\sqrt{\frac{\epsilon}{\beta - \alpha(\beta)}}\left(\frac{F(\beta) - F(\omega_{1;\lambda})}{(\beta - \omega_{1;\lambda})^2}\right)^{-\frac{3}{2}}\frac{1}{z^{\frac{1}{2}}}$$
$$+ \frac{C_1}{2}\sqrt{\frac{\epsilon}{\beta - \alpha(\beta)}}\left(\frac{\epsilon}{\omega_{1;\lambda} - \alpha(\beta)}\right)^{\frac{3}{2}}\left(\frac{F(\alpha(\beta)) - F(\omega_{1;\lambda})}{(\omega_{1;\lambda} - \alpha(\beta))^2}\right)^{-\frac{3}{2}}\frac{1}{(1-z)^{\frac{1}{2}}}.$$

We note it holds

$$\frac{C_1}{2}\sqrt{\frac{\epsilon}{\beta - \alpha(\beta)}}\left(\frac{F(\beta) - F(\omega_{1;\lambda})}{(\beta - \omega_{1;\lambda})^2}\right)^{-\frac{3}{2}} \to \frac{C_1}{2}\sqrt{\frac{1}{2}}\left(\frac{1}{2}F_{\omega\omega}(\omega_{1;\lambda})\right)^{-\frac{3}{2}},$$
$$\frac{C_1}{2}\sqrt{\frac{\epsilon}{\beta - \alpha(\beta)}}\left(\frac{\epsilon}{\omega_{1;\lambda} - \alpha(\beta)}\right)^{\frac{3}{2}}\left(\frac{F(\alpha(\beta)) - F(\omega_{1;\lambda})}{(\omega_{1;\lambda} - \alpha(\beta))^2}\right)^{-\frac{3}{2}} \to \frac{C_1}{2}\sqrt{\frac{1}{2}}\left(\frac{1}{2}F_{\omega\omega}(\omega_{1;\lambda})\right)^{-\frac{3}{2}},$$

as $\epsilon \to +0$. Thus it holds for sufficiently small $\epsilon_0 > 0$ that

$$\left|\frac{d}{d\beta}\left(\frac{\beta-\alpha(\beta)}{\sqrt{F(\beta)-F(\beta-(\beta-\alpha(\beta))z)}}\right)\right| \leq 2C_1 F_{\omega\omega}(\omega_{1;\lambda})^{-\frac{3}{2}}\left(\frac{1}{z^{\frac{1}{2}}}+\frac{1}{(1-z)^{\frac{1}{2}}}\right) \in L^1(0,1).$$

Hence from Lebesgue dominated convergence theorem, we have

$$\frac{dT^{(IV)}_{p;\lambda,d(\beta)}}{d\beta} = 2p\int_0^1 \frac{d}{d\beta}\left(\frac{\beta-\alpha(\beta)}{\sqrt{F(\beta)-F(\beta-(\beta-\alpha(\beta))z)}}\right)dz. \tag{42}$$

Next, we show the continuity of $dT^{(IV)}_{p;\lambda,d(\beta)}/d\beta$ on $(\omega_{1;\lambda}-\epsilon_0, \omega_{1;\lambda}+\epsilon_0)$. For simplicity, we put

$$\Psi(z,\beta) = 2p\frac{d}{d\beta}\left(\frac{\beta-\alpha(\beta)}{\sqrt{F(\beta)-F(\beta-(\beta-\alpha(\beta))z)}}\right).$$

Then, for $\epsilon_1 > 0$ sufficiently small we have

$$\begin{aligned}
&\left|\left.\frac{dT^{(IV)}_{p;\lambda,d(\beta)}}{d\beta}\right|_{\beta=\beta_0} - \left.\frac{dT^{(IV)}_{p;\lambda,d(\beta)}}{d\beta}\right|_{\beta=\beta_1}\right| \\
&\leq \int_0^{\epsilon_1} |\Psi(z,\beta_0)-\Psi(z,\beta_1)|\,dz + \int_{1-\epsilon_1}^1 |\Psi(z,\beta_0)-\Psi(z,\beta_1)|\,dz \\
&\quad + \int_{\epsilon_1}^{1-\epsilon_1} |\Psi(z,\beta_0)-\Psi(z,\beta_1)|\,dz \\
&\leq 4C_1 F_{\omega\omega}(\omega_{1;\lambda})^{-\frac{3}{2}}\left\{\int_0^{\epsilon_1}\frac{1}{z^{\frac{1}{2}}}+\frac{1}{(1-z)^{\frac{1}{2}}}dz + \int_{1-\epsilon_1}^1 \frac{1}{z^{\frac{1}{2}}}+\frac{1}{(1-z)^{\frac{1}{2}}}dz\right\} \\
&\quad + \int_{\epsilon_1}^{1-\epsilon_1} |\Psi(z,\beta_0)-\Psi(z,\beta_1)|\,dz
\end{aligned}$$

Again by Lebesgue dominated convergence theorem, we have

$$\int_{\epsilon_1}^{1-\epsilon_1} |\Psi(z,\beta_0)-\Psi(z,\beta_1)|\,dz \to 0$$

as $\beta_1 \to \beta_0$. Thus for $\epsilon_1 > 0$ sufficiently small, there exists $\delta_1 > 0$ and $C_2 > 0$ such that if $|\beta_0 - \beta_1| < \delta_1$, then

$$\left|\left.\frac{dT^{(IV)}_{p;\lambda,d(\beta)}}{d\beta}\right|_{\beta=\beta_0} - \left.\frac{dT^{(IV)}_{p;\lambda,d(\beta)}}{d\beta}\right|_{\beta=\beta_1}\right| \leq C_2\epsilon_1$$

holds. This shows the continuity of $dT^{(IV)}_{p;\lambda,d(\beta)}/d\beta$ on $(\omega_{1;\lambda}-\epsilon_0, \omega_{1;\lambda}+\epsilon_0)$.

To show C^2 differentiability, we put

$$\frac{d^2}{d\beta^2}\left(\frac{\beta-\alpha(\beta)}{\sqrt{F(\beta)-F(\beta-(\beta-\alpha(\beta))z)}}\right) = \frac{\varphi_2(\omega)}{4(\beta-\alpha(\beta))\,(F(\beta)-F(\omega))^{\frac{5}{2}}}$$

and show there exist a constant $C_3 > 0$ such that

$$|\varphi_2(\omega)| \le \begin{cases} C_3\epsilon^4(\beta-\omega)^2, & \omega \in [\omega_{1;\lambda}, \beta] \\ C_3\epsilon^4(\omega-\alpha(\beta))^2, & \omega \in [\alpha(\beta), \omega_{1;\lambda}]. \end{cases} \tag{43}$$

In the following we have to perform some complex computation and we use *Mathematica* for these computation. First we obtain the expression of $\varphi_2(\omega)$ as:

$$\begin{aligned}
&\varphi_2(\omega)\\
&=\Big\{(\beta-\alpha(\beta))^2\left(2F(\omega)F''(\beta)-2F(\beta)F''(\beta)+3F'(\beta)^2\right)\\
&\quad+6(\beta-\alpha(\beta))F'(\beta)F'(\omega)\left(\omega\alpha'(\beta)-\beta\alpha'(\beta)+\alpha(\beta)-\omega\right)\\
&\quad+\left(\omega\alpha'(\beta)-\beta\alpha'(\beta)+\alpha(\beta)-\omega\right)^2\left(2F(\beta)F''(\omega)-2F(\omega)F''(\omega)+3F'(\omega)^2\right)\\
&\quad-4(\beta-\alpha(\beta))\left(F(\beta)^2+F(\omega)^2\right)\alpha''(\beta)\\
&\quad+2(F(\beta)-F(\omega))F'(\omega)\\
&\qquad\cdot\left(4\left(\alpha'(\beta)-1\right)\left((\omega-\beta)\alpha'(\beta)+\alpha(\beta)-\omega\right)-2(\beta-\alpha(\beta))(\omega-\beta)\alpha''(\beta)\right)\\
&\quad+4(\beta-\alpha(\beta))(F(\beta)-F(\omega))F'(\beta)\left(\alpha'(\beta)-1\right)\Big\}.
\end{aligned}$$

We can check

$$\varphi_2(\beta)=0,\quad \varphi_2'(\beta)=0. \tag{44}$$

Next we estimate $\varphi_2''(\omega)$. Similar to (35), substituting $\beta=\omega_{1;\lambda}+\epsilon$, we obtain from Taylor series expansion for ω and ϵ that

(45)

$$\begin{aligned}
&\varphi_2''(\omega)\\
&=\frac{2}{3}\left(\left(-3F'''(\omega_{1;\lambda})^2+F''(\omega_{1;\lambda})F^{(4)}(\omega_{1;\lambda})\right)\epsilon^4+O(\epsilon^5)\right)\\
&\quad+(\omega-\omega_{1;\lambda})\left(\frac{8}{3}F'''(\omega_{1;\lambda})^2\epsilon^3+O(\epsilon^4)\right)+(\omega-\omega_{1;\lambda})^2\left(4F''(\omega_{1;\lambda})F^{(4)}(\omega_{1;\lambda})\epsilon^2+O(\epsilon^3)\right)\\
&\quad+(\omega-\omega_{1;\lambda})^3\left(-\frac{40}{9}F'''(\omega_{1;\lambda})^2\epsilon+O(\epsilon^2)\right)+O\left((\omega-\omega_{1;\lambda})^4\right).
\end{aligned}$$

Since $|\omega-\omega_{1;\lambda}|\le 2\epsilon$ for $\omega\in[\alpha(\beta),\beta]$, (45) means that there exists a constant $C_4>0$ such that

$$\left|\varphi_2''(\omega)\right|\le C_4\epsilon^4. \tag{46}$$

Hence from (44) and (46), we obtain the following estimate for $\omega\in[\omega_{1;\lambda},\beta]$,

$$\left|\varphi_2'(\omega)\right|\le\int_\omega^\beta\left|\varphi_2''(u)\right|du\le\int_\omega^\beta C_4\epsilon^4du=C_4\epsilon^4(\beta-\omega). \tag{47}$$

Thus, again from (44) and (47), we obtain

$$|\varphi_2(\omega)| \leq \frac{C_4\epsilon^4}{2}(\beta-\omega)^2 = C_3\epsilon^4(\beta-\omega)^2. \tag{48}$$

Using *Mathematica*, we can check

$$\varphi_2(\alpha(\beta)) = 0, \quad \varphi_2'(\alpha(\beta)) = 0. \tag{49}$$

So, again from (47), we have

$$|\varphi_2(\omega)| \leq C_3\epsilon^4(\omega-\alpha(\beta))^2$$

for $\omega \in [\alpha(\beta), \omega_{1;\lambda}]$. The remaining part of the proof can be shown as in the proof of C^1 differentiability. $\square$

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