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On the stable complexity and the stable presentation length for 3-manifolds

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1 Introduction

This article is a survey of the stable complexity introduced by Francaviglia, Frigerio, and Martelli [4] and the stable presentation length introduced by Yoshida [19].

We will consider some invariants for a 3-manifold. We assume that a 3-manifold is oriented, compact, and possibly with boundary consisting of tori, unless otherwise stated. We define a finite volume hyperbolic 3-manifold to be a compact 3-manifold whose interior admits a complete metric of constant sectional curvature -1 and finite volume.

Perelman [14, 15] proved the geometrization of a 3-manifold. A closed 3-manifold admits the prime decomposition, i.e. the maximal decomposition by connected sums. After performing the prime decomposition, each component is an irreducible manifold or $S^1\times S^2$. An irreducible 3-manifold admits the JSJ decomposition, which is a decomposition along essential tori. The geometrization implies that each piece after the JSJ decomposition is a Seifert fibered manifold or a finite volume hyperbolic manifold.

Milnor and Thurston [12] considered some characteristic numbers of manifolds. An invariant C of manifolds is a characteristic number if $C(N)=d\cdot C(M)$ for any d-sheeted covering $N\rightarrow M$. For example, Milnor and Thurston introduced the following characteristic number, which is called the stable Δ-complexity by Francaviglia, Frigerio, and Martelli [4]. The Δ-complexity $\sigma(M)$ of a n-manifold M is the minimal number of n-simplices in a triangulation of M. The stable Δ-complexity is defined by

$$\sigma_{\infty}(M)=\inf_{N\rightarrow M}\frac{\sigma(N)}{\deg(N\rightarrow M)},$$

where the infimum is taken for the finite sheeted coverings of M. The stable Δ-complexity of a 3-manifold is almost same as the stable complexity.

Gromov [5] introduced the simplicial volume of a manifold, and showed that the simplicial volume of a hyperbolic manifold is proportional to its volume. In particular, the simplicial volume $\|M\|$ of a hyperbolic 3-manifold is equal to $\mathrm{vol}(M)/V_3$, where V_3 is the volume of an ideal regular tetrahedron in the hyperbolic 3-space. Soma [17] showed that the simplicial volume is additive for the connected sum and the JSJ decomposition. Therefore, the simplicial volume $\|M\|$ a closed 3-manifold M is the sum of the ones of the hyperbolic pieces after the geometrization.

In fact, the simplicial volume of a closed 3-manifold is uniquely determined by the fundamental group. This follows from the following theorems. Kneser's conjecture proved by Stallings states that if the fundamental group of a 3-manifold is decomposed as a free product, the manifold can be decomposed by a connected sum corresponding to the free product. We refer Hempel [8] for a proof. This reduces the statement to the case of irreducible 3-manifolds. Waldhausen [18] showed that a Haken 3-manifold is detemined by its fundamental group. The geometrization implies that a non-Haken irreducible 3-manifold is elliptic or hyperbolic. The simplicial volume of an elliptic 3-manifold vanishes. Mostow rigidity [13] states that a finite volume hyperbolic 3-manifold is determined by its fundamental group.

In Section 2 and Section 3, we review the stable complexity and the stable presentation length in parallel. In Section 4, we compare the simplicial volume, the stable complexity, and the stable presentation length for 3-manifolds.

2 Stable complexity

We review the stable complexity of a 3-manifold introduced by Francaviglia, Frigerio, and Martelli [4].

For a 3-manifold M, let $c(M)$ denote the (Matveev) complexity, which is defined as the minimal number of the vertices in a simple spine of M. If M is closed and irreducible, and not $S^3, \mathbb{R}P^3$ or the Lens space $L(3,1)$, $c(M)$ coincides with the minimal number of the tetrahedra in a triangulation of M. If M is a non-closed hyperbolic manifold of finite volume, $c(M)$ coincides with the minimal number of the tetrahedra in an ideal triangulation of M. Here we take a triangulation as a cell complex decomposition whose 3-simplices are tetrahedra. We refer Matveev [11] for details.

The complexity c is an upper volume in the sense of Reznikov [16]. Namely, if N is a d-sheeted covering of a 3-manifold M, it holds that $c(N) \leq d \cdot c(M)$. This allows us to define the stable complexity of a 3-manifold M by

$$c_\infty(M) = \inf_{N \to M} \frac{c(N)}{\deg(N \to M)},$$

where the infimum is taken for the finite sheeted coverings of M. The stable complexity c_∞ is a characteristic number in the sense of Milnor and Thurston [12]. Namely, if N is a d-sheeted covering of a 3-manifold M, it holds that $c_\infty(N) = d \cdot c_\infty(M)$.

The following example is obtained by constructing explicit triangulations or spines.

Proposition 2.1. [4, Proposition 5.11] *For a Seifert fibered 3-manifold M, it holds that*

$$c_\infty(M) = 0.$$

Let M_0 denote the Figure-eight knot complement. M_0 is a hyperbolic 3-manifold obtained by gluing two ideal regular tetrahedra. Since the ideal regular tetrahedron has the largest volume of the geodesic tetrahedra in the hyperbolic 3-space, we obtain the explicit value of the stable complexity of M_0.

Proposition 2.2. [4, Proposition 5.14]

$$c_\infty(M_0) = 2.$$

The stable complexity has additivity like the simplicial volume.

Theorem 2.3. [4, Corollary 5.3] *For 3-manifolds M_1 and M_2, suppose that $M = M_1 \# M_2$ is the connected sum. Then it holds that*

$$c_\infty(M) = c_\infty(M_1) + c_\infty(M_2).$$

Theorem 2.4. [4, Proposition 5.10] *Let M be an irreducible 3-manifold. Suppose that $M_1, \ldots, M_n$ are the components after the JSJ decomposition of M. Then it holds that*

$$c_\infty(M) = c_\infty(M_1) + \cdots + c_\infty(M_n).$$

Proposition 2.1, Theorem 2.3 and Theorem 2.4 implies that the stable complexity of a closed 3-manifold is the sum of the ones of the hyperbolic pieces after the geometrization.

In order to prove the above additivity, we use the following estimate of complexity. An essential surface in a closed 3-manifold M is an embedded sphere which does not bound a ball, or an embedded surface of at least genus 1 whose fundamental group injects to $\pi_1(M)$ by the induced map.

Theorem 2.5. [11, Section 4] *Let M be a closed 3-manifold, and let $S_1, \ldots, S_n$ be disjoint essential surfaces in M. Let $M_1, \ldots, M_m$ denote the components after decomposing M along $S_1, \ldots, S_n$. Then it holds that*

$$c(M) \geq c(M_1) + \cdots + c(M_m).$$

In order to prove the additivity for the JSJ decomposition, we need glue arbitrary finite coverings of decomposed pieces by taking larger coverings. The following theorem essentially by Hamilton [7].

Theorem 2.6. [4, Proposition 5.7] *Let M be an irreducible 3-manifold. Suppose that $M_1, \ldots, M_n$ are the components after the JSJ decomposition of M. Let $f_i \colon \widetilde{M_i} \to M_i$ be finite coverings for $1 \leq i \leq n$. Then there exist a natural number p independent of i and finite coverings $g_i \colon N_i \to \widetilde{M_i}$ such that each $f_i \circ g_i \colon N_i \to M_i$ is a p-characteristic covering, i.e. the restriction of the covering on each component of the boundary is the covering corresponding to $p\mathbb{Z} \times p\mathbb{Z} \leq \mathbb{Z} \times \mathbb{Z}$.*

The complexity is also additive for the connected sum, but is not additive for the JSJ decomposition. The latter follows from a finiteness of the complexity. Namely, the number of the irreducible 3-manifold whose complexity is a given number is finite. Indeed, there are only finite ways to glue tetrahedra of a given number. Let M_1 and M_2 be 3-manifolds with torus boundary. Since there are infinitely many ways to glue M_1 and M_2 along the boundary, it is impossible that all the complexities of the obtained manifolds coincide.

3 Stable presentation length

At first, we review the presentation length of a finitely presentable group (also known as Delzant's T-invariant) introduced by Delzant [3].

Definition 3.1. Let G be a finitely presentable group. We define the *presentation length* $T(G)$ of G by

$$T(G) = \min_{\mathcal{P}} \sum_{i=1}^{m} \max\{0, |r_i| - 2\},$$

where the minimum is taken for the presentations such as $\mathcal{P} = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ of G, and let $|r_i|$ denote the word length of r_i.

Delzant [3] also introduced a relative version of the presentation length. We need this in order to estimate the presentation length under a decomposition of group.

Definition 3.2. Let G be a finitely presentable group. Suppose that $C_1, \ldots, C_n$ are subgroups of G. A *(relative) presentation complex* P for $(G; C_1, \ldots, C_n)$ is a 2-dimensional cell complex satisfying the following conditions:

- P consists of triangles, bigons, edges and n vertices marked with $C_1, \ldots, C_n$.
- P is an orbihedron in the sense of Haefliger [6], with isotropies $C_1, \ldots, C_n$ on the vertices.
- The fundamental group $\pi_1^{\mathrm{orb}}(P)$ of P as an orbihedron is isomorphic to G. This isomorphism makes the isotropies $C_1, \ldots, C_n$ be the subgroups of G up to conjugacy.

We define the *(relative) presentation length* $T(G; C_1, \ldots, C_n)$ as the minimal number of triangles in a relative presentation complex for $(G; C_1, \ldots, C_n)$.

The presentation length depends only on G and the conjugacy classes of $C_1, \ldots, C_n$ in G. For an orbihedron P as above, there is an universal covering $\widetilde{P}$ of P as an orbihedron. The group G acts on the cell complex $\widetilde{P}$ simplicially, and the istropy groups of the vertices are the conjugacy classes of $C_1, \ldots, C_n$ in G. For example, the 2-skeleton of an ideal triangulation of a hyperbolic 3-manifold M with cusps $S_1, \ldots, S_n$ is a presentation complex for $(\pi_1(M); \pi_1(S_1), \ldots, \pi_1(S_1))$.

We associate the presentation complex P to a presentation $\mathcal{P} = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ of G. P is the 2-dimensional cell complex consisting of a single 0-cell, 1-cells and 2-cells corresponding to the generators and relators such that $\pi_1(P)$ is isomorphic to G. By dividing a k-gon of a presentation complex into $k-2$ triangles for $k \geq 3$, we obtain $T(G) = T(G; \{1\})$. Hence Definition 3.2 is a generalization of Definition 3.1.

The presentation length is an upper volume.

Proposition 3.3. [19, Proposition 3.1] *Let G be a finitely presentable group, and suppose that $C_1, \ldots, C_n$ are subgroups of G. Let H be a finite index subgroup of G, and let $d = [G : H]$ denote the index of H in G. Then it holds that*

$$T(H) \leq d \cdot T(G),$$
$$T(H; \{gC_ig^{-1} \cap H\}_{1 \leq i \leq n, g \in G}) \leq d \cdot T(G; C_1, \ldots, C_n).$$

We write $T(H; \{gC_ig^{-1} \cap H\}_{1 \leq i \leq n, g \in G}) = T(H; C'_1, \ldots, C'_{n'})$, where $C'_1, \ldots, C'_{n'}$ are representatives for the conjugacy classes of $gC_ig^{-1} \cap H$ in H. We remark that $\{gC_ig^{-1} \cap H\}_{1 \leq i \leq n, g \in G}$ is a finite family of subgroups up to conjugacy in H.

This allows us to define the stable presentation length.

Definition 3.4. Let $G, C_1, \ldots, C_n$ and H be as above. We define the *stable presentation length* of G by

$$T_\infty(G) = \inf_{H \leq G} \frac{T(H)}{[G : H]},$$

and the *(relative) stable presentation length* of $(G; C_1, \ldots, C_n)$ by

$$T_\infty(G; C_1, \ldots, C_n) = \inf_{H \leq G} \frac{T(H; \{gC_ig^{-1} \cap H\}_{1 \leq i \leq n, g \in G})}{[G : H]},$$

where the infima are taken for the finite index subgroups H of G.

The stable presentation length of the fundamental group of a surface coincides with its simplicial volume.

Theorem 3.5. [19, Theorem A.1] *Let Σ_g denote the closed orientable surface of genus $g \geq 1$. Then*

$$\begin{aligned} T_\infty(\pi_1(\Sigma_g)) &= 4g - 4 \\ &= -2\chi(\Sigma_g). \end{aligned}$$

Theorem 3.6. [19, Theorem A.2] *Let $\Sigma_{g,b}$ denote the compact orientable surface of genus g whose boundary components are $S_1, \ldots, S_b$. Suppose that $b > 0$ and $2g - 2 + b > 0$. Then*

$$\begin{aligned} T_\infty(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b)) &= T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b)) \\ &= 4g - 4 + 2b \\ &= -2\chi(\Sigma_{g,b}). \end{aligned}$$

We consider the stable presentation length of the fundamental group of a 3-manifold. For a 3-manifold M, we write

$$\begin{aligned} T(M) &= T(\pi_1(M)), \\ T_\infty(M) &= T_\infty(\pi_1(M)), \\ T(M; \partial M) &= T(\pi_1(M); \pi_1(S_1), \ldots, \pi_1(S_n)), \\ T_\infty(M; \partial M) &= T_\infty(\pi_1(M); \pi_1(S_1), \ldots, \pi_1(S_n)), \end{aligned}$$

where $S_1, \ldots, S_n$ are the components of ∂M. Thus the stable presentation length is a characteristic number for 3-manifolds.

We have the following examples by constructing explicit presentations.

Proposition 3.7. [19, Theorem 5.2] *For a Seifert fibered 3-manifold M, it holds that*

$$T_\infty(M) = 0.$$

Proposition 3.8. [19, Proposition A.3 and Proposition A.4] *Let M_0 denote the Figure-eight knot complement, and W_0 denote the Whitehead link complement. Then it holds that*

$$T_\infty(M_0) \leq 1, \quad T_\infty(W_0) \leq 2.$$

The stable presentation length also has additivity.

Theorem 3.9. [19, Theorem 5.1] *For finitely presentable groups G_1 and G_2, suppose that $G = G_1 * G_2$ is the free product. Then it holds that*

$$T_\infty(G) = T_\infty(G_1) + T_\infty(G_2).$$

In particular, it holds that

$$T_\infty(M_1 \# M_2) = T_\infty(M_1) + T_\infty(M_2)$$

for 3-manifolds M_1 and M_2.

Theorem 3.10. [19, Theorem 5.3] *Let M be an irreducible 3-manifold. Suppose that $M_1, \dots, M_n$ are the components after the JSJ decomposition of M. Then it holds that*

$$T_\infty(M) = T_\infty(M_1) + \cdots + T_\infty(M_n).$$

Proposition 3.8, Theorem 3.9 and Theorem 3.10 implies that the stable presentation length of a closed 3-manifold is the sum of the ones of the hyperbolic pieces after the geometrization.

In order to prove the above additivity, we use the following estimate of presentation length under decomposition along essential surfaces. We need also Theorem 2.6.

Theorem 3.11. [3, Theorem II and Proposition I.6.1] *Let M be a closed 3-manifold, and let $S_1, \dots, S_n$ be disjoint essential surfaces in M. Let $M_1, \dots, M_m$ denote the components after decomposing M along $S_1, \dots, S_n$. Then it holds that*

$$T(M) \geq T(M_1; \partial M_1) + \cdots + T(M_m; \partial M_m) \geq T(\pi_1(M); \pi_1(S_1), \dots, \pi_1(S_n)).$$

In the same manner as the complexity, the presentation length is additive for the free product, but is not additive for the JSJ decomposition.

We need consider the relative presentation length to use Theorem 3.11. In fact, the stable presentation length coincides with the relative version in the case of a 3-manifold with boundary consisting of incompressible tori. A group G is residually finite if for any element $g \in G - \{1\}$, there is a finite index subgroup of G which does not contain g. The fundamental group of a 3-manifold is residually finite by Hempel [9] and the geometrization.

Theorem 3.12. [19, Theorem 4.2] *Let G be a finitely presentable group, and let $C_1, \dots, C_n$ be free abelian subgroups of G whose ranks are at least two. Suppose that G is residually finite. Then it holds that*

$$T_\infty(G; C_1, \dots, C_n) = T_\infty(G).$$

Corollary 3.13. *Let M be a compact 3-manifold. Suppose that the boundary ∂M consists of incompressible tori. Then it holds that*

$$T_\infty(M; \partial M) = T_\infty(M).$$

4 Comparison of invariants

We present some inequalities between simplicial volume, stable complexity and stable presentation length for 3-manifolds. The additivity implies that the following inequalities are reduced to the case for the hyperbolic 3-manifolds. The three invariants are bounded by constant multiples of each other after all.

Proposition 4.1. *For a 3-manifold M, it holds that*

$$\|M\| \leq c_{\infty}(M).$$

For a hyperbolic 3-manifold M, $c(M)$ is the minimal number of the tetrahedra in a triangulation of M. It holds that $\|M\| \leq c(M)$ by definition of the simplicial volume. This implies Proposition 4.1.

Proposition 4.2. [19, Corollary 4.7] *For a 3-manifold M, it holds that*

$$T_{\infty}(M) \leq c_{\infty}(M).$$

For a hyperbolic 3-manifold M, take an (ideal) triangulation τ of M with $c(M)$ tetrahedra. We can obtain a presentation complex from the 2-skeleton of τ. Since we can remove $c(M) - 1$ triangles from the 2-skeleton without changing the fundamental group, it holds that $T(M; \partial M) \leq c(M) + 1$. This inequality and Corollary 3.13 imply Proposition 4.2.

Proposition 4.3. *For a 3-manifold M, it holds that*

$$\|M\| \leq \frac{\pi}{V_3} T_{\infty}(M).$$

Proposition 4.3 follows from Cooper's inequality [2] that $\mathrm{vol}(M) < \pi \cdot T(M)$ for a hyperbolic 3-manifold M. Cooper's inequality is obtained from an isoperimetric inequality for an image of a presentation complex to M.

Theorem 4.4. *There is a constant $K > 0$ such that*

$$c_{\infty}(M) \leq K\|M\|$$

for a 3-manifold M.

Theorem 4.4 is essentially due to the fact by Jørgensen and Thurston that a thick part of a hyperbolic 3-manifold can be decomposed to uniformly large simplices. We refer Kobayashi-Rieck [10] and Breslin [1] for detailed proofs of this fact.

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