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Problems on Low-dimensional Topology, 2015

Edited by T. Ohtsuki[1]

This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 20–22, 2015.

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[1]Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, JAPAN
Email: tomotada@kurims.kyoto-u.ac.jp
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1 The slice-ribbon conjecture and Akbulut-Kirby's conjecture for 0-surgeries along knots

(Tetsuya Abe, Keiji Tagami)

The slice-ribbon conjecture asks whether any slice knot in S^3 bounds a ribbon disk in the standard 4-ball B^4 ([12]). On the other hand, Akbulut and Kirby conjectured the following:

Conjecture 1.1 ([17, Problem 1.19]). *If 0-surgeries on two knots give the same 3-manifold, then the knots with relevant orientations are concordant.*

In our paper [1], we proved that if the slice-ribbon conjecture is true, Conjecture 1.1 is false. More precisely, we proved the following: Let K_0 and K_1 be the knots depicted in Figure 1. Orient K_0 and K_1 arbitrarily. Then, we see

- $K_0 \sharp \overline{K_1}$ is not a ribbon knot, and
- K_0 and K_1 admit the same 0-surgery,

where $\overline{K_1}$ is the mirror image of K_1. If the slice-ribbon conjecture is true, $K_0 \sharp \overline{K_1}$ is not a slice knot. Namely, K_0 and K_1 are not concordant. In particular, Conjecture 1.1 is not true.

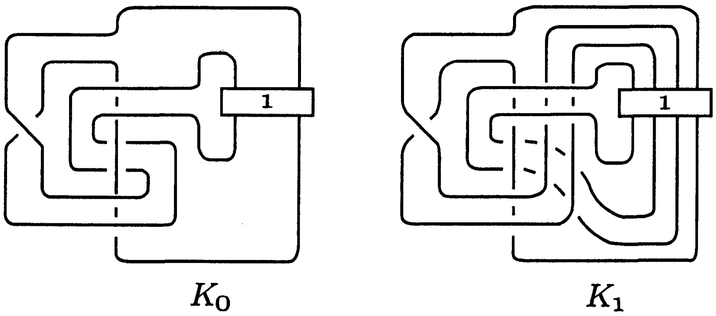


Figure 1: K_0 and K_1. Each rectangle labeled 1 implies a right-handed full twist.

Here, we give the following question.

Question 1.2 (T. Abe, K. Tagami). *Is $K_0 \sharp \overline{K_1}$ slice?*

If the answer of Question 1.2 is "YES", we see that $K_0 \sharp \overline{K_1}$ is a counterexample of the slice-ribbon conjecture. If the answer is "NO", we see that $K_0 \sharp \overline{K_1}$ is a counterexample of Conjecture 1.1.

Remark. Recently, in private communications, Kouichi Yasui showed us infinitely many counterexamples of Conjecture 1.1 by utilizing cork twists and satellite maps.

2 Random links

(Kazuhiro Ichihara)[2]

In Knot theory, and larger, in the low-dimensional topology, there have been several studies on random links (manifolds) recently. For example, in [19, 20], Jiming Ma introduced some models of random links, the random braid model and the random bridge presentation model. In the following of this section, as in [19], a *random link* via the random braid model (resp. the random bridge presentation model) is a link obtained as the closure (resp. the plat closure) of the braid induced from a random walk of length k on the braid group (resp. the mapping class group on a punctured sphere), and we consider the behavior of an invariant (or a property) of such a link at $k \to \infty$.

Based on his results, the following problems can be considered.

Problem 2.1 (K. Ichihara). *Does the probability of hyperbolic random links go to 1? This was already answered in [20] for the random braid model. How about for the bridge model? The author and Jiming Ma are considering this as an ongoing joint work.*

Problem 2.2 (K. Ichihara). *For some N, does the probability of random links of N components go to 1? Or, for any N, does the probability go to 0? The expected values of the number of components goes to the infinity, as shown in [19], with respect to the standard generators. How about the other generating sets (i.e., the other probability distribution)?*

Problem 2.3 (K. Ichihara). *Does the expected value of the (simplicial or hyperbolic) volume for random link diverge? If this is true, what is the growth late? For any fixed V, does the probability of the ones with the volume at most V go to 0?*

Problem 2.4 (K. Ichihara). *Does the probability of the sufficiently large random links (i.e., containing closed essential surfaces in the exterior) go to 1 or 0 ?*

Problem 2.5 (K. Ichihara). *Can we consider all the problems above for the random links in general 3-manifolds? For the random braid model, in [15], Ito introduced an generalization to the links in general 3-manifolds by considering open book decompositions.*

Problem 2.6 (K. Ichihara). *What is the behavior of knot invariants for random links? For example, can we say something Alexander polynomials for random links?*

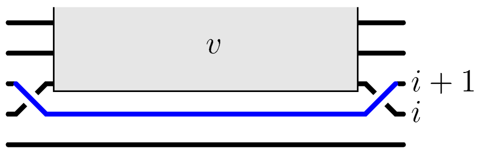
[2]The author would like to thank Jiming Ma for useful discussions and advises to make this note.

3 Braid ordering

(Patrick Dehornoy)

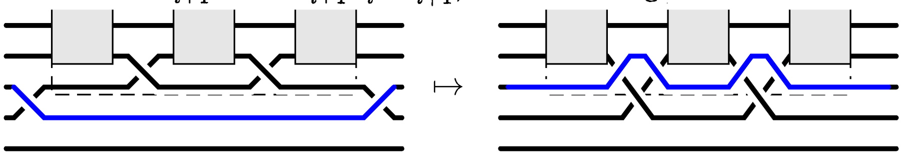
Termination of handle reduction

Let B_n be Artin's n-strand braid group. An n-strand braid word is any word in the alphabet $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$. For $1 \leqslant i < n$, define a *σ_i-handle* to be an n-strand braid word of the form $\sigma_i^e v \sigma_i^{-e}$ with $e = \pm 1$ and v an n-strand braid word that contains no letter $\sigma_j^{\pm 1}$ with $j \leqslant i$.



For instance, $\sigma_2 \sigma_3^{-1} \sigma_2^{-1}$ is a σ_2-handle, but neither $\sigma_2 \sigma_3^{-1}$ nor $\sigma_2 \sigma_1^{-1} \sigma_2^{-1}$ is.

Call a σ_i-handle $\sigma_i^e v \sigma_i^{-e}$ *reducible* if at most one of $\sigma_{i+1}, \sigma_{i+1}^{-1}$ occurs in v and, in this case, define the *reduct* of $\sigma_i^e v \sigma_i^{-e}$ to be the word obtained from v by replacing every occurrence of $\sigma_{i+1}^{\pm 1}$ with $\sigma_{i+1}^{-e} \sigma_i^{\pm 1} \sigma_{i+1}^{e}$, as in the diagram



Finally, say that a braid word w *reduces to* another braid word w' if w' is obtained from w by replacing a reducible handle by its reduct, that is, we have $w = w_1 w_2 w_3$ and $w' = w_1 w_2' w_3$ with w_2 a reducible σ_i-handle and w_2' the corresponding reduct.

Conjecture 3.1 (P. Dehornoy). *For every $n \geqslant 2$, there exists a constant C_n such that no more than $C_n \ell^2$ reduction steps may be performed starting from an n-strand braid word of length ℓ.*

Conjecture 3.1 is known to be true for $n \leqslant 3$ [8, Chap. V]. It is also known that, for every n, no infinite sequence of reductions may start from an n-strand word [5], but, for $n \geqslant 4$, the only upper bound on the number of reductions proved so far is exponential in ℓ, namely $O(2^{n^4 \ell})$. Computer experiments provide a strong evidence for Conjecture 3.1, but the only currently known proof, which combines algebraic and order arguments in a nontrivial way, seems hard to improve. This suggests that the "true" proof is still missing. Note the connection with the Word Problem for B_n: an n-strand braid word w represents 1 in B_n if some/any sequence of reductions starting from w leads to the empty word. Conjecture 3.1 is also connected with the standard braid ordering (see next Subsection), since, in an irreducible braid word, the generator σ_i with least index appears only positively, or only negatively.

The μ_n function

For β, β' in the n-strand braid group B_n, declare $\beta <_{\mathrm{D}} \beta'$ if $\beta^{-1}\beta'$ admits at least one expression in terms of the generators $\sigma_i^{\pm 1}$ in which the generator σ_i with least index occurs only positively (no σ_i^{-1}). It is known [8] that the relation $<_\mathrm{D}$ is a linear order on B_n that is compatible with multiplication on the left ($\beta <_\mathrm{D} \beta'$ implies $\gamma\beta <_\mathrm{D} \gamma\beta'$ for every γ). A deep result of R. Laver [18] states that, for every n, the restriction of $<_\mathrm{D}$ to the submonoid B_n^+ of B_n generated by $\sigma_1, ..., \sigma_{n-1}$ ("positive braids") is a well-ordering, that is, every nonempty subset of B_n^+ has a $<_\mathrm{D}$-smallest element. In particular, if β is a positive braid, there exists a $<_\mathrm{D}$-smallest positive braid conjugate to β.

Question 3.2 (P. Dehornoy). *Can one effectively compute the function $\mu_n : B_n^+ \to B_n^+$ defined by $\mu_n(\beta) = \min\{\beta' \in B_n^+ \mid \beta'$ is conjugate to $\beta\}$?*

Question 3.2 is open for $n \geqslant 3$. Note that any method for computing μ_n would lead to a solution of the Conjugacy Problem for the monoid B_n^+, which, by using the fact that multiplying an arbitrary braid by a sufficient power of the central braid Δ_n^2 yields a positive braid, would in turn lead to a solution of the Conjugacy Problem for the group B_n. The alternating normal form of [6] or the rotating normal form of [14] might be useful for answering Question 3.2. A natural step toward a solution could be the following computational formula, extensively tested on examples.

Conjecture 3.3 (Dehornoy, Fromentin, Gebhardt). *For every β in B_3^+, one has*

$$\mu_3(\beta\Delta_3^2) = \sigma_1\sigma_2^2\sigma_1 \cdot \mu_3(\beta) \cdot \sigma_2^2.$$

Exotic orders and presentations

The Artin presentation of the braid group B_n in terms of Artin's generators $\sigma_1, ..., \sigma_{n-1}$ is standard. It is equally well-known that B_n can be presented simply in terms of the $\binom{n}{2}$ band generators $a_{i,j} = \sigma_{j-1}^{-1} \cdots \sigma_{I+1}^{-1}\sigma_i\sigma_{i+1} \cdots \sigma_{j-1}$ with $1 \leqslant i < j \leqslant n$ [2]. Other natural generating families can be considered.

Question 3.4 (P. Dehornoy). *Put $\tau_i = \sigma_i\sigma_{i-1} \cdots \sigma_1$ for $1 \leqslant i < n$. Does B_n admit a finite presentation in terms of the generators $\tau_1, ..., \tau_{n-1}$?*

The answer is positive for $n = 3$: writing a for τ_1 and b for τ_2, one easily checks that B_3 admits the finite presentation $\langle a, b \mid aba = b^2\rangle$, which provides an alternative "Garside structure" on B_3. The general case is open (and seemingly not easy; a negative answer seems likely).

Similar questions arise in connection with the Dubrovina–Dubrovin ordering of braids [10].

Question 3.5 (P. Dehornoy). *Put $\rho_i = (\sigma_1\sigma_2 \cdots \sigma_i)^{(-1)^i}$ for $1 \leqslant i < n$. Does B_n admit a finite presentation in terms of the generators $\rho_1, ..., \rho_{n-1}$?*

Again the answer is positive for $n = 3$. Writing a for ρ_1 and b for ρ_2, one sees that B_3 admits the presentation $\langle a, b \mid b = ab^2a\rangle$. The submonoid of B_n generated by $\rho_1, ..., \rho_{n-1}$ is the positive cone of the Dubrovina-Dubrovin order on B_n, a variant of the left-invariant ordering $<_\mathrm{D}$ with the interesting property that it is an isolated point in the space of orderings on B_n.

Similar questions may be raised for the signed version $a_{i,j}^{(-1)^i}$ of the band generators, or for that $\Delta_i^{(-1)^i}$ of Garside fundamental braids, see [7] for further results and comments on these (strange and interesting) questions.

4 Relation between knots in S^3 and knots in $\mathbb{R}P^3$

(Sergei Matveev)

A knot in $\mathbb{R}P^3$ is the image of a smooth embedding of S^1 into $\mathbb{R}P^3$. When a knot K in $\mathbb{R}P^3$ is not null-homotopic in $\mathbb{R}P^3$, the preimage of K with respect to the natural projection $S^3 \to \mathbb{R}P^3$ is a knot in S^3.

Question 4.1 (S. Matveev). *Do there exist non-equivalent knots in $\mathbb{R}P^3$ such that their lifts to S^3 are equivalent knots?*

Motivation: The answer would clarify the relation between the theory of knots in S^3 and the one in $\mathbb{R}P^3$.

5 Surgery presentations of genus two 3-manifolds

(Sergei Matveev, Vladimir Tarkaev)

A *genus g Heegaard splitting* of a closed orientable 3-manifold M is a decomposition of M into a union of two handlebodies of genus g. We say that a closed 3-manifold M is of *genus g* if M has a genus g Heegaard splitting.

Problem 5.1 (S. Matveev, V. Tarkaev). *Prove that any closed 3-manifold of genus two can be obtained by a rational surgery along a framed 4-component link in S^3.*

Motivation: Using 3-manifold Recognizer (an interactive computer tool), we have tested several hundreds of genus two 3-manifolds and for all of them have found such a presentation.

6 Tight contact structures on 3-manifolds with open book decompositions

(Keiko Kawamuro)

Let S be a compact surface with non-empty boundary. Let $\phi : S \to S$ be a diffeomorphism that fixes the boundary ∂S of S pointwise. Let $M_{(S,\phi)}$ be the 3-manifold obtained by gluing the mapping torus of ϕ and solid tori in such a way that a meridian of a solid torus is glued to $\{\text{point}\} \times S^1 \subset \partial S \times S^1$. If a 3-manifold M is diffeomorphic to $M_{(S,\phi)}$ we say that M admits an open book decomposition (S, ϕ). Due to Giroux, there is a correspondence between (isotopy classes of) contact structures on a closed oriented 3-manifold M and (certain equivalence classes of) open book decompositions of M.

Question 6.1 (T.Ito, K. Kawamuro). *Let $\phi : S \to S$ be a diffeomorphism of a compact surface S that fixes the boundary of the surface pointwise. If the fractional Dehn twist coefficient $c(\phi, C) > 1$ for all the boundary components $C \subset \partial S$ then does the open book (S, ϕ) support a tight contact structure?*

In [16], it is proven that the answer is "yes" for planar open books (that is, S has genus 0).

7 Random 3-manifolds and orderability

(Tetsuya Ito)

It is conjectured that for a compact 3-manifold M, the following three conditions are equivalent (L-space conjecture, [3]):

Non-LO: $\pi_1(M)$ is not left-orderable, *i.e.*, there is no total ordering $<$ on $\pi_1(M)$ that is invariant under the left action of $\pi_1(M)$ itself.

L-space: M is an L-space. (Here we adopt the convention that if M is not a rational homology sphere, it is automatically not an L-space)

No Taut: M admits no co-oriented taut foliation.

We consider a random walk on $MCG(S)$, the mapping class group of a surface with connected boundary S, with respect to a probability measure μ on $MCG(S)$. We denote by $\phi_k \in MCG(S)$ the position of the random walk at time k.

Recently it is shown that for a random open book (S, ϕ), the corresponding 3-manifold $M_{(S,\phi)}$ admits a co-oriented taut foliation, and is not an L-space [15] with asymptotic probability one. That is, the probability that $M_{(S,\phi_k)}$ has these properties goes to one as $k \to \infty$, under reasonable assumptions on μ. (In the current version of [15], the proof of a key assertion on the Nielsen-Thurston ordering of ϕ_k (Theorem 1) has an error. Fortunately this assertion follows from the theorem of Malyutin [21] so the main results in [15] is unaffected).

Thus in the light of the aforementioned conjecture, we ask:

Question 7.1 (T. Ito). *Is $\pi_1(M_{(S,\phi)})$ left-orderable with asymptotic probability one?*

One may ask a similar question for different models (*e.g.* random Heegaard splitting) of random 3-manifolds:

Question 7.2 (T. Ito). *For a random 3-manifold M (obtained by random Heegaard splitting, or, other means – see [11] for precise formulation of random 3-manifold), does M have the property* **Non-LO, L-space, No Taut** *(with probability one/zero)?*

These questions can be seen as a "probabilistic approach" to disprove the L-space conjecture. It should be noted that a (density model of) random group is *not* left-orderable [4].

8 Stable complexity and stable presentation length for 3-manifolds

(Ken'ichi Yoshida)

For a 3-manifold M, let $\|M\|$ denote the simplicial volume of M, and let $c(M)$ denote the complexity of M [22]. The complexity of a hyperbolic 3-manifold is the minimal number of tetrahedra in an (ideal) triangulation of the manifold. The stable complexity of M is defined as $c_\infty(M) = \inf_N c(N)/\deg(N \to M)$, where the infimum is taken for the finite coverings N of M [13].

Question 8.1 (S. Francaviglia, R. Frigerio, B. Martelli [13]). *Does the equality*

$$\|M\| = c_\infty(M)$$

hold for a 3-manifold M?

For a finitely presentable group G, let $T(G)$ denote the presentation length of G [9]. The stable presentation length of G is defined as $T_\infty(G) = \inf_H T(H)/[G : H]$, where the infimum is taken for the finite index subgroups H of G [27].

Question 8.2 (K. Yoshida [27]). *Does the equality*

$$T_\infty(\pi_1(M)) = \frac{1}{2} c_\infty(M)$$

hold for a 3-manifold M? In particular, does it hold that $T_\infty(\pi_1(M_0)) = 1$ for the figure-eight knot complement M_0?

The above questions are reduced to considering for hyperbolic 3-manifolds, since the simplicial volume, the stable complexity and the stable presentation length are additive for the decomposition of geometrization.

The explicit values of stable complexity of hyperbolic 3-manifolds are known only for the hyperbolic manifolds commensurable with the figure-eight knot complement. No explicit value of stable presentation length is known for the fundamental group of a hyperbolic 3-manifold.

Question 8.3 (K. Yoshida). *Suppose that W is a hyperbolic 3-manifold obtained by gluing n ideal regular octahedra. Then does it hold that $c(W) = 4n$?*

If Question 8.3 is true for arbitrary W, it implies $c_\infty(W) = 4n$ and a counterexample of Question 8.1.

Question 8.4 (K. Yoshida). *Suppose that a hyperbolic 3-manifold M' is obtained by a Dehn filling from a hyperbolic 3-manifold M. Then does it holds that $c_\infty(M') < c_\infty(M)$ and $T_\infty(\pi_1(M')) < T_\infty(\pi_1(M))$?*

It holds that $c_\infty(M') \leq c(M)$ and $T_\infty(\pi_1(M')) \leq T(\pi_1(M))$.

Ehrenpreis asked a problem whether any two closed Riemann surfaces of genus ≥ 2 have finite coverings with arbitrarily small distance in an appropriate sense; it is known that this problem was solved affirmatively. Francaviglia, Frigerio and Martelli [13] proposed to generalize this problem to any dimension, and call it an *Ehrenpreis problem.* They observed that Question 8.1 would be proved from a positive answer to an appropriate version of such a problem.

To formulate an Ehrenpreis problem for hyperbolic 3-manifolds, it is a problem to give an appropriate "metric" on the set of hyperbolic 3-manifolds. Let $\mathcal{H}$ and $\mathcal{C}$ denote the isometry classes and the commensurability classes of finite volume hyperbolic 3-manifolds, respectively. The set $\mathcal{H}$ admits the "geometric topology", which is the standard topology by Gromov-Hausdorff convergence. A metric on $\mathcal{H}$ gives an Ehrenpreis problem, which asks if for $M, N \in \mathcal{H}$, there are finite coverings $\widetilde{M}, \widetilde{N}$ of M, N such that the distance between $\widetilde{M}$ and $\widetilde{N}$ is arbitrarily small. If an Ehrenpreis problem in the above sense holds, the quotient pseudometric on $\mathcal{C}$ is identically zero. We expect a result that an Ehrenpreis problem does not hold, since the claim of Question 8.3 is likely to hold.

Problem 8.5 (K. Yoshida). *Find a natural metric on $\mathcal{H}$ whose quotient pseudometric on $\mathcal{C}$ is a metric.*

We remark that the topology induced by a quotient pseudometric does not coincide with the quotient topology in general. However, we have a following necessary condition for Problem 8.5.

Question 8.6 (K. Yoshida). *Is the quotient topology of $\mathcal{C}$ Hausdorff?*

9 Invariants of knots and 3-manifolds with representations of their fundamental groups

(Toshie Takata)

In late 1980s, quantum invariants of closed oriented 3-manifolds have been introduced, motivated by the Chern–Simons path integral. This path integral itself is a formal integral over G connections on a 3-manifold M in mathematical physics (where G is a compact Lie group), but there are two ways to obtain invariants of 3-manifolds from the path integral: the operator formalism and the perturbative expansion. The operator formalism of the Chern–Simons path integral induces a mathematically rigorous definition of quantum invariants of M. The perturbative expansion of the Chern–Simons path integral gives a sum of contributions from flat G connections, where each contribution gives a power series invariant of M. By arithmetic expansion of quantum invariants, we obtain the perturbative invariant of M, which can be regarded as a mathematical construction of the contribution from the trivial G connection. For details, see *e.g.* [23]. It would be a problem to give a mathematical construction of a contribution from a non-trivial flat G connection, noting that a flat G connection can be regarded as a representation $\pi_1(M) \to G$. The above observation suggests that there might exist many such invariants.

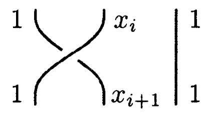
Problem 9.1 (T. Ohtsuki, T. Takata). *Let G be a Lie group.*
(1) *Construct concrete invariants of a closed oriented 3-manifold M with a repre-*

sentation $\pi_1(M) \to G$.
(2) *Construct concrete invariants of a knot* K *with a representation* $\pi_1(S^3 - K) \to G$.

So far, there have been some attempts to construct such invariants, such as "homotopy field theory" [25, 26]. This theory describes a formulation to construct such invariants in terms of G tensor categories, but it might be a problem to find concrete interesting examples of such categories. We also note that, if G was a finite group, Dijkgraaf–Witten invariant is such an invariant. In the same way as the construction of this invariant, we can also construct "Dijkgraaf–Witten invariant" for a Lie group G by giving the discrete topology to G and finding a 3-cocycle of it, but it might be a problem to find concrete interesting 3-cocycles of such G.

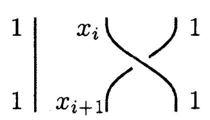
The volume conjecture suggests another aspect of asymptotic expansions of quantum invariants. The Kashaev invariant $\langle L \rangle_N \in \mathbb{C}$ of a link L is defined for $N = 2, 3, \cdots$ by using the quantum dilogarithm at $q = e^{2\pi\sqrt{-1}/N}$. Kashaev conjectured that, for any hyperbolic link L, the hyperbolic volume of $S^3 - L$ appears in the asymptotic behavior of $\langle L \rangle_N$ at $N \to \infty$ (Kashaev conjecture). H. Murakami and J. Murakami showed that the Kashaev invariant is equal to the N-colored Jones polynomial evaluated at $q = e^{2\pi\sqrt{-1}/N}$, where the N-colored Jones polynomial is the quantum invariant associated with the N-dimensional irreducible representation of the quantum group $U_q(sl_2)$. They reformulated the Kashaev conjecture in terms of the N-colored Jones polynomial (the volume conjecture). From the viewpoint of mathematical physics, the asymptotic behavior of the problem can be regarded as the asymptotic expansion of the $\mathrm{SL}_2\mathbb{C}$ Chern–Simons path integral at the $\mathrm{SL}_2\mathbb{C}$ flat connection corresponding to the holonomy representation of the hyperbolic structure, noting that we can formally calculate the asymptotic behavior of the problem by formally applying the saddle point method to the space of SU(2) connections in the space of $\mathrm{SL}_2\mathbb{C}$ connections. It is conjectured that, for a hyperbolic knot K, the semi-classical part $\omega(K)$ of this expansion is equal to a scalar multiple of the square root of the twisted Reidemeister torsion $\tau(K)$ of K. For details, see [24]. In [24], for any hyperbolic two-bridge knot K, $\omega(K)^2$ and $\tau(K)$ are calculated in terms of "representations" Ψ and Φ of parameterized 3-braids, which are given by

$$\frac{x_{i+1}}{x_i}\begin{pmatrix} -\frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i - x_{i+1}}{x_{i+1}} & -\frac{x_i-1}{x_{i+1}-1} \end{pmatrix} \xleftarrow{\ \Psi\ }$$



$$\xmapsto{\ \Phi\ } x_{i+1}\begin{pmatrix} 1 & 2x_{i+1} & 1 \\ 0 & -x_{i+1} & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\frac{x_{i+1}}{x_i}\begin{pmatrix} \frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i - x_{i+1}}{x_{i+1}} & \frac{x_i-1}{x_{i+1}-1} \end{pmatrix} \xleftarrow{\ \Psi\ }$$



$$\xmapsto{\ \Phi\ } x_{i+1}\begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_{i+1} & 0 \\ 1 & 2x_{i+1} & 1 \end{pmatrix}.$$

Here, we recall that the hyperbolic structure of a knot complement (or, more generally, a knot K with a parabolic representation $\pi_1(S^3 - K) \to \mathrm{SL}_2\mathbb{C}$) can be described by a knot diagram with hyperbolicity parameters, which are solutions of hyperbolicity equations.

Problem 9.2 (T. Ohtsuki, T. Takata).
(1) *Construct concrete "representations" of parameterized braids (i.e., braids with*

hyperbolicity parameters).
(2) *Construct concrete invariants of a knot K with a parabolic representation $\pi_1(S^3 - K) \to \mathrm{SL}_2\mathbb{C}$.*

As shown in [24], the above mentioned "representations" Ψ and Φ give the same invariant of hyperbolic 2-bridge knot, though they themselves do not look equivalent in a usual sense of equivalent representations. It might be a problem to develop a theory to analyze "representations" of parameterized braids.

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