数理解析研究所講究録
第 1960 巻 2015 年 37-45

Removing local extrema of surfaces in open book decompositions

Keiko Kawamuro
Department of Mathematics, University of Iowa

1. Open book foliations

In this section we recall some terminologies in the open book foliation package that are needed for this note. Open book foliations are defined by Ito and Kawamuro in [14]. The idea of open book foliations came from Bennequin's work [1] and Birman and Menasco's braid foliations studied in the series of papers [2, 3, 4, 5, 6, 7, 8, 9, 10]. Many terminologies of open book foliations are taken from those of braid foliations.

Let $S = S_{g,r}$ be a compact oriented genus g surface with r (> 0) boundary components. Let $\phi : S \to S$ be an orientation preserving diffeomorphism that fixes the boundary ∂S point-wise. Let $M_{(S,\phi)}$ denote the 3-manifold given by the (abstract) open book (S, ϕ). That is $M_{(S,\phi)}$ is obtained from the mapping torus M_ϕ of ϕ with solid tori glued trivially along all the boundary tori of M_ϕ. (See, for example, Etnyre's survey article [12] for more detail.) We denote the binding of the open book by B and the pages by S_t where $t \in [0, 1]$.

Consider a compact oriented surface F embedded in $M_{(S,\phi)}$. The surface F may or may not have boundary. If F has boundary we require that the boundary ∂F is in a braid position with respect to the open book (S, ϕ). That is, ∂F is transverse to the pages positively, is never tangent to pages, and does not intersect the binding B. Let $\mathcal{F}_{ob}(F)$ be a singular foliation on the surface F given by the intersection of F and the pages $\{S_t \mid t \in [0, 1]\}$. We call $\mathcal{F}_{ob}(F)$ the *open book foliation* on F with respect to the open book (S, ϕ).

Up to isotopy we may assume that all the singularities of $\mathcal{F}_{ob}(F)$ are elliptic or hyperbolic. In particular no local extrema exist. An elliptic point is a transverse intersection of F and the binding B. A hyperbolic point is a saddle tangency of F and a page S_t. See Figure 1.

Non-singular leaves of $\mathcal{F}_{ob}(F)$ are classified into three types: a-arcs, b-arcs, and c-circles. See Figure 2. An a-arc is a clasp-intersection of F and a page S_t and it joins a point on the binding B and a point on the braid ∂F. A b-arc is a ribbon-intersection of F and a page S_t and it joins two points of B. A c-circle is a simple closed curve where F and S_t intersect.

2. Main result

The main result we focus in this note is the following:

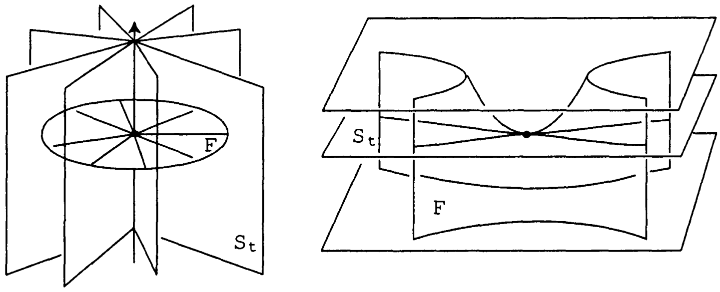

FIGURE 1. An elliptic point (left) and a hyperbolic point (right).

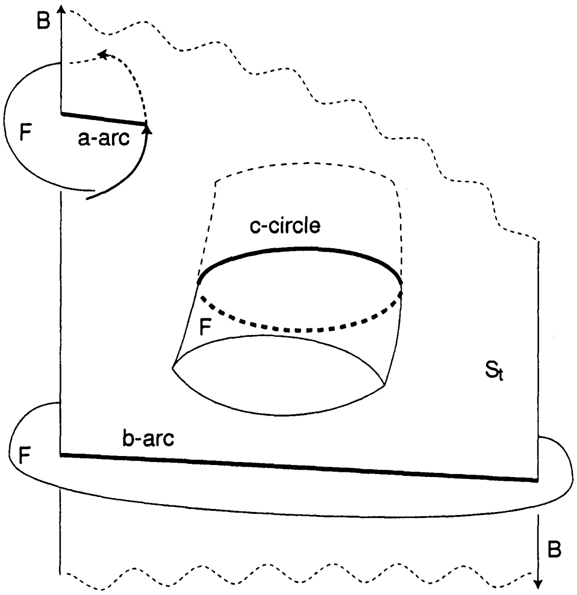

FIGURE 2. a-arc, b-arc, and c-circle.

Theorem 2.1. [15] *Suppose that a surface $F \subset M_{(S,\phi)}$ is incompressible of an essential sphere. One can find a surface $F' \subset M_{(S,\phi)}$ such that F and $F'\#S^2\#\ldots\#S^2$ (the connect sum of F' and possibly empty essential spheres) are isotopic and:*

> $(*)$: *All the b-arcs of $\mathcal{F}_{ob}(F')$ are essential arcs in each punctured page $S_t \setminus (S_t \cap \partial F')$.*

If the above condition $(*)$ is satisfied we say that the open book foliation is *essential.* See Figure 3.

This technical theorem has interesting applications to topology and geometry of the open book manifold $M_{(S,\phi)}$ (cf. [15]), braids in open books (cf. [16]), and the contact structure supported by (S,ϕ) (cf. [17]). We will discuss some of them in Section 3.

2.1. **Sketch of the proof of Theorem 2.1.** A detailed proof of Theorem 2.1 can be found in [15]. Our proof consists of two steps.

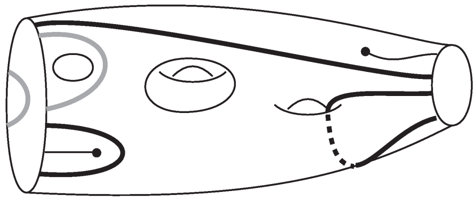

FIGURE 3. Thick black (resp. gray) arcs are essential (resp. non-essential) b-arcs.

(**Step 1**): We may assume that the surface F admits an open book foliation $\mathcal{F}_{ob}(F)$ (we actually allow local extremal points at this moment).

If the foliation contains a non-essential b-arc in S_t (i.e., a boundary parallel arc in the punctured page $S_t \setminus (S_t \cap \partial F)$ then we push a neighborhood of the b-arc along the cobounded disc with B. See Figure 4. We apply this operation to all the non-essential b-arcs.

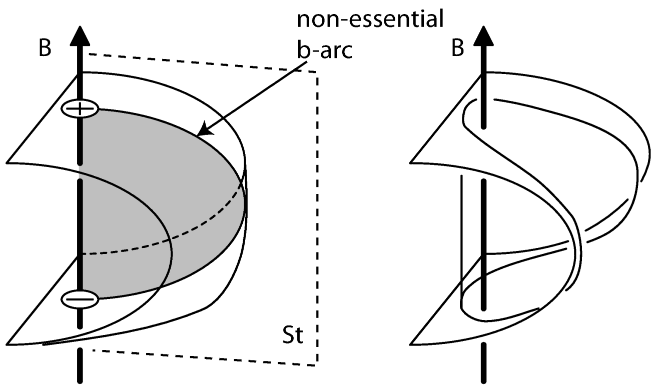

FIGURE 4

As shown in Figure 5 the operation removes a pair of elliptic points from $\mathcal{F}_{ob}(F)$ but introduces a pair of local minimum and maximum at the same time. In the following

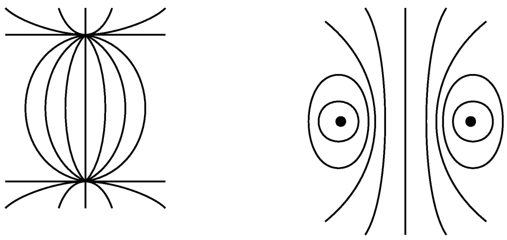

FIGURE 5

we modify F by isotopy away from the binding so that $\mathcal{F}_{ob}(F)$ has no local extrema and every tangency of F and S_t is a saddle point. That is, $\mathcal{F}_{ob}(F)$ becomes an essential open book foliation. This concludes the theorem.

One may think that Thurston's general position theorem [18] can be applied for our purpose. However we cannot directly apply Thurston's argument. Let us compare Thurston's theorem and our setting.

Theorem 2.2 (Thurston's general position theorem). [18, 11] *Assume that $\mathcal{F}$ is a taut foliation on a compact oriented 3-manifold M and transverse to ∂M. Let $F \hookrightarrow M$ be an admissible surface (i.e., F is an incompressible surface or an essential sphere). Then, through admissible imbeddings, F is isotopic to a leaf of $\mathcal{F}$ or is isotopic to a surface all of whose tangencies with $\mathcal{F}$ are saddles.*

See [11, Definition 9.5.2] for the precise definition of "admissible".

In our case the family of the pages $\{\mathrm{Int}(S_t) \mid t \in [0,1]\}$ gives a taut foliation on $M \setminus B$. We assume that $F \subset M$ is incompressible or an essential sphere, which does not immediately guarantee that $(F \setminus (F \cap B)) \subset (M \setminus B)$ is incompressible. Therefore we cannot directly apply Thurston's result to our case.

(**Step 2**): In this step we remove all the local minima of $\mathcal{F}_{ob}(F)$ by isotopy and desumming essential spheres. Our method will not introduce new non-essential b-arcs.

Local minima can be classified into the following three types: Let p be a local minimum. As the time parameter t increases we see a smooth family of c-circles arising from p until it forms a saddle point q_1. Each c-circle bounds a disc which we denote by $X = X_t$ ($\subset S_t$).

Type I: If a describing arc of the saddle point q_1 joins a point of the c-circle ∂X and another leaf then we say that p is a Type I local minimum and q_1 is a Type I saddle. See Figure 6.

Type II: If a describing arc of the saddle point q_1 joins two points of ∂X and is embedded in the complement of X then we say that p is a Type II local minimum and q_1 is a Type II saddle.

Type III: If a describing arc of the saddle point q_1 is an embedded arc in the region X then we say that p is a Type III local minimum and q_1 is a Type III saddle.

Suppose that q_i is a Type II saddle point arising from a local minimum p. Let $\{X_t\}$ be the family of connected regions bounded by the c-circles that arise from p. The next saddle point q_{i+1} is called

- Type I if a describing arc for q_{i+1} joins a point of ∂X and another leaf,
- Type II if a describing arc for q_{i+1} joins two points of ∂X and is embedded in the complement of X,
- Type III if a describing arc for q_{i+1} is embedded in the region X.

For each local minimum p we get a family of connected regions $\{X_t\}$ arising from p and ending at a region containing a Type I or Type III saddle. We say that p is *innermost* if the family $\{\mathrm{Int}(X_t)\}$ contains no local minimum other than p.

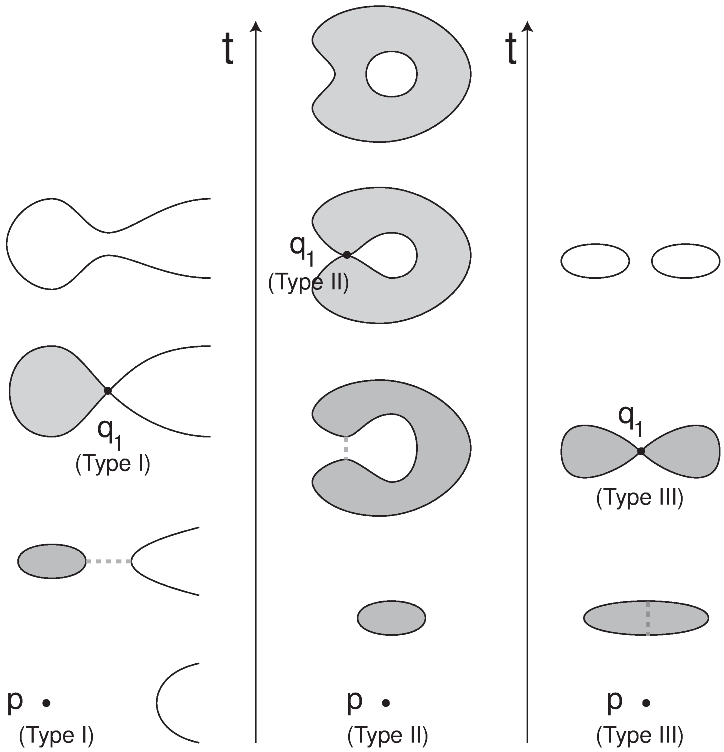

FIGURE 6. Movie presentations of Type I, II, III local minima p and saddles q_1. Shaded regions represent $X = X_t$. Dashed arcs are describing arcs of the saddle points.

We start with an innermost local minimum p. (The case where p is not innermost is discussed in [15]). We have three cases to study.

(**Case A**): If p is of Type I, by isotopy, we flatten the bump that reduces the number of local minima by one.

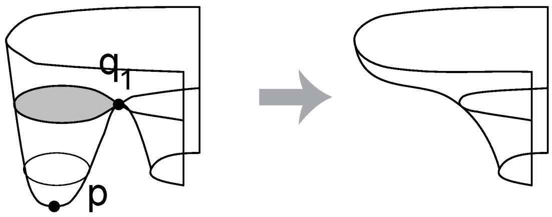

FIGURE 7. Case A

(**Case B**): If p is of Type III we compress the surface along the vertical disc (shaded in Figure 8). Since our surface is incompressible this gives a connected sum $F = F'\#S^2$ of an incompressible surface (or an essential sphere), F', and an essential sphere S^2. Though F and F' have the same number of local minima we note that the number of saddles of F' is smaller than that of F. Thus F' has less complexity than F.

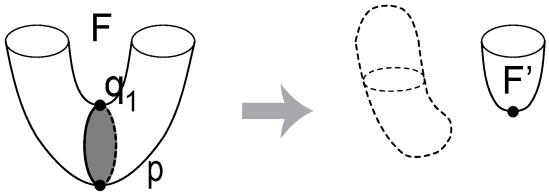

FIGURE 8. Case B

(**Case C**): Now we assume that p is of Type II and followed by a sequence of Type II saddles, $q_1, \ldots, q_{k-1}$, and ended with a Type I or III saddle, q_k. We have two observations:

(1) The order of consecutive Type II and Type I saddles (eg. q_{k-1} and q_k) can be changed by a local isotopy.
(2) If q_k is of Type III then, with a local isotopy, p becomes Type III. In other words, we can find a new sequence of saddles $q'_1, \ldots, q'_k$ such that q'_1 is of Type III and $q'_2, \ldots, q'_k$ are of Type II. Figure 9 gives such an example.

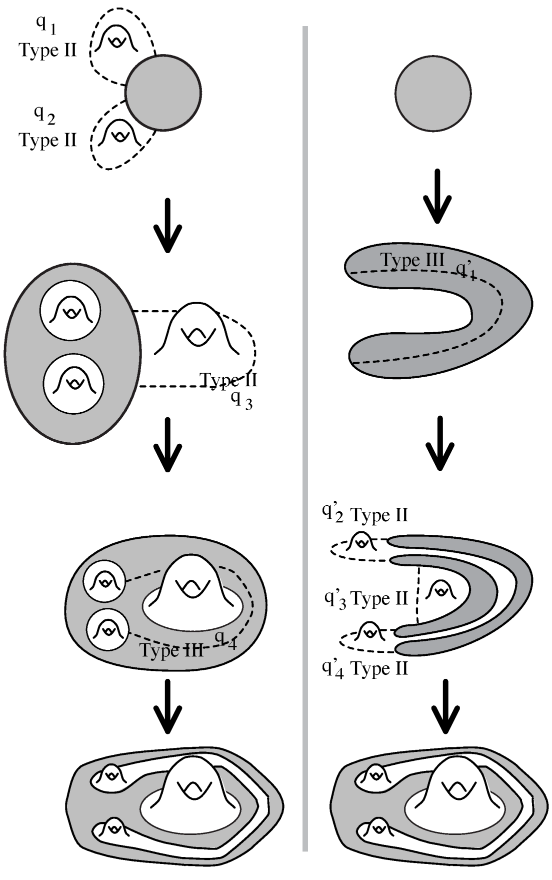

FIGURE 9. Observation (2)

The observation (1) holds because describing arcs of consecutive Type II and Type I saddles (strictly speaking, projected onto a same page) are disjoint.

The observation (2) needs more care. A parallel statement to (1) does not hold in general. That is, there is an example (like the one in Figure 9) where the order of q_{k-1} and q_k is not changeable.

The observations (1) and (2) imply that each Type II local minimum can become Type I or Type III by isotopy. Hence (Case C) is reduced to (Case A) or (Case B).

3. Applications of Theorem 2.1

Among the above mentioned applications of Theorem 2.1, here we give two corollaries. Let $c(\phi, C)$ denote the fractional Dehn twist coefficient (FDTC) of a diffeomorphism $\phi : S \to S$ with respect to the boundary component $C \subset \partial S$. See [13] for the definition of FDTC.

Corollary 3.1. [15] *Let $\phi : S \to S$ be a diffeomorphism that is freely isotopic to a periodic or pseudo-Anosov homeomorphism. If*

(1) $|c(\phi, C)| > 4$ *for all the boundary components $C \subset \partial S$, or*
(2) *the boundary ∂S is connected and $|c(\phi, \partial S)| > 1$*

then the manifold $M_{(S,\phi)}$ is irreducible and atoroidal.

The next is a corollary of Corollary 3.1.

Corollary 3.2. [15] *Supposed that $\phi : S \to S$ satisfies the conditions (1) and (2) in Corollary 3.1. Then we have the following:*

- *$M_{(S,\phi)}$ is toroidal if and only if ϕ is reducible.*
- *$M_{(S,\phi)}$ is hyperbolic if and only if ϕ is freely isotopic to a pseudo-Anosov homeomorphism.*
- *$M_{(S,\phi)}$ is a Seifert fibered space if and only if ϕ is freely isotopic to a periodic homeomorphism.*

Without the conditions (1) and (2) this corollary does not hold. For example, if $\phi = id_S$ (i.e., periodic) and ∂S is connected then $c(\phi, \partial S) = 0$ and $M_{(S,\phi)} = \overset{2g}{\#} (S^1 \times S^2)$, which is not Seifert fibered.

3.1. Proof of Corollary 3.1.

Here is a lemma for Corollary 3.1.

Lemma 3.3. [15] *Let $F \subset M_{(S,\phi)}$ be a closed, genus g surface satisfying the condition $(*)$ of Theorem 2.1 and intersecting the binding B in $2n$ (> 0) points.*

- *Suppose that S has connected boundary. Then*

$$|c(\phi, \partial S)| \le \begin{cases} 1 & \text{if } g = 1 \\ g & \text{if } g > 1. \end{cases}$$

- *Suppose that $|\partial S| > 1$. Then there exists a component $C \subset \partial S$ such that*
$$|c(\phi, C)| \leq \begin{cases} 3 & \textit{if } g = 1 \\ 4 + \lfloor \frac{4g-4}{n} \rfloor & \textit{if } g > 1. \end{cases}$$

Proof of Corollary 3.1. We first show irreducibility of $M_{(S,\phi)}$. Assume that there exists an essential sphere $F \subset M_{(S,\phi)}$. By Theorem 2.1 the condition $(*)$ is satisfied. Assumptions (1) and (2) of Corollary 3.1 and Lemma 3.3 imply $n = 0$. The Euler characteristic satisfies $2 = \chi(F) = e - h$, where e (≥ 0) (resp. h (≥ 0)) is the number of elliptic (resp. hyperbolic) points in the open book foliation $\mathcal{F}_{ob}(F)$. Recall that elliptic points are exactly the intersection of F and B. Thus $2n = e = 2 + h > 0$. This contradicts the above fact that $n = 0$. Therefore $M_{(S,\phi)}$ contains no essential spheres.

Next we show atoroidality of $M_{(S,\phi)}$. Assume that there exists an incompressible torus $F \subset M_{(S,\phi)}$. By Theorem 2.1 the condition $(*)$ is satisfied. Assumptions (1) and (2) of Corollary 3.1 and Lemma 3.3 imply that $n = 0$. The Euler characteristic satisfies $0 = \chi(F) = e - h$, that is $h = e = 2n = 0$. This means that the open book foliation $\mathcal{F}_{ob}(F)$ has no singularities and all the leaves are c-circles. Since we identify the page S_0

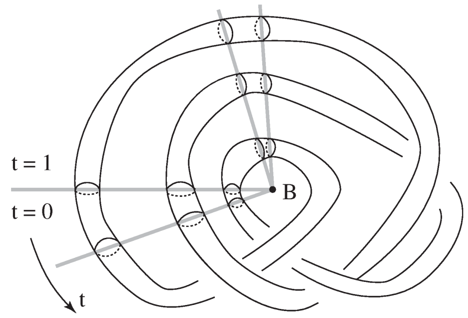

FIGURE 10. A torus foliated only by c-circles. A half line represents a page.

with S_1 under ϕ we have an equation: $F \cap S_0 = \phi(F \cap S_1)$. On the other hand, the sets $F \cap S_1$ and $F \cap S_0$ are isotopic through the smooth family $\{F \cap F_t \mid t \in [0,1]\}$. Thus we have $\phi(F \cap S_1) = F \cap S_1$ and ϕ preserves the set of c-circles $F \cap S_1$. That is, ϕ is reducible. This contradicts the assumption that ϕ is periodic or pseudo-Anosov. Therefore $M_{(S,\phi)}$ contains no incompressible tori. $\square$

ACKNOWLEDGEMENT

Part of this work was done during the author's stay at Kyoto University. The author thanks Professor Ohtsuki for partial support of her stay. She was also partially supported by NSF grant DMS-1206770.

REFERENCES

[1] D. Bennequin, *Entrelacements et équations de Pfaff,* Astérisque, 107-108, (1983) 87-161.
[2] J. Birman and W. Menasco, *Studying links via closed braids. IV. Composite links and split links.* Invent. Math. **102** (1990), no. 1, 115-139.

[3] J. Birman and W. Menasco, *Studying links via closed braids. II. On a theorem of Bennequin.* Topology Appl. **40** (1991), no. 1, 71-82.

[4] J. Birman and W. Menasco, *Studying links via closed braids. V. The unlink.* Trans. Amer. Math. Soc. **329** (1992), no. 2, 585-606.

[5] J. Birman and W. Menasco, *Studying links via closed braids. I. A finiteness theorem.* Pacific J. Math. **154** (1992), no. 1, 17-36.

[6] J. Birman and W. Menasco, *Studying links via closed braids. VI. A nonfiniteness theorem.* Pacific J. Math. **156** (1992), no. 2, 265-285.

[7] J. Birman and W. Menasco, *Studying links via closed braids. III. Classifying links which are closed 3-braids.* Pacific J. Math. **161** (1993), no. 1, 25-113.

[8] J. Birman and W. Menasco, *Special positions for essential tori in link complements.* Topology. **33** (1994), no.3, 525-556.

[9] J. Birman and W. Menasco, *Stabilization in the braid groups. I. MTWS.* Geom. Topol. **10** (2006), 413-540.

[10] J. Birman and W. Menasco, *Stabilization in the braid groups. II. Transversal simplicity of knots.* Geom. Topol. **10** (2006), 1425-1452.

[11] A. Candel and L. Conlon, *Foliations II* AMS Graduate Studies in Mathematics Series Vol 60, 2003.

[12] J. Etnyre, *Lectures on open book decompositions and contact structures* Clay Math. Proc. 5 (the proceedings of the "Floer Homology, Gauge Theory, and Low Dimensional Topology Workshop"), (2006), 103–141.

[13] K. Honda, W. Kazez, and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary,* Invent. math. **169**, No.2 (2007), 427-449.

[14] T. Ito and K. Kawamuro, *Open book foliations,* Geom. Topol. **18** (2014), 1581-1634.

[15] T. Ito and K. Kawamuro, *Essential open book foliations and fractional Dehn twist coefficient*, preprint.

[16] T. Ito and K. Kawamuro, *Operations on open book foliations* Algebr. Geom. Topol. **14** (2014) 2983-3020.

[17] T. Ito and K. Kawamuro, *Overtwisted discs in planar open books*, Intern. J. Math. **26**, No. 2 (2015), 1550027 (29 pages).

[18] W. Thurston, *A norm on the homology of three-manifolds*, Mem. Amer. Math. Soc. **59** (1986), 99-130.

Department of Mathematics
University of Iowa
Iowa City 52242
U.S.A
E-mail address: `kawamuro@iowa.uiowa.edu`

川室圭子