数理解析研究所講究録
第 1961 巻 2015 年 26-28

On Kato's inequality for the relativistic Schrödinger operators with magnetic fields *

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This lecture deals with whether *Kato's inequality* holds for the *magnetic* relativistic Schrödinger operator H_A with vector potential A(x) and mass $m \geq 0$ associated with the classical relativistic Hamiltonian symbol $\sqrt{(\xi - A(x))^2 + m^2}$ such as

$$\mathrm{Re}[(\mathrm{sgn}\, u)H_A u] \geq \sqrt{-\Delta + m^2}\,|u|, \tag{1}$$

in the distribution sense, for u is in $L^2(\mathbf{R}^d)$ with $H_A u$ in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$.

In the literature there are three magnetic relativistic Schrödinger operators associated with the classical symbol (1) (e.g. [I 12], [I 13]). The first two $H_A^{(1)}$ and $H_A^{(2)}$ are to be defined as pseudo-differential operators: for $f \in C_0^\infty(\mathbf{R}^d)$,

$$(H_A^{(1)} f)(x) := \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\Big(\xi - A(\frac{x+y}{2})\Big)^2 + m^2}\, f(y)dyd\xi, \tag{2}$$

$$(H_A^{(2)} f)(x) := \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\Big(\xi - \int_0^1 A((1-\theta)x + \theta y)d\theta\Big)^2 + m^2}\, f(y)dyd\xi. \tag{3}$$

The third $H_A^{(3)}$ is defined as the square root of the nonnegative selfadjoint (nonrelativistic Schrödinger) operator $(-i\nabla - A(x))^2 + m^2$ in $L^2(\mathbf{R}^d)$:

$$H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2}. \tag{4}$$

$H_A^{(1)}$ is the so-called *Weyl* pseudo-differential operator ([ITa 86], [I 89]). $H_A^{(2)}$ is a modification of $H_A^{(1)}$ given in [IfMP 07], and $H_A^{(3)}$ used in [LSei 10] to discuss *relativistic stability of matter.*

All these three operators are nonlocal operators, and, under suitable condtion on $A(x)$, become selfadjoint. For $A = 0$ we put $H_0 = \sqrt{-\Delta + m^2}$, where $-\Delta$ is the *minus-signed* Laplacian in $\mathbf{R}^d$. $H_A^{(2)}$ and $H_A^{(3)}$ are gauge-covariant, but not $H_A^{(1)}$.

Inequality (1) for $H_A^{(1)}$ has been shown in [I 89], [ITs 76], and similarly will be for $H_A^{(2)}$.

For $H_A^{(3)}$, we assume that $d \geq 2$, as in case $d = 1$ any magnetic vector potential can be removed by a gauge tranformation. We want to show

Theorem 1 (Kato's inequality). *Let $m \geq 0$ and assume that $A \in [L^2_{\mathrm{loc}}(\mathbf{R}^d)]^d$. Then if u is in $L^2(\mathbf{R}^d)$ with $H_A^{(3)} u$ in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$, then the distributional inequality holds:*

$$\mathrm{Re}[(\mathrm{sgn}\, u)H_A^{(3)} u] \geq \sqrt{-\Delta + m^2}\,|u|, \tag{5}$$

*Talk at RIMS 研究集会「量子場の数理とその周辺」(2014/10/6–8)

or

$$\mathrm{Re}[(\mathrm{sgn}\, u) H_A^{(3)} u] \geq [\sqrt{-\Delta + m^2} - m]\, |u|. \tag{6}$$

Here $(\mathrm{sgn}\, u)(x) := \overline{u(x)}/|u(x)|, \text{if } u(x) \neq 0; \quad = 0, \text{if } u(x) = 0.$

From Theorem 1 follows the following corollary.

Corollary (Diamagnetic inequality) (cf. [FLSei 08], [HILo 12, 13]) *Let $m \geq 0$ and assume that $A \in [L^2_{\mathrm{loc}}(\mathbf{R}^d)]^d$. Then $f, g \in L^2(\mathbf{R}^d)$*

$$|(f, e^{-t[H_A^{(3)} - m]} g)| \leq (|f|, e^{-t[H_0 - m]} |g|). \tag{7}$$

Once Theorem 1 is established, we can apply it to show the following theorem on essential selfadjointness of the relativistic Schrödinger operator with both vector and scalar potentials $A(x)$ and $V(x)$:

$$H := H_A^{(3)} + V. \tag{8}$$

Theorem 2. *Let $m \geq 0$ and assume that $A \in [L^2_{\mathrm{loc}}(\mathbf{R}^d)]^d$. If $V(x)$ is in $L^2_{\mathrm{loc}}(\mathbf{R}^d)$ with $V(x) \geq 0$ a.e., then $H = H_A^{(3)} + V$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ and its unique selfadjoint extension is bounded below by m.*

The characteristic feature is that, unlike $H_A^{(1)}$ and $H_A^{(2)}$, $H_A^{(3)}$ is, since being defined as an operator square root (4), neither an integral operator nor a pseudo-differential operator associated with a certain tractable symbol. $H_A^{(3)}$ is, under the condition of the theorem, essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ so that $H_A^{(3)}$ has domain

$$D[H_A^{(3)}] = \{u \in L^2(\mathbf{R}^d)\, ;\, (i\nabla + A(x))u \in L^2(\mathbf{R}^d)\},$$

which contains $C_0^\infty(\mathbf{R}^d)$ as an operator core. Although we can know the domain of $H_A^{(3)}$ is determined, the point which becomes crucial is in how to derive regularity of the weak solution $u \in L^2(\mathbf{R}^d)$ of equation

$$H_A^{(3)} u \equiv \sqrt{(-i\nabla - A(x))^2 + m^2}\, u = f, \quad \text{for given } f \in L^1_{\mathrm{loc}}(\mathbf{R}^d).$$

We shall show inequality (5)/(6), modifying the method used in the case ([I 89], [ITs 92]) for the Weyl pseudo-differential operator $H_A^{(1)}$, basically along the idea of Kato's original proof for the magnetic nonrelativistic Schrödinger operator $\frac{1}{2}(-i\nabla - A(x))^2$ in [K 72]. However, the present case seems to be not so simple as to need much further modification within "operator theory *plus alpha*".

References

[FLSei 08] R.L. Frank, E.H. Lieb and R. Seiringer: Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators, *J. Amer. Math. Soc.* **21**, 925–950 (2008).

[HILo 12] F. Hiroshima, T. Ichinose and J. Lőrinczi: Path integral representation for Schrödinger operators with Bernstein functions of the Laplacian, *Rev. Math. Phys.* **24**, 1250013(40 pages) (2012).

[HILo 13] F. Hiroshima, T. Ichinose and J. Lőrinczi: Probabilistic representation and fall-off of bound states of relativistic Schrödinger operators with spin 1/2, *Publ. RIMS Kyoto University* **49**, 189–214 (2013).

[I 89] T. Ichinose: Essential selfadjointness of the Weyl quantized relativistic Hamiltonian, *Ann. Inst. Henri Poincaré, Phys. Théor.*, **51**, 265–298(1989).

[I 12] T. Ichinose: On three magnetic relativistic Schrödinger operators and imaginary-time path integrals, *Lett. Math. Phys.* **101**, 323–339 (2012).

[I 13] T. Ichinose: Magnetic relativistic Schrödinger operators and imaginary-time path integrals, *Mathematical Physics, Spectral Theory and Stochastic Analysis*, Operator Theory: Advances and Applications 232, pp. 247–297, Springer/Birkhäuser 2013.

[ITa 86] T. Ichinose and Hiroshi Tamura: Imaginary-time path integral for a relativistic spinless particle in an electromagnetic field, *Commun. Math. Phys.* **105**, 239–257 (1986).

[ITs 76] T. Ichinose and T. Tsuchida: On Kato's inequality for the Weyl quantized relativistic Hamiltonian, *Manuscripta Math.* **76**, 269–280 (1992).

[IfMP 07] V. Iftimie, M. Măntoiu and R. Purice: Magnetic pseudodifferential operators, *Publ. RIMS Kyoto Univ.* **43**, 585–623 (2007).

[K 72] T. Kato: Schrödinger operators with singular potentials, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), *Israel J. Math.* **13**, 135–148 (1973).

[LSei 10] E.H. Lieb and R. Seiringer: *The Stability of Matter in Quantum Mechanics*, Cambridge University Press 2010.

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