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Some properties of eigenvalues for fully nonlinear operators

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1 Introduction

This note is based on joint work [8] with Professor Hitoshi Ishii and provides some results in [8].

In this note, we consider the following eigenvalue problem:

$$\text{(1)} \qquad \begin{cases} F(D^2u, Du, u, x) + \mu u = 0 & \text{in } \Omega, \\ B(Du, u, x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ is a finite open interval (a, b) if $N = 1$ and, otherwise, an open ball B_R centered at the origin with radius $R > 0$. The function $F : \mathbf{S}^N \times \mathbf{R}^N \times \mathbf{R} \times \Omega \to \mathbf{R}$ is given where $\mathbf{S}^N$ denotes a set of all $N \times N$ symmetric matrices and B has the form of

$$\text{(2)} \qquad \begin{aligned} B(p, u, x) &:= s_{a,b}(x)\sigma(x)p + \tau(x)u \quad \text{if } N = 1, \\ B(p, u, x) &:= \sigma_R\langle p, \nu(x)\rangle + \tau_R u \qquad \text{if } N \geq 2. \end{aligned}$$

Here $(\sigma(x), \tau(x)), (\sigma_R, \tau_R) \in \mathbf{R}^2 \setminus \{(0,0)\}$, $s_{a,b}(a) = -1$, $s_{a,b}(b) = 1$ and $\nu(x)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$. Remark that the sign of $s_{a,b}(x)$ corresponds to the outer unit normal derivative. Note also that the Robin boundary conditions (2) include the zero Dirichlet ($(\sigma, \tau) = (0, 1)$) and the zero Neumann ($(\sigma, \tau) = (1, 0)$) boundary condition. The pair $(\mu, u) \in \mathbf{R} \times W^{2,q}(\Omega)$ is unknown and called *eigenpair of* (1) provided $u \not\equiv 0$.

Many researchers consider the eigenvalue problem for fully nonlinear elliptic operators, and study the existence of eigenpairs and their properties. Here we refer to [2, 4, 5, 6, 7, 9, 10, 11, 12]. Among those results, in [5] and [6], the authors prove the existence of sequences of eigenparis of (1) in the one-dimensional or the radially symmetric problem with the zero Dirichlet boundary condition ($(\sigma(x), \tau(x)) = (0, 1) = (\sigma_R, \tau_R)$), and these results are extended into the L^q framework in [7]. In [8] and this note, we treat other boundary conditions and show some properties of eigenpairs of (1).

Next, we introduce our assumptions on F. For this purpose, we first give the definition of the Pucci operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u)$ where $0 < \lambda \leq \Lambda < \infty$. For two positive constants $0 < \lambda \leq \Lambda < \infty$ and $M \in \mathbf{S}^N$, define $\mathcal{M}^{\pm}_{\lambda,\Lambda}(M)$ by

$$\begin{aligned} \mathcal{M}^{+}_{\lambda,\Lambda}(M) &:= \Lambda \sum_{i=1}^{N} (\mu_i(M))_+ - \lambda \sum_{i=1}^{N} (\mu_i(M))_-, \\ \mathcal{M}^{-}_{\lambda,\Lambda}(M) &:= \lambda \sum_{i=1}^{N} (\mu_i(M))_+ - \Lambda \sum_{i=1}^{N} (\mu_i(M))_-, \end{aligned}$$

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where $\mu_1(M) \le \cdots \le \mu_N(M)$ stand for eigenvalues of M and $a_\pm := \max\{\pm a, 0\}$. Hence, when $N = 1$, one observes that $\mathcal{M}^+_{\lambda,\Lambda}(m) = \Lambda m$ if $m \ge 0$ and $\mathcal{M}^+_{\lambda,\Lambda}(m) = \lambda m$ if $m < 0$.

Now let us state the assumptions on F.

(F1) The function $F : \mathbf{S}^N \times \mathbf{R}^N \times \mathbf{R} \times \Omega \to \mathbf{R}$ is a Carathéodory function. This means that the function $x \mapsto F(M,p,u,x)$ is measurable for any $(M,p,u) \in \mathbf{S}^N \times \mathbf{R}^{N+1}$ and the function $(M,p,u) \mapsto F(M,p,u,x)$ is continuous for a.a. $x \in \Omega$.

(F2) There exist $0 < \lambda \le \Lambda < \infty$, $q \in [1,\infty]$ and functions $\beta, \gamma \in L^q(\Omega)$ such that

$$\begin{aligned} &F(M_1,p_1,u_1,x) - F(M_2,p_2,u_2,x) \\ &\le \mathcal{M}^+_{\lambda,\Lambda}(M_1 - M_2) + \beta(x)|p_1 - p_2| + \gamma(x)|u_1 - u_2| \end{aligned}$$

for all $(M_1,p_1,u_1), (M_2,p_2,u_2) \in \mathbf{S}^N \times \mathbf{R}^{N+1}$ and a.a. $x \in \Omega$.

(F3) $F(tM,tp,tu,x) = tF(M,p,u,x)$ for all $t \ge 0$, all $(M,p,u) \in \mathbf{S}^N \times \mathbf{R}^{N+1}$ and a.a. $x \in \Omega$.

(F4) When $N \ge 2$, the function F is radially symmetric in the sense that for any $(m,l,q,u) \in \mathbf{R}^4$ and a.a. $r \in (0,R)$, the function

$$\omega \mapsto F(m\omega \otimes \omega + l(I_N - \omega \otimes \omega), q\omega, u, r\omega)$$

is constant on the unit sphere $S^{N-1} \subset \mathbf{R}^N$ where $x \otimes x$ denotes the matrix in $\mathbf{S}^N$ with the (i,j) entry given by $x_i x_j$ for $x \in \mathbf{R}^N$.

Remark that (F4) is only assumed in the case $N \ge 2$.

2 Results

2.1 Existence of sequences of eigenpairs of (1)

We first state the existence result of eigenpairs of (1) when $N = 1$.

Theorem 2.1 ([8]). *Let $N = 1$, $\Omega = (a,b)$ and suppose $(\sigma(x), \tau(x)) \in \mathbf{R}^2 \setminus \{(0,0)\}$ for $x = a,b$ and* (F1)–(F3) *with $q \in [1,\infty]$. Then there exist sequences $\{(\mu_n^\pm, \varphi_n^\pm)\}_{n=0}^\infty \subset \mathbf{R} \times W^{2,q}(a,b)$ of eigenpairs of* (1) *and sequences $\{x_{n,k}^\pm\}_{k=0}^{n+1}$ with $a = x_{n,0}^\pm < \cdots < x_{n,n+1}^\pm = b$ such that*
(i) $\pm(-1)^k \varphi_n^\pm > 0$ *in* $(x_{n,k}^\pm, x_{n,k+1}^\pm)$ *for all* $0 \le k \le n$, $\max\{\pm\varphi_n^\pm(a), \pm(\varphi_n^\pm)'(a)\} > 0$ *and* $\max\{\pm(-1)^n\varphi_n^\pm(b), \mp(-1)^n(\varphi_n^\pm)'(b)\} > 0$.
(ii) *If $(\mu, \varphi) \in \mathbf{R} \times W^{2,q}(a,b)$ is an eigenpair of* (1)*, there exist $n \in \mathbf{N}$ and $\theta > 0$ such that either $(\mu, \varphi) = (\mu_n^+, \theta\varphi_n^+)$ or $(\mu, \varphi) = (\mu_n^-, \theta\varphi_n^-)$ holds.*

From Theorem 2.1, we can find all eigenpairs of (1) whose eigenvalues are real. Moreover, combining Proposition 3.2 below, we also observe that each eigenvalue is simple.

Next, we consider the case $N \ge 2$. In this case, for $q \in [1,\infty]$, let $W^{2,q}_{\mathrm{r}}(0,R)$ be a set of functions $\varphi \in W^{2,q}(B_R)$ which are radially symmetric. In what follows, we identify any function $f \in W^{2,q}_{\mathrm{r}}(0,R)$ with a function g on $[0,R]$ such that $f(x) = g(|x|)$ for a.a. $x \in B_R$ and use the standard abuse of notation: $f(x) = f(|x|)$ for $x \in B_R$. Finally, set $\lambda_* = \lambda/\Lambda$ and $q_* = N/(\lambda_* N + 1 - \lambda_*)$. Notice that $0 < \lambda_* \le 1$ and $q_* \in [1, N)$.

Theorem 2.2 ([8]). *Let $N \geq 2$, $\Omega = B_R$, and assume* (F1)-(F4), $(\sigma_R, \tau_R) \in \mathbf{R}^2 \setminus \{(0,0)\}$, $q \in (\max\{N/2, q_*\}, \infty]$ *and* $\beta \in L^N(B_R)$ *if* $q < N$. *Then there exist sequences* $\{(\mu_n^\pm, \varphi_n^\pm)\}_{n=0}^\infty \subset \mathbf{R} \times W_r^{2,q}(0,R)$ *of eigenpairs of* (1) *and sequences* $\{r_{n,k}^\pm\}_{k=0}^{n+1}$ *with* $0 = r_{n,0}^\pm < \cdots < r_{n,n+1}^\pm = R$ *such that*
(i) $\pm(-1)^k \varphi_n^\pm > 0$ *in* $(r_{n,k}^\pm, r_{n,k+1}^\pm)$ *for any* $0 \leq k \leq n$, $\varphi_n^-(0) < 0 < \varphi_n^+(0)$ *and* $\max\{\pm(-1)^n \varphi_n^\pm(R), \mp(-1)^n (\varphi_n^\pm)'(R)\} > 0$.
(ii) *Let* $(\mu, \varphi) \in \mathbf{R} \times W_r^{2,q}(0,R)$ *be an eigenpair of* (1). *Then there exist* $n \in \mathbf{N}$ *and* $\theta > 0$ *such that either* $(\mu, \varphi) = (\mu_n^+, \theta\varphi_n^+)$ *or* $(\mu, \varphi) = (\mu_n^-, \theta\varphi_n^-)$.

As in Theorem 2.1, we find all eigenpairs of (1) whose eigenvalues are real and eigenfunctions radially symmetric. Furthermore, by Theorem 2.2 and Proposition 3.2, radial eigenvalues are simple in $W_r^{2,q}(0,R)$.

2.2 Monotonicity of eigenvalues on domains

Next, we observe the monotonicity properties of eigenvalues obtained in Theorems 2.1 and 2.2 with respect to domains under the zero Dirichlet boundary condition. To be more precise, we first remark that when $N = 1$, for every subinterval $[c,d] \subset [a,b]$, we can consider the eigenvalue problem (1) on $[c,d]$ instead of $[a,b]$ and Theorem 2.1 may be applied. We denote these eigenvalues by $\mu_n^\pm(c,d)$ to emphasize the dependence of eigenvalues on domains $[c,d]$. Similarly, if $N \geq 2$, we use the notation $\mu_n^\pm(0,\hat{R})$ for the eigenvalues of F on $B_{\hat{R}}$ where $\hat{R} \leq R$.

For the zero Dirichlet boundary condition, we have the following monotonicity of $\mu_n^\pm(c,d)$ on $[c,d]$.

Proposition 2.3. *Assume* $N = 1$ *and* $(\sigma(a), \tau(a)) = (0,1)$ (*resp.* $(\sigma(b), \tau(b)) = (0,1)$). *Then for each* $n \in \mathbf{N}$ *and* $a \leq c_1 < c_2 < d \leq b$ (*resp.* $a \leq c < d_1 < d_2 \leq b$), *the inequalities* $\mu_n^\pm(c_1,d) < \mu_n^\pm(c_2,d)$ (*resp.* $\mu_n^\pm(c,d_1) < \mu_n^\pm(c,d_2)$) *hold. Similarly, when* $N \geq 2$ *and* $(\sigma_R, \tau_R) = (0,1)$, *for every* $0 < R_1 < R_2 \leq R$, *the inequalities* $\mu_n^\pm(0,R_2) < \mu_n^\pm(0,R_1)$ *hold.*

Proposition 2.3 will be proved in Section 3. On the other hand, when $N = 1$ and we replace the zero Dirichlet boundary condition by the Robin boundary condition of the form $B(p,u) = p\cos\theta + u\sin\theta$ at $x = b$ where $0 < \theta < \pi/2$, we show that $\mu_0^+(a,c)$ fails to be monotone on c by the arguments based on the strong maximum principle. See Section 4.

2.3 Solvability of inhomogeneous equation

Finally, we give characterizations of $\mu_n^\pm(a,b)$ and $\mu_n^\pm(0,R)$ by the solvability of the following inhomogeneous equations: For $n \in \mathbf{N}$,

$$
(3) \qquad \begin{cases} F(u''(x), u'(x), u(x), x) + \mu u(x) + \operatorname{sgn}(u(x)) f(x) = 0 & \text{in } (a,b), \\ B(u'(x), u(x), x) = 0 & \text{for } x = a, b, \\ \text{there exist } a = x_{n,0} < x_{n,1} < \cdots < x_{n,n+1} = b \text{ such that} \\ u \neq 0 \text{ in } (x_{n,i}, x_{n,i+1}) \text{ for all } 0 \leq i \leq n \text{ and } u(x_{n,j}) = 0 \text{ for all } 1 \leq j \leq n \end{cases}
$$

where $\mu \in \mathbf{R}$, $f \in L^q(a,b)$, $f \geq 0, \not\equiv 0$ and $\operatorname{sgn}(s) = 1$ if $s \geq 0$, $\operatorname{sgn}(s) = -1$ if $s < 0$. When $N \geq 2$, we assume that $f \in L^q(B_R)$ is radial and consider radial solutions of

$$(4) \quad \begin{cases} F(D^2u(x), Du(x), u(x), x) + \mu u(x) + \operatorname{sgn}(u) f(x) = 0 \quad \text{in } B_R, \\ B(Du(x), u(x), x) = 0 \quad \text{on } \partial\Omega, \\ \text{there exist } 0 = r_{n,0} < r_{n,1} < \cdots < r_{n,n+1} = R \text{ such that } u(x) = u(|x|) \text{ satisfies} \\ u \neq 0 \text{ in } (r_{n,i}, r_{n,i+1}) \text{ for all } 0 \leq i \leq n \text{ and } u(r_{n,j}) = 0 \text{ for all } 1 \leq j \leq n. \end{cases}$$

When $n = 0$, the relationship between $\mu_0^\pm$ and the solvability of (3) and (4) is studied, for instance, in [1, 3, 12]. For other settings, see [2, 10].

Regarding the solvability of (3) and (4), we have

Theorem 2.4 ([8]). *Assume $N = 1$, (F1)–(F3) with $q \in [1, \infty]$, $(\sigma(x), \tau(x)) \in \mathbf{R}^2 \setminus \{(0,0)\}$ for $x = a, b$, $n \in \mathbf{N}$ and $f \in L^q(a,b)$ with $f \geq 0$, $f \not\equiv 0$ in (a,b). Then*
(i) *If $\mu < \mu_n^+$ (resp. $\mu < \mu_n^-$), then (3) admits a solution u (resp. v) such that $\max\{u(a), u'(a)\} > 0$ (resp. $\max\{-v(a), -v'(a)\} > 0$).*
(ii) *If $\mu \geq \mu_n^+$ (resp. $\mu \geq \mu_n^-$), then (3) has no solution satisfying $\max\{u(a), u'(a)\} > 0$ (resp. $\max\{-v(a), -v'(a)\} > 0$).*
(iii) *When $n = 0$, the solution obtained in (i) is unique.*

When $N \geq 2$, we have

Theorem 2.5 ([8]). *Suppose $N \geq 2$, (F1)–(F4) with $q \in (\max\{N/2, q_*\}, \infty]$, $\beta \in L^N(B_R)$ provided $q < N$, $(\sigma, \tau) \in \mathbf{R}^2 \setminus \{(0,0)\}$, $n \in \mathbf{N}$ and that $f \in L^q(B_R)$ is radial and satisfies $f \geq 0$, $f \not\equiv 0$ in $(0, R)$. Then*
(i) *If $\mu < \mu_n^+$ (resp. $\mu < \mu_n^-$), then (4) has a solution u (resp. v) such that $u(0) > 0$ (resp. $-v(0) > 0$).*
(ii) *If $\mu \geq \mu_n^+$ (resp. $\mu \geq \mu_n^-$), then (3) has no solution satisfying $u(0) > 0$ (resp. $-v(0) > 0$).*
(iii) *When $n = 0$, the solution obtained in (i) is unique.*

Form Theorems 2.4 and 2.5, we see that (3) and (4) have a unique solution when $n = 0$ and $\mu < \mu_0^\pm$. However, for the case $n \geq 1$, the uniqueness of solutions of (3) and (4) may fail. In fact, we give an example in Section 4 such that (3) has infinitely many solutions for $n = 1$.

In the rest of this note, we shall prove Proposition 2.3 and state a monotonicity of $\{\mu_n^\pm\}_{n=0}^\infty$ on n in Section 3. See Proposition 3.2. In Section 4, we give some examples related Proposition 2.3 and Theorems 2.4 and 2.5.

3 Proof of Proposition 2.3 and some remarks

In this section, we shall give a proof of Proposition 2.3, namely, the monotonicity of eigenvalues on domains provided the boundary condition is the zero Dirichlet boundary condition. To this end, we need the following proposition:

Proposition 3.1. (i) *Assume that $N = 1$ and (F1)–(F3) with $q \in [1, \infty]$. Let $u, v \in W^{2,1}(a,b)$ satisfy*

$$F[v](x) \leq F[u](x) \quad \text{and} \quad u \leq v \quad \text{in } (a,b)$$

where $F[w](x) := F(w''(x), w'(x), w(x), x)$. *Then either* $u \equiv v$ *in* $[a,b]$ *or else* $u < v$ *in* (a,b), $\max\{(v-u)(a), (v-u)'(a)\} > 0$ *and* $\max\{(v-u)(b), -(v-u)'(b)\} > 0$.
(ii) *Suppose* $N \geq 2$, (F1)-(F4) *with* $q \in (\max\{N/2, q_*\}, \infty]$ *and* $\beta \in L^N(B_R)$ *provided* $q < N$. *Let* $u, v \in W^{2,q}_{\mathrm{r}}(0,R)$ *satisfy*

$$F[v](x) \leq F[u](x) \quad \text{and} \quad u \leq v \quad \text{in } B_R. \tag{5}$$

Then either $u \equiv v$ *in* B_R *or* $u < v$ *in* B_R *and* $\max\{(v-u)(R), -(v-u)'(R)\} > 0$.

Proof. For a proof of Proposition 3.1 (i), see a proof of Theorem 2.6 in [7]. Concerning statement (ii), we give a sketchy proof. First, arguing as in [7, Section 6], thanks to (F2) and (F4), we may find a $\omega_0 \in S^{N-1}$ such that

$$\begin{aligned} &\mathcal{F}(m_1, l_1, p_1, u_1, r) - \mathcal{F}(m_2, l_2, p_2, u_2, r) \\ &\leq \mathcal{P}^+_{\lambda,\Lambda}(m_1 - m_2, l_1 - l_2) + \beta(r\omega_0)|p_1 - p_2| + \gamma(r\omega_0) \end{aligned} \tag{6}$$

for all $(m_i, l_i, p_i, u_i) \in \mathbf{R}^4$ and a.a. $r \in (0,R)$ where

$$\begin{aligned} \mathcal{F}(m,l,p,u,r) &:= F\left(m\omega_0 \otimes \omega_0 + l(I_N - \omega_0 \otimes \omega_0), p\omega_0, u, r\omega_0\right), \\ \mathcal{P}^+_{\lambda,\Lambda}(m,l) &:= \mathcal{M}^+_{\lambda,\Lambda}\left(m\omega_0 \otimes \omega_0 + l(I_N - \omega_0 \otimes \omega_0)\right), \\ \mathcal{P}^-_{\lambda,\Lambda}(m,l) &:= -\mathcal{P}^+_{\lambda,\Lambda}(-m,-l). \end{aligned}$$

Moreover, we may assume that functions defined by $\bar{\beta}(r) := \beta(r\omega_0)$ and $\bar{\gamma}(r) := \gamma(r\omega_0)$ satisfy

$$\bar{\beta}, \bar{\gamma} \in L^q_{\mathrm{r}}(0,R) := \left\{ f \in L^q(0,R) \;\middle|\; \int_0^R |f(r)|^q r^{N-1} dr < \infty \right\}$$

Since one has

$$Du(x) = u'(|x|)\frac{x}{|x|}, \quad D^2u(x) = u''(|x|)P_x + \frac{u'(|x|)}{|x|}\left(I_N - P_x\right), \quad P_x := \frac{x}{|x|} \otimes \frac{x}{|x|}$$

for any radial function u, it follows from (5), (6), (F2) and (F4) that $w(r) := v(r) - u(r)$ satisfies

$$\mathcal{P}^-(w'', w'/r) - \bar{\beta}|w'| - \bar{\gamma}w \leq 0, \quad 0 \leq w \quad \text{in } (0,R).$$

Hence, applying [7, Theorem 7.7], we observe that statement (ii) holds except for the last assertion. Noting

$$\begin{aligned} 0 &\geq \mathcal{P}^-_{\lambda,\Lambda}(w'', w'/r) - \bar{\beta}|w'| - \bar{\gamma}w \\ &\geq \mathcal{M}^-_{\lambda,\Lambda,1\mathrm{d}}(w'') - \left(\bar{\beta} + \frac{\Lambda(N-1)}{r}\right)|w'| - \bar{\gamma}w \quad \text{in } (R/2, R) \end{aligned}$$

where $\mathcal{M}^-_{\lambda,\Lambda,1\mathrm{d}}$ denotes the one-dimensional Pucci operator, we can apply statement (i) on $(R/2, R)$ with

$$F(m,p,u,r) = \mathcal{M}^-_{\lambda,\Lambda,1\mathrm{d}}(m) - \left(\bar{\beta}(r) + \frac{\Lambda(N-1)}{r}\right)p - \bar{\gamma}(r)u$$

and obtain the last assertion in statement (ii). Thus we complete the proof. □

Using Proposition 3.1, we prove Proposition 2.3.

Proof of Proposition 2.3. We consider the case $N = 1$ and $n = 0$. We first treat the case where $a \le c_1 < c_2 < d \le b$, $(\sigma(a), \tau(a)) = (0, 1)$. Set

$$(\mu_1, \varphi_1(x)) := (\mu_0^+(c_1, d), \varphi_0^+(c_1, d)(x)), \quad (\mu_2, \varphi_2(x)) := (\mu_0^+(c_2, d), \varphi_0^+(c_2, d)(x))$$

and we shall prove $\mu_1 < \mu_2$. We argue indirectly and suppose $\mu_2 \le \mu_1$. Put

$$\theta := \sup_{x \in (c_2, d)} \frac{\varphi_2(x)}{\varphi_1(x)}.$$

By Theorem 2.1 (i), $c_1 < c_2$ and $\varphi_1 > 0$ in (c_1, d), we notice that $\max\{\varphi_i(c_2), \varphi_i'(c_2)\} > 0$ and $\max\{\varphi_i(d), -\varphi_i'(d)\} > 0$ for $i = 1, 2$. Hence, when $\sigma(b) \ne 0$, it is easy to see $\varphi_i(d) > 0$ and $0 < \theta < \infty$ from the boundary condition. On the other hand, if $\sigma(b) = 0$, then $\varphi_1(d) = 0 = \varphi_2(d)$ and l'Hôpital's rule asserts

$$\lim_{x \nearrow d} \frac{\varphi_2(x)}{\varphi_1(x)} = \frac{\varphi_2'(d)}{\varphi_1'(d)} < \theta.$$

Thus we have $0 < \theta < \infty$.

Put $\psi_1 := \theta\varphi_1$. Then it follows from (F3), $\mu_2 \le \mu_1$ and $\psi_1, \varphi_2 > 0$ in (c_2, d) that

$$F[\psi_1] + \mu_2\psi_1 \le F[\psi_1] + \mu_1\psi_1 = \theta\,(F[\varphi_1] + \mu_1\varphi_1) = 0 = F[\varphi_2] + \mu_2\varphi_2 \quad \text{in } (c_2, d)$$

and $\varphi_2 \le \psi_1$ in (c_2, d). Hence, Proposition 3.1 implies that either $\varphi_2 \equiv \psi_1$ in $[c_2, d]$ or else $\varphi_2 < \psi_1$ in (c_2, d), $\max\{(\psi_1 - \varphi_2)(c_2), (\psi_1 - \varphi_2)'(c_2)\} > 0$ and $\max\{(\psi_1 - \varphi_2)(d), -(\psi_1 - \varphi_2)'(d)\} > 0$. Recalling $c_1 < c_2$ and $\psi_1(c_2) = \theta\varphi_1(c_2) > 0$, the latter case occurs. Noting

$$\frac{\varphi_2(x)}{\varphi_1(x)} < \theta \quad \text{for all } x \in [c_2, d),$$

we observe that

$$\lim_{x \nearrow d} \frac{\varphi_2(x)}{\varphi_1(x)} = \theta,$$

which yields $(\psi - \varphi_2)(d) = 0$ and $\varphi_2'(d) > \psi_1'(d)$. Hence, when $\sigma(b) \ne 0$, this contradicts

$$\sigma(b)\varphi_2'(d) + \tau(b)\varphi_2(d) = 0 = \sigma(b)\psi_1'(d) + \tau(b)\psi_1(d).$$

When $\sigma(b) = 0$, we have $\varphi_2(d) = 0 = \psi_1(d)$ and l'Hôpital's rule yields

$$1 = \lim_{x \nearrow d} \frac{\varphi_2(x)}{\psi_1(x)} = \frac{\varphi_2'(d)}{\psi_1'(d)} < 1,$$

which is a contradiction. Thus, $\mu_1 < \mu_2$ holds.

For the case where $a \le c < d_1 < d_2 \le b$ and $(\sigma(b), \tau(b)) = (0, 1)$, we introduce the following function:

$$\hat{F}(m, p, u, y) := F(m, -p, u, -y) \quad \text{for every } (m, p, u, y) \in \mathbf{R}^3 \times (-b, -a).$$

Notice that $\hat{F}$ satisfies (F1)–(F3) with the same constants λ, Λ, q and functions $\beta, \gamma \in L^q$. Setting

$$\mu_i := \mu_0^+(c, d_i) \quad \text{for } i = 1, 2, \quad \psi_i(y) := \varphi_0^+(c, d_i)(-y) \quad \text{for } y \in (-d_i, -c),\ i = 1, 2,$$

we may observe that

$$\begin{cases} \hat{F}[\psi_i](y) + \mu_i \psi_i = 0 \quad \text{in } (-d_i, -c), \quad \psi_i > 0 \quad \text{in } (-d_i, -c), \\ -\sigma(b)\psi_i'(-d_i) + \tau(b)\psi_i(-d_i) = 0 = \sigma(a)\psi_i'(-c) + \tau(a)\psi_i(-c). \end{cases}$$

By Theorem 2.1, we obtain $\mu_i = \mu_0^+(-d_i, -c)$. Recalling $-d_2 < -d_1$ and $(\sigma(b), \tau(b)) = (0, 1)$, the case is reduced into the previous case and one has $\mu_2 < \mu_1$. Therefore, Proposition 2.3 holds for the case $N = 1$ and (μ_0^+, φ_0^+).

Next we prove that the case (μ_0^-, φ_0^-) can be reduced into the previous case. In fact, we set

$$F^-(m, p, u, x) := -F(-m, -p, -u, x) \quad \text{for } (m, p, u, x) \in \mathbf{R}^3 \times (a, b),$$
$$\psi_i(x) := -\varphi_0^-(c_i, d)(x).$$

It is easily seen that F^- also satisfies (F1)–(F3) with constants λ, Λ, q and functions $\beta, \gamma \in L^q(a, b)$. Furthermore, since

$$\begin{cases} F^-[\psi_i] + \mu_0^-(c_i, d)\psi_i = 0, \quad \psi_i > 0 \quad \text{in } (a, b), \\ -\sigma(a)(\psi_i)'(c_i) + \tau(a)\psi_i(c_i) = 0 = \sigma(b)\psi_i'(b) + \tau(b)\psi_i(b), \end{cases}$$

we observe that $(\mu_0^-(c_i, d), \psi_i)$ are positive eigenvalues of F^-. Hence, by the previous result, we obtain $\mu_0^-(c_1, d) < \mu_0^-(c_2, d)$. In a similar way, we can also get $\mu_0^-(c, d_2) < \mu_0^-(c, d_1)$. Thus Proposition 2.3 holds for the case $N = 1$ and $n = 0$.

Next we consider the case $N = 1$ and $n \geq 1$. As in the above, by $\hat{F}$ and F^-, it is enough to show $\mu_n^+(c, d_1) < \mu_n^+(c, d_2)$ for $a \leq c < d_1 < d_2 \leq b$. Set $(\mu_i, \varphi_i) := (\mu_n^+(c, d_i), \varphi_n^+(c, d_i))$, let $c < x_{n,1}^i < \cdots < x_{n,n}^i < d_i$ be zeroes of φ_i and put $c := x_{n,0}^i$, $d_i := x_{n,n+1}^i$. Since $x_{n,n+1}^1 = d_1 < d_2 = x_{n,n+1}^2$ and $x_{n,0}^1 = c = x_{n,0}^2$, we put

$$k := \sup\left\{\ell \in \{0, \ldots, n+1\} \mid x_{n,\ell}^2 \leq x_{n,\ell}^1\right\} \in \{0, 1, \ldots, n\}.$$

Then, one has $(x_{n,k}^1, x_{n,k+1}^1) \subset (x_{n,k}^2, x_{n,k+1}^2)$ and $(x_{n,k}^1, x_{n,k+1}^1) \neq (x_{n,k}^2, x_{n,k+1}^2)$. Notice that

$$F[\varphi_i] + \mu_i \varphi_i \doteq 0 \quad \text{in } (x_{n,k}^i, x_{n,k+1}^i)$$

and that φ_i $(i = 1, 2)$ have the same sign and satisfy the zero Dirichlet boundary condition at $x = x_{n,k}^i, x_{n,k+1}^i$ if $k \geq 1$ and $0 = -\sigma(a)u'(x) + \tau(a)u(x)$ at $x = x_{n,0}^i = c$ if $k = 0$. Therefore, by the uniqueness of eigenvalues for $n = 0$, we get $\mu_i = \mu_0^+(x_{n,k}^i, x_{n,k+1}^i)$ or $\mu_i = \mu_0^-(x_{n,k}^i, x_{n,k+1}^i)$. Since $(x_{n,k}^1, x_{n,k+1}^1) \neq (x_{n,k}^2, x_{n,k+1}^2)$, we may apply the result in the case $n = 0$ and obtain $\mu_2 < \mu_1$. When $N = 1$, Proposition 2.3 holds.

When $N \geq 2$, we proceed in a similar way to the case $N = 1$. In fact, for $\mu_0^\pm(0, R_i)$, we can prove our assertion using Proposition 3.1 (ii) instead of Proposition 3.1 (i). For $(\mu_n^+(0, R_i), \varphi_n^+(0, R_i)) =: (\mu_i, \varphi_i)$ with $n \geq 1$, let $0 < r_{n,1}^i < \cdots < r_{n,n}^i < R_i$ be zeroes of φ_i and set $r_{n,0}^i := 0$, $r_{n,n+1}^i := R_i$ and

$$k := \sup\left\{\ell \in \{0, \ldots, n+1\} \mid r_{n,\ell}^2 \leq r_{n,\ell}^1\right\} \in \{0, 1, \ldots, n\}.$$

Then $(r^1_{n,k}, r^1_{n,k+1}) \subset (r^2_{n,k}, r^2_{n,k+1})$, $(r^1_{n,k}, r^1_{n,k+1}) \neq (r^2_{n,k}, r^2_{n,k+1})$ and φ_i $(i = 1, 2)$ have the same sign in $(r^1_{n,k}, r^1_{n,k+1})$.

If $k = 0$, then we have

$$F[\varphi_i](x) + \mu_i \varphi_i = 0, \quad \varphi_i > 0 \quad \text{in } B_{R_i}, \quad \varphi_i(R_i) = 0.$$

Thus, applying the result in the case $n = 0$, we have $\mu_2 < \mu_1$.

On the other hand, if $1 \leq k$, then setting

$$\widetilde{\mathcal{F}}(m, p, u, r) := \mathcal{F}(m, p/r, p, u, r) : \mathbf{R}^3 \times (0, R) \to \mathbf{R},$$

one sees that

$$\widetilde{\mathcal{F}}[\varphi_i] + \mu_i \varphi_i = 0 \quad \text{in } (r^i_{n,k}, r^i_{n,k+1}), \quad \varphi_i(r^i_{n,k}) = 0 = \varphi_i(r^i_{n,k+1}). \tag{7}$$

Since $0 < r^2_{n,k} \leq r^1_{n,k}$, regarding $\widetilde{\mathcal{F}}$ as a function on $\mathbf{R}^3 \times (r^2_{n,k}, r^2_{n,k+1})$, we also observe that $\widetilde{\mathcal{F}}$ satisfies (F1)–(F3). Therefore, we can apply the result in the case $N = 1$ and it follows from (7) that $\mu_2 < \mu_1$. Thus we complete the proof. □

A similar argument is also useful to prove the monotonicity of eigenvalues on n, namely, the number of zeroes of corresponding eigenfunctions. In fact, we have

Proposition 3.2. *Suppose that the assumptions of Theorem 2.1 or Theorem 2.2 hold. Then*

$$\max\{\mu^+_n, \mu^-_n\} < \min\{\mu^+_{n+1}, \mu^-_{n+1}\}.$$

To prove Proposition 3.2, we shall use some characterizations of $\mu^\pm_n$ from [8]. Before stating the result, we need preparations. First, for any $(\sigma(a), \tau(a)), (\sigma(b), \tau(b)) \in \mathbf{R}^2 \setminus \{(0,0)\}$, notice that $(\sigma(a), \tau(a))$ and $-(\sigma(a), \tau(a))$ (resp. $(\sigma(b), \tau(b))$ and $-(\sigma(b), \tau(b))$) give the same boundary condition. Therefore, replacing $(\sigma(a), \tau(a))$ (resp. $(\sigma(b), \tau(b))$) by $-(\sigma(a), \tau(a))$ (resp. $-(\sigma(b), \tau(b))$) if necessary, we may find $\theta_a, \theta_b \in (-\pi/2, \pi/2]$ such that

$$(\sigma(a), \tau(a)) \in \ell(\theta_a), \quad (\sigma(b), \tau(b)) \in \ell(\theta_b)$$
$$\text{where } \ell(\theta) := \{\alpha(\cos\theta, \sin\theta) \in \mathbf{R}^2 \mid \alpha \geq 0\}.$$

Remark that $\theta_a, \theta_b \in (-\pi/2, \pi/2]$ are uniquely determined by $(\sigma(a), \tau(b)), (\sigma(b), \tau(b)) \in \mathbf{R}^2 \setminus \{(0,0)\}$. Thus, it is clear that for the Robin boundary conditions of the form (2), giving $(\sigma(a), \tau(a)), (\sigma(b), \tau(b)) \in \mathbf{R}^2 \setminus \{(0,0)\}$ is equivalent to giving $\theta_a, \theta_b \in (-\pi/2, \pi/2]$. Similarly, in the case $N \geq 2$, for the Robin boundary condition of the form (2), to give $(\sigma_R, \tau_R) \in \mathbf{R}^2 \setminus \{(0,0)\}$ is equivalent to giving $\theta_R \in (-\pi/2, \pi/2]$, and these quantities are related in the following sense: $(\sigma_R, \tau_R) \in \ell(\theta_R)$.

In what follows, instead of $(\sigma(x), \tau(x)) \in \mathbf{R}^2 \setminus \{(0,0)\}$ for $x = a, b$, we consider $\theta_a, \theta_b \in (-\pi/2, \pi/2]$. Under these conventions, we have the following characterizations of $\mu^\pm_n$.

Theorem 3.3 ([8]). (i) *Let $N = 1$ and the assumptions in Theorem 2.1 hold. Assume that $\theta_{i,a}, \theta_{i,b} \in (-\pi/2, \pi/2]$ $(i = 1, 2)$ satisfy $\theta_{1,a} \leq \theta_{2,a}$ and $\theta_{1,b} \leq \theta_{2,b}$. Let $\mu^\pm_{i,n}$ $(i = 1, 2)$ denote eigenvalues of F under the boundary condition of the form (2) corresponding to $(\theta_{i,a}, \theta_{i,b})$. Then $\mu^\pm_{1,n} \leq \mu^\pm_{2,n}$.*
(ii) *Let $N \geq 2$ and the assumptions of Theorem 2.2 hold. Suppose that $\theta_{i,R} \in (-\pi/2, \pi/2]$ $(i = 1, 2)$ satisfy $\theta_{1,R} \leq \theta_{2,R}$. Then the corresponding eigenvalues $\mu^\pm_{i,n}$ satisfy $\mu^\pm_{1,n} \leq \mu^\pm_{2,n}$.*

Remark 3.4. In [8], we use different notation. However, using the results in [8], it is not difficult to see that Theorem 3.3 holds.

With the aid of Proposition 2.3 and Theorem 3.3, we prove Proposition 3.2.

Proof of Proposition 3.2. We first consider the case $N = 1$. Let $(\mu_n^{\pm}, \varphi_n^{\pm})$ denote the sequences of eigenvalues of (1), $(x_{n,k}^{\pm})_{k=1}^{n}$ zeroes of $\varphi_n^{\pm}$, and set $x_{n,0}^{\pm} = a$ and $x_{n,n+1}^{\pm} = b$.

We prove $\mu_n^+ < \mu_{n+1}^+$. Since

$$a = x_{n,0}^+ < x_{n,1}^+ < \cdots < x_{n,n+1}^+ = b, \quad a = x_{n+1,0}^+ < x_{n+1,1}^+ < \cdots < x_{n+1,n+2}^+ = b,$$

as in the proof of Proposition 2.3, set

$$k := \sup\{\ell \in \{0, 1, \ldots, n+1\} \mid x_{n,\ell}^+ \leq x_{n+1,\ell}^+\} \in \{0, \ldots, n\}.$$

Then it is easily seen that

$$(x_{n+1,k}^+, x_{n+1,k+1}^+) \subset (x_{n,k}^+, x_{n,k+1}^+), \quad (x_{n+1,k}^+, x_{n+1,k+1}^+) \neq (x_{n,k}^+, x_{n,k+1}^+).$$

When $k = 0$, we have $(a, x_{n+1,1}^+) \subset (a, x_{n,1}^+)$ and $x_{n+1,1}^+ < x_{n,1}^+$. Since φ_n^+ (resp. φ_{n+1}^+) is a positive eigenfunction in $(a, x_{n,1}^+)$ (resp. $(a, x_{n+1,1}^+)$), satisfies the zero Dirichlet boundary condition at $x = x_{n,1}^+$ (resp. $x = x_{n+1,1}^+$) and μ_n^+ (resp. μ_{n+1}^+) corresponds to the positive eigenvalue of F, we may apply Proposition 2.3 to obtain $\mu_n^+ < \mu_{n+1}^+$.

When $1 \leq k < n$, since the boundary conditions are the zero Dirichlet boundary condition, μ_n^+ (resp. μ_{n+1}^+) is either a positive or negative eigenvalue of F in $(x_{n,k}^+, x_{n,k+1}^+)$ (resp. $(x_{n+1,k}^+, x_{n+1,k+1}^+)$) and their signs are equal, i.e., $\varphi_n^+\varphi_{n+1}^+ > 0$ in $(x_{n+1,k}^+, x_{n+1,k+1}^+)$, Proposition 2.3 yields $\mu_n^+ < \mu_{n+1}^+$.

When $k = n$, we have $(x_{n+1,n}^+, x_{n+1,n+1}^+) \subset (x_{n,n}^+, b)$ and $x_{n+1,n+1}^+ < b$. Remark that μ_n^+ and μ_{n+1}^+ are positive (resp. negative) eigenvalues of F in $(x_{n,n}^+, b)$ and $(x_{n+1,n}^+, x_{n+1,n+1}^+)$ provided n is even (resp. odd) and φ_{n+1}^+ satisfies the zero Dirichlet boundary condition at $x = x_{n+1,n}^+, x_{n+1,n+1}^+$. Hence, let ν be a positive (resp. negative) eigenvalue of F in $(x_{n,n}^+, b)$ under the zero Dirichlet boundary conditions at $x = x_{n,n}^+, b$ if n is even (resp. odd). Then it follows from Proposition 2.3 that $\nu < \mu_{n+1}^+$. On the other hand, since the zero Dirichlet condition corresponds to the case $\theta_b = \pi/2$ in the notation above to Theorem 3.3, we also obtain $\mu_n^+ \leq \nu$ by Theorem 3.3. Thus $\mu_n^+ < \mu_{n+1}^+$ holds. Moreover, by F^- as in the proof of Proposition 2.3, we also observe that $\mu_n^- < \mu_{n+1}^-$.

Now we prove $\mu_n^- < \mu_{n+1}^+$. We notice that φ_n^- and φ_{n+1}^+ have the same sign in $(\max\{x_{n,n}^-, x_{n+1,n+1}^+\}, b)$, namely,

$$\varphi_n^-\varphi_{n+1}^+ > 0 \quad \text{in } (\max\{x_{n,n}^-, x_{n+1,n+1}^+\}, b).$$

Hence, using $\hat{F}$ as in the proof of Proposition 2.3, the case can be reduced into a proof of $\mu_n^+ < \mu_{n+1}^+$ or $\mu_n^- < \mu_{n+1}^-$. Thus, by the previous result, we get $\mu_n^- < \mu_{n+1}^+$.

By F^- and $\hat{F}$, we also see that $\max\{\mu_n^+, \mu_n^-\} < \mu_{n+1}^-$, which implies that

$$\max\{\mu_n^+, \mu_n^-\} < \min\{\mu_{n+1}^+, \mu_{n+1}^-\}.$$

Thus when $N = 1$, Proposition 3.2 holds.

Next, we treat the case $N \geq 2$. Let $\{(\mu_n^{\pm}, \varphi_n^{\pm})\}_{n=0}^{\infty}$ be eigenpairs, $0 < r_{n,1}^{\pm} < \cdots < r_{n,n+1}^{\pm} < R$ zeroes of $\varphi_n^{\pm}$, and set $r_{n,0}^{\pm} = 0$ and $r_{n,n+1}^{\pm} = R$. We first notice that by

Proposition 2.3, one can prove $\mu_n^+ < \mu_{n+1}^+$ and $\mu_n^- < \mu_{n+1}^-$ in a similar way to the case $N = 1$. Now we prove $\mu_n^- < \mu_{n+1}^+$. For this purpose, we set

$$k := \inf\{\ell \in \{0, 1, \dots, n+1\} \mid r_{n+1,\ell+1}^+ \le r_{n,\ell}^-\}.$$

and remark that $0 < k \le n+1$ since $r_{n+1,1}^+ > 0 = r_{n,0}^-$. By the definition of k and properties of $\varphi_n^\pm$, we obtain $(r_{n+1,k}^+, r_{n+1,k+1}^+) \subset (r_{n,k-1}^-, r_{n,k}^-)$, $(r_{n+1,k}^+, r_{n+1,k+1}^+) \ne (r_{n,k-1}^-, r_{n,k}^-)$ and $\varphi_n^- \varphi_{n+1}^+ > 0$ in $(r_{n+1,k}^+, r_{n+1,k+1}^+)$. When $k \ge 2$, noting $r_{n,k-1}^- > 0$, we can prove $\mu_n^- < \mu_{n+1}^+$ in a similar way to the case $N = 1$ with the aid of $\mathcal{F}$.

When $k = 1$, we remark that $\varphi_n^-(r_{n+1,1}^+) < 0$, hence, for sufficiently small $\varepsilon > 0$, we may find $\theta_\varepsilon \in (-\pi/2, \pi/2)$ so that

$$-(\varphi_n^-)'(r_{n+1,1}^+ - \varepsilon)\cos\theta_\varepsilon + \varphi_n^-(r_{n+1,1}^+ - \varepsilon)\sin\theta_\varepsilon = 0. \tag{8}$$

Since we may also find $\theta \in (-\pi/2, \pi/2]$ so that

$$(\varphi_n^-)'(r_{n+1,2}^+)\cos\theta + \varphi_n^-(r_{n+1,2}^+)\sin\theta = 0, \tag{9}$$

we observe that (μ_n^-, φ_n^-) is a negative eigenpair of $\mathcal{F}$ on $(r_{n+1,1}^+ - \varepsilon, r_{n+1,2}^+)$ under the boundary conditions (8) and (9). Let ν_ε be a negative eigenfunction of $\mathcal{F}$ under the zero Dirichlet boundary condition on $(r_{n+1,1}^+ - \varepsilon, r_{n+1,2}^+)$. Notice that $\nu_0 = \mu_{n+1}^+$ and $\nu_\varepsilon < \mu_{n+1}^+$ thanks to Proposition 2.3. On the other hand, applying Theorem 3.3, we obtain $\mu_n^- \le \nu_\varepsilon$, which implies $\mu_n^- < \mu_{n+1}^+$. Hence, $\max\{\mu_n^+, \mu_n^-\} < \mu_{n+1}^+$ when $N \ge 2$. Using F^-, we may also show $\max\{\mu_n^+, \mu_n^-\} < \mu_{n+1}^-$ and Proposition 3.2 holds in the case $N \ge 2$. □

4 Examples

In this section, we give examples related to Proposition 2.3 and Theorems 2.4 and 2.5. More precisely, regarding Proposition 2.3, we give examples in which the monotonicity of eigenvalues on domains may fail when we replace the zero Dirichlet boundary condition by the Robin boundary condition. On the other hand, about Theorems 2.4 and 2.5, we provide examples in which (3) and (4) have infinitely many solutions when $n = 1$.

4.1 Example about the monotonicity of eigenvalues

We first consider an example in which the monotonicity of eigenvalues on domains fails when we change the zero Dirichlet boundary condition. We prove this fact by the argument based on the strong maximum principle, namely, Proposition 3.1. We only treat the case $N = 1$. Fix a $\theta_2 \in (0, \pi/2)$. Then we consider the following boundary condition of the form (2):

$$B_1(p, u) := u, \quad B_2(p, u) := p\cos\theta_2 + u\sin\theta_2.$$

Remark that

$$0 = B_2(-\sin\theta_2, \cos\theta_2).$$

Next, select $p > 0$ so that

$$\cos\theta_2 > \frac{\sin\theta_2}{p}$$

and set

$$v(x) := \cos\theta_2 \cosh\left\{p\left(x - \theta_2 - \frac{\pi}{2}\right)\right\} - \frac{\sin\theta_2}{p}\sinh\left\{p\left(x - \theta_2 - \frac{\pi}{2}\right)\right\}.$$

From the choice of p and the definition of v, it is easily seen that

$$v(x) > 0 \quad \text{in } \mathbf{R}, \quad \lim_{x\to\infty} v(x) = \infty, \quad v''(x) = p^2 v(x), \quad v' \text{ is increasing in } \mathbf{R}.$$

Next, put $a := 0$ and $b := \theta_2 + \pi/2 \in (\pi/2, \pi)$. Then $v(x)$ is rewritten as

$$v(x) = \cos\theta_2 \cosh\{p(x-b)\} - \frac{\sin\theta_2}{p}\sinh\{p(x-b)\}.$$

Since $v'(b) = -\sin\theta_2 < 0$ and $v(b) = \cos\theta_2 > 0$, there is a $c \in (b,\infty)$ such that $v(c) = \cos\theta_2 = v(b)$. Here we remark that $v'(c) > 0$ and $v(c) > 0$, hence, we may find $\theta_3 \in (-\pi/2, 0)$ such that

$$B_3(v'(c), v(c)) := v'(c)\cos\theta_3 + v(c)\sin\theta_3 = 0.$$

Now we define $F(m,p,u,x)$ by

$$F(m,p,u,x) := m - (p^2+1)\chi_{(b,c)}(x)u, \quad \psi(x) := \begin{cases} \sin x & \text{if } x \in [a,b], \\ v(x) & \text{if } x \in (b,c]. \end{cases}$$

It is immediate to see that $\psi \in W^{2,\infty}(a,c)$ satisfies

$$\begin{cases} F[\psi] + \psi = 0 \quad \text{in } (a,c), \quad \psi > 0 \quad \text{in } (a,c), \\ B_1(\psi'(a), \psi(a)) = B_2(\psi'(b), \psi(b)) = B_3(\psi'(c), \psi(c)) = 0. \end{cases}$$

Thus, $(1,\psi)$ is a positive eigenpair of F in (a,b) and (a,c) under the boundary conditions B_1, B_2 and B_1, B_3.

Let $\mu_{a,c}$ be a positive eigenvalue of F under the boundary conditions B_1 and B_2. Our aim here is to prove $\mu_{a,c} > 1$. If this is true, then we may observe that the dependence of positive eigenvalue of F on domains under the boundary conditions B_1 and B_2 is not monotone.

Now we prove $\mu_{a,c} > 1$. We first notice that $(1,\psi)$ is a positive eigenpair of F on (a,c) under the boundary conditions B_1 and B_3. Recalling $\theta_3 \in (-\pi/2, 0)$, $\theta_2 \in (0,\pi/2)$ and Theorem 3.3, we observe that $1 \le \mu_{a,c}$. Let us suppose $\mu_{a,c} = 1$ and φ is a positive eigenfunction of F corresponding to $\mu_{a,c}$. Define

$$\rho := \sup_{(a,c)} \frac{\varphi(x)}{\psi(x)} \in (0,\infty).$$

Then $\varphi(x) \le \rho\psi(x)$ in $[a,c]$. Moreover, we have

$$F[\rho\psi] + \rho\psi = \rho\left(F[\psi] + \psi\right) = 0 = F[\varphi] + \varphi \quad \text{in } (a,c).$$

Applying Proposition 3.1 with $F(m,p,u,x) + u$, we obtain either $\rho\psi \equiv \varphi$ in (a,c) or else $\rho\psi > \varphi$, $\max\{(\rho\psi - \varphi)(a), (\rho\psi - \varphi)'(a)\} > 0$ and $\min\{(\rho\psi - \varphi)(c), -(\rho\psi - \varphi)'(c)\} > 0$.

Next, we remark that the first case does not occur since it follows from $\theta_2 \in (0, \pi/2)$ and $\psi(c), \psi'(c) > 0$ that $B_2(\rho\psi'(c), \rho\psi(c)) > 0$. Thus the latter case occurs. By definition of θ, we observe that either $\rho\psi(a) = \varphi(a)$ or else $\rho\psi(c) = \varphi(c)$. However, if $\rho\psi(c) = \varphi(c)$, then noting that $\rho\psi'(c) > 0 > \varphi'(c)$ due to $-\pi/2 < \theta_3 < 0 < \theta_2 < \pi/2$ and $B_2(\varphi'(c), \varphi(c)) = 0 = B_3(\rho\psi'(c), \rho\psi(c))$, one obtains $-(\rho\psi - \varphi)'(c) < 0$. This contradicts $\max\{(\rho\psi - \varphi)(c), -(\rho\psi - \varphi)'(c)\} > 0$ by $\rho\psi(c) = \varphi(c)$. Thus $\rho\psi(c) > \varphi(c)$ and $\rho\psi(a) = 0 = \varphi(a)$. Since $0 \leq \varphi'(a) < \rho\psi'(a)$, we obtain

$$\lim_{x \to a} \frac{\varphi(x)}{\psi(x)} = \frac{\varphi'(a)}{\psi'(a)} < \rho.$$

Recalling $\varphi < \rho\psi$ in $(a, c]$, we have a contradiction. Thus we have $\mu_{a,c} > 1$.

4.2 Example related to the multiplicity of solutions of (3) and (4)

Finally, we give examples in which (3) and (4) have infinitely many solutions for $n = 1$. First, we consider the case $N = 1$. Set

$$(a, b) := (0, 7), \quad F_0(m, p, u, x) := m + \frac{\pi^2}{4}\chi_{(6,7)}(x)u, \quad f(x) := \chi_{(1,3)}(x) + \chi_{(4,6)}(x).$$

Then F_0 satisfies (F1)–(F3) with $\lambda = 1 = \Lambda$, $q = \infty$, $\beta = 0$ and $\gamma(x) = \pi^2/4$. Let us consider the equation:

$$(10) \qquad F_0[u] + \operatorname{sgn}(u) f = 0 \quad \text{in } (0,7), \quad u'(0) = 0 = u(7), \quad u \text{ has one zero in } (0,7).$$

Remark that this corresponds to the case $\mu = 0$ and $n = 1$ in (3).

Next we claim that $\mu_0^\pm = 0 < \mu_1^\pm$. Indeed, it is easy to check that a function defined by

$$\varphi_0(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq 6, \\ \cos\left(\frac{\pi}{2}(x-6)\right) & \text{if } 6 < x \leq 7 \end{cases}$$

satisfies

$$F_0[\varphi_0] = 0 \quad \text{in } (0,7), \quad \varphi_0 > 0 \quad \text{in } (0,7), \quad \varphi_0'(0) = 0 = \varphi_0(7).$$

Hence, we observe that $(0, \varphi_0)$ is a positive eigenpair of F_0. Noting that $\mu_k^+ = \mu_k^-$ holds for all k since F_0 is linear, Proposition 3.2 asserts $\mu_0^\pm = 0 < \mu_1^\pm$.

Now for $0 \leq t \leq 2$, set

$$u_t(x) := \begin{cases} 2 + t & \text{if } 0 \leq x < 1, \\ (2+t) - (x-1)^2/2 & \text{if } 1 \leq x < 3, \\ t - 2(x-3) & \text{if } 3 \leq x < 4, \\ t - 2 - 2(x-4) + (x-4)^2/2 & \text{if } 4 \leq x < 6, \\ (t-4)\cos(\pi(x-6)/2) & \text{if } 6 \leq x < 7. \end{cases}$$

Then by direct calculations, one sees that u_t $(t \in [0, 2])$ is a solution of (10). Hence, the uniqueness of solutions of (10) fails.

Lastly, we treat the case $N \geq 2$. Let $\Omega := B_7(0)$ and set

$$F(M,p,u,x) := \mathrm{Tr}(M) - \chi_{1,7}(x)\frac{N-1}{|x|}\left\langle \frac{x}{|x|}, p\right\rangle - \frac{\pi^2}{4}\chi_{6,7}(x)u,$$
$$f(x) := \chi_{1,3}(x) + \chi_{4,6}(x)$$

where $\chi_{i,j}(x)$ denotes the characteristic function of annulus $\{x \in \mathbf{R}^N \mid i < |x| < j\}$. Notice that F satisfies (F1)–(F4) with $q = \infty$. Note also that for radial functions u, we have

$$F(D^2u) = \Delta u \quad \text{in } B_1(0), \quad \mathcal{F}(u''(r), u'(r), u(r), r) = F_0(u''(r), u'(r), u(r), r) \quad \text{in } (1,7)$$

where F_0 appears in the above example.

Now one can check that a function $\psi_0(r) := \varphi_0(r) \in W^{2,\infty}_{\mathrm{r}}(0,7)$ satisfies $\mathcal{F}[\psi_0] = 0$, $\psi_0 > 0$ in $[0,7)$ and $\psi_0(7) = 0$. In this case, by the linearity of F and Proposition 3.2, one has $0 = \mu_0^{\pm} < \mu_1^{\pm}$. Setting $v_t(r) := u_t(r) \in W^{2,\infty}_{\mathrm{r}}(0,7)$ $(t \in [0,2])$, it is not hard to check that v_t satisfies

$$\mathcal{F}[v_t](r) + \mathrm{sgn}(v_t)f(r) = 0 \quad \text{in } (0,7), \quad v_t \text{ has exaxtly one zero in } (0,7), \quad v_t(7) = 0.$$

Since this equation corresponds to (4) with $n = 1$ and $\mu = 0 < \mu_1^{\pm}$. Thus the uniqueness of solutions of (4) does not hold.

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