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Local Gauge Invariance and Maxwell Equation in Categorical QFT*

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Abstract

From the viewpoint of "geometry of symmetry breaking", universal roles played by holonomy terms have been found in relation with Élie Cartan's characterization of symmetric spaces: they can be regarded as geometric templates in the physical emergence processes of Macro classical objects from Micro quantum dynamics. In view of the essential roles played by natural transformations here, the logical essence of the emergences can be found in the local gauge invariance, which entails the validity of Maxwell type equations.

1 Introduction

To clarify the close relationship among symmetry breaking, local gauge invariance and Maxwell-type equations, we discuss the following basic pointts:

1) Quadrality scheme [1, 2] as a framework for going back & forth between ***Macro*** and ***Micro*** levels of nature:

Visible phenomenogical Macro data
$\rightleftarrows$ ***theory of invisible Micro processes***;

2) In algebraic & categorical QFT we show how ***local gauge invariance*** arises from symmetry breaking;

3) From the viewpoint of broken symmeries & local gauge invariance, basic ingredients of the formalism are reviewed, in which symmetric space structure is found in the sector-classifying space.

2 Basic Concepts: Quadrality Scheme & Micro-Macro Duality based on Sectors

Quadrality Scheme,

↗	Spec	
States	⇆↿ (Rep) ⇂⇆	Alg
	Dyn	↗

, for describing physical

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phenomena is composed of the following four basic ingredients:

Alg(ebra of physical variables)/ ***States*** (as expectation functionals)/

Spec (as a classifying space of sectors)/ ***Dyn***(amics),

forming ***Micro-Macro Duality*** [3]:

Macro: $\mathcal{A}$		left adjoint F
	$\searrow\nwarrow$	
right adjoint E		$\mathcal{X}$: Micro

1) Its Micro-Macro boundary is defined in terms of ***sectors*** and

2) the Macro side is epigenetically due to the ***emergence process*** of

3) Spec= sector-classifying space from Micro dynamics, to form a categorical adjunction:

$\mathcal{A}(a \leftarrow Fx)$		$\varepsilon_a F(-)$
	$\searrow\nwarrow$	
$E(-)\eta_x$		$\mathcal{X}(Ea \leftarrow x)$

with unit $\eta : I_{\mathcal{X}} \dot{\to} T := EF$ intertwining $\mathcal{X}$ to monad $T \curvearrowright \mathcal{X}$ as Micro dynamics and with counit $\varepsilon : S := FE \dot{\to} I_{\mathcal{A}}$ to $\mathcal{A}$ from comonad $S \curvearrowright \mathcal{A}$ as dual of monad T.

Micro-Macro duality as categorical adjunction:

	comonad $S = FE$: Spec	
emergence $\nearrow$	Arveson spec $V \uparrow\downarrow I$ spectral subspace	$\searrow$ quantm fields
States $\mathcal{A}$	$\overset{F}{\longleftarrow}$ $\leftrightarrows$ Bimodule of adjoint pair $\leftrightarrows$ $\underset{E}{\longrightarrow}$	$\searrow$ Algebra $\mathcal{X}$
	$\uparrow\downarrow$: Gal	$\nearrow$ co-emergence
	Dyn = monad $T = EF$	

3 Sectors & Spec = sector-classifying space

Basic ingredients of the formalism [1, 2] are defined as follows:

1) ***Sectors= pure phases*** parametrized by ***order parameter***[= central observables $\mathfrak{Z}_\pi(\mathcal{X}) = \pi(\mathcal{X})'' \cap \pi(\mathcal{X})'$ commuting with all physical variables $\pi(\mathcal{X})''$ in a generic representation π of algebra $\mathcal{X}$ of physical variables]:

Mathematically, a ***sector***(= ***pure phase***) $\overset{\text{def}}{=}$ a ***quasi-equivalence class of factor*** states (& representations π_γ) of (C*-)algebra $\mathcal{X}$ of physical variables, as a *minimal unit* of representations characterized by ***trivial centre*** $\pi_\gamma(\mathcal{X})'' \cap \pi_\gamma(\mathcal{X})' =: \mathfrak{Z}_{\pi_\gamma}(\mathcal{X}) = \mathbb{C}1$.

2) The roles of ***sectors as Micro-Macro boundary*** can be seen in ***Micro-Macro duality*** as a mathematical formulation of "***Quantum-Classical correpsondence***" between microscopic ***intra-sectorial*** & macroscopic ***inter-sectorial*** levels described by geometrical structures on central spectrum $Sp(\mathfrak{Z}) := Spec(\mathfrak{Z}_\pi(\mathcal{X}))$:

Micro-Macro Duality of Intra- vs. Inter-sectorial levels

$\longleftarrow$	Visible *Macro*	of	***Spec*** $=$	***classifying space***	$\longrightarrow$	***Inter-sectorial***
$\cdots$	γ_N	$\cdots$	***sectors*** γ	γ_2	γ_1	$Sp(\mathfrak{Z})$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	***Intra-sectorial***
$\cdots$	π_{γ_N}	$\cdots$	π_γ	π_{γ_2}	π_{γ_1}	$\|$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	invisible ***Micro***

Different sectors: mutually ***disjoint*** with respect to ***unbroken*** symmetry, and ***connected*** by the actions of ***broken*** symmetries

As explained later, this contrast is shared even by D(H)R theory of *unbroken* symmetry!

4 Emergence of Macro Spec & Symmetry Breaking

3a) ***Emergence process*** [Macro $\Longleftarrow$ Micro] of Spec = sector-classifying space via ***forcing*** along (generic) filters

Mathematically this is controlled by ***Tomita theorem*** of integral decomposition of a Hilbert bimodule ${}_{\pi(\mathcal{X})''}\widetilde{\mathcal{X}}_{L^\infty(E_\mathcal{X})} := \pi(\mathcal{X})'' \otimes L^\infty(E_\mathcal{X})$ with left $\pi(\mathcal{X})''$ & right $L^\infty(E_\mathcal{X}, \mu)$ actions, via ***central measure*** μ supported by ***Spec***$=$ supp$(\mu) = Sp(\mathfrak{Z}) \subset F_\mathcal{X}$: factor states in state space $E_\mathcal{X}$ of $\mathcal{X}$.

$\Longrightarrow$ Applications to statistical inference based on large deviation principle [4] and to derivation of Born rule [5].

3b) ***Symmetry Breaking & Emergence of Classifying Space***

Sector-classifying space emerges typically from spontaneous breakdown of symmetry of a dynamical system $\mathcal{X} \curvearrowleft G$ with action of a group G ("spontaneous" = no changes in dynamics of the system).

4.1 Symmetry breaking & classifying space

Criterion for Symmetry Breaking (SB criterion, for short) [1, 2]: judged by non-triviality of ***central*** dynamical system $\mathfrak{Z}_\pi(\mathcal{X}) \curvearrowleft G$ associated with the original one $\mathcal{X} \curvearrowleft G$.

I.e., symmetry G is ***broken in sectors*** $\in Sp(3)$ ***with non-trivial responses to central*** ***G-action.***

G-transitivity assumption with ***unbroken*** subgroup H in broken G leads to sector-classifying space in a specific form of homogeneous space G/H.
$\Longrightarrow$ ***Classical geometric*** structure on G/H arises physically from ***emergence*** process via ***condensation*** of a family of ***degenerate vacua***, each of which is mutually distinguished by condensed values $\in Sp(3) = G/H$.

In this way, ∞-number of low-energy quanta are condensed into geometry of classical Macro objects $\in G/H$.

4.2 Sector bundle & logical extension from constants to variables

In combination with sector structure $\hat{H}$ of unbroken symmetry H (à la DHR-DR theory), total sector structure due to this symmetry breaking is described by a ***sector bundle*** $G \underset{H}{\times} \hat{H}$ with fiber $\hat{H}$ over base space G/H consisting of "***degenerate vacua***" [1, 2].

When this geometric structure is established, all the physical quantities are ***parametrized by condensed values of order parameters*** $\in G/H$
$\Longrightarrow$ "***Logical extension***" [6] of ***constants*** (= global objects) into ***sector-dependent function objects*** (: origin of ***local gauge*** structures)

5 G/H as Symmetric Space

This homogeneous space G/H is shown to be a ***symmetric space*** with Cartan involution as follows [IO, in preparation].

Lie-bracket relations $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$ hold for Lie structures $\mathfrak{g},\mathfrak{h},\mathfrak{m}$ of $G, H, M := G/H$. If $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$ is verified, M becomes a symmetric space (at least, locally) equipped with Cartan involution $\mathcal{I}$ with eigenvalues $\mathcal{I}\restriction_{\mathfrak{h}}= +1$ & $\mathcal{I}\restriction_{\mathfrak{m}}= -1$:

This property $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$ follows from the relation: $[\mathfrak{m},\mathfrak{m}]=$ ***holonomy*** associated with an infinitesimal loop in ***inter-sectorial space*** $M = Sp(3)$ along ***broken direction.*** Since $[\mathfrak{m},\mathfrak{m}]=$ effect of ***broken*** G transformation along an infinitesimal loop $\overset{\gamma}{\circlearrowleft}$ on M starting from and returning to the same point $\gamma \in M$. Thus, $\mathfrak{m}$-component in $[\mathfrak{m},\mathfrak{m}]$ is absent by the above SB criterion, and hence, $M = G/H = Sp(3)$ is a symmetric space (at least, locally).
Example 1): Lorentz boosts
Typical example of this sort can be found for Lorentz group $\mathcal{L}_+^\uparrow =: G$, rotation group $SO(3) =: H$, $G/H = M \cong \mathbb{R}^3$: symmetric space of Lorentz frames connected by Lorentz boosts.

For $\mathfrak{h} := \{M_{ij}; i,j = 1,2,3, i < j\}$, $\mathfrak{m} := \{M_{0i}; i = 1,2,3\}$,

$[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h},\mathfrak{m}] = \mathfrak{m}, [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$: verified by the basic Lie algebra structure:

$[iM_{\mu\nu}, iM_{\rho\sigma}] = -(\eta_{\nu\rho}iM_{\mu\sigma} - \eta_{\nu\sigma}iM_{\mu\rho} - \eta_{\mu\rho}iM_{\nu\sigma} + \eta_{\mu\sigma}iM_{\nu\rho})$.

In contrast to the usual interpretation of Lorentz invariance, ***unbroken*** Lorentz boosts $\mathfrak{m}$ is ***speciality of the vacuum situation***, due to such results as Borcher-Arveson thm (: Poincaré generators can be physical observables only in vacuum representation) & spontaneous breakdown of Lorentz boosts at $T \neq 0K$ [7]. In this sense, Lorentz frames $M \cong \mathbb{R}^3$ with [boost, boost] = rotation, give a typical example of symmetric space structure emerging from symmetry breaking.

Example 2): Along this line, ***chiral symmetry*** with current algebra structure $[V,V] = V; [V,A] = A, [A,A] = V$ and ***conformal symmetry*** also provide typical examples.

Example 3): 2nd Law of Thermodynamics

Physically more interesting example can be found in ***thermodynamics***:

1st law of thermodynamics $\Longrightarrow \Delta'Q \hookrightarrow \Delta E = \Delta'Q + \Delta'W \twoheadrightarrow \Delta'W$: exact sequence corresponding to $\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$.

With respect to Cartan involution with + assigned to heat production $\Delta'Q$ and − to macroscopic work $\Delta'W$, the holonomy $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$ corresponding to a loop in the space M of thermodynamic variables becomes just

Kelvin's version of 2nd law of thermodynamics

namely, holonomy $[\mathfrak{m},\mathfrak{m}]$ in the cyclic process with $\Delta E = \Delta'Q + \Delta'W = 0$, describes heat production $\Delta'Q \geq 0$: $-\Delta'W = -[\mathfrak{m},\mathfrak{m}] = \Delta'Q > 0$ (from system to outside)

6 Origin of Symmetric Space: Disjointness vs. Quasi-equivalence

As far as symmetry beaking is formulated in the sector-classifying space, consistent description of its spectrum necessarily reduces to a symmetric space, as seen above. One may have, however, a question why a *non-symmetric* homogeneous space G/H is not possible as a choice of (reductive) pair (G,H) of Lie groups G and $H(\subset G)$ with H describing unbroken symmetry and G broken one. While we cannot exclude such a case as an abstract possibility, we can see that the appearance of $\mathfrak{m}$-component in $[\mathfrak{m},\mathfrak{m}]$ induces an infinitesimal shift of the end point of a loop on G/H, which causes an instability in the sector structure under the broken symmetry. Through stabilization under this perturbation, therefore, a *non-symmetric* homogeneous space G/H should be extended to its *symmetric-space completion* $(G/H)^{\ddot{\circ}\ddot{\circ}}$, which is to be discussed in the following.

To consider this problem in relation with quasi-equivalence and centre, we focus on the *universal representation* $\pi_u = (\pi_u, \mathfrak{H}_u) \in Rep_{\mathcal{X}}$ in C*-category $Rep_{\mathcal{X}}$ of representations of a C*-algebra $\mathcal{X}$, with such universality that it contains any representation $\forall\pi = (\pi, \mathfrak{H}_\pi) \in Rep_{\mathcal{X}}$ of $\mathcal{X}$ as its subrepresentation: $\pi \leq \pi_u$. Such a $(\pi_u, \mathfrak{H}_u)$ is well known to be realized concretely as the direct sum $(\pi_u, \mathfrak{H}_u) := \bigoplus_{\omega \in E_{\mathcal{X}}} (\pi_\omega, \mathfrak{H}_\omega)$ of all GNS representations, resulting in universal enveloping von Neumann algebra $\mathcal{X}'' \cong \mathcal{X}^{**} \cong \pi_u(\mathcal{X})'' \curvearrowright \mathfrak{H}_u$.

We now define "disjoint complement" $\pi^{\overset{|}{\circ}}$ of a representation $\pi \in Rep_{\mathcal{X}}$, by maximal representation disjoint from π: $\pi^{\overset{|}{\circ}} := \sup\{\rho \in Rep_{\mathcal{X}}; \rho \overset{|}{\circ} \pi\}$, where disjointness means $\rho \overset{|}{\circ} \pi \Longleftrightarrow Rep_{\mathcal{X}}(\pi \leftarrow \rho) = \{0\}$: i.e., no non-zero intertwiners.

Then, we see (IO2004, unpublished):

i) $P(\pi^{\overset{|}{\circ}}) = c(\pi)^{\perp}$,
$P(\pi^{\overset{||}{\circ\circ}}) = c(\pi)^{\perp\perp} = c(\pi) := \bigvee_{u \in \mathcal{U}(\pi(\mathcal{X})')} u P_\pi u^* \in \mathcal{P}(\mathfrak{Z}(\pi_u(\mathcal{X})''))$,
where $P(\pi) \in \pi_u(\mathcal{X})'$: projection corresponding to $(\pi, \mathfrak{H}_\pi)$ in $\mathfrak{H}_u$ and $c(\pi)$: central support of $P(\pi)$ defined by the minimal central projection majorizing $P(\pi)$ in centre $\mathfrak{Z}(\mathcal{X}'') := \mathcal{X}'' \cap \mathcal{X}'$ of $\mathcal{X}''$.

ii) $\pi_1^{\overset{||}{\circ\circ}} = \pi_2^{\overset{||}{\circ\circ}} \Leftrightarrow \pi_1 \approx \pi_2$ (: quasi-equivalence= unitary equivalence up to multiplicity $\Leftrightarrow \pi_1(\mathcal{X})'' \simeq \pi_2(\mathcal{X})'' \Leftrightarrow c(\pi_1) = c(\pi_2) \Leftrightarrow (\pi_1(\mathcal{X}))''_* = \pi_2(\mathcal{X}))''_*$)

6.1 Quasi-equivalence & modular structure

iii) Representation $(\pi^{\overset{||}{\circ\circ}}, c(\pi)\mathfrak{H}_u)$ of von Neumann algebra $\pi(\mathcal{X})'' \simeq \pi^{\overset{||}{\circ\circ}}(\mathcal{X})''$ in $c(\pi)\mathfrak{H}_u$ gives the ***standard form*** of $\pi(\mathcal{X})''$ equipped with normal faithful semifinite weight φ and the associated Tomita-Takesaki ***modular structure*** $(J_\varphi, \Delta_\varphi)$, whose universality is characterized by adjunction,

$$Std(\sigma \leftarrow \pi^{\overset{||}{\circ\circ}}) \simeq Rep_{\mathcal{X}}(\sigma \leftarrow \pi).$$

Namely, any intertwiner $T \in Rep_{\mathcal{X}}(\sigma \leftarrow \pi)$ to a standard form representation $(\sigma, \mathfrak{H}_\sigma)$ of $\sigma(\mathcal{X})''$ is uniquely factored $T = T^{\overset{||}{\circ\circ}} \circ \eta_\pi$ through the canonical homotopy $\eta_\pi \in Rep_{\mathcal{X}}(\pi^{\overset{||}{\circ\circ}} \leftarrow \pi)$ with $\exists! T^{\overset{||}{\circ\circ}} \in Rep_{\mathcal{X}}(\sigma \leftarrow \pi^{\overset{||}{\circ\circ}})$.

6.2 Quasi-equivalence & sector-classifying groupoid

Modular structure of von Neumann algebra $\pi(\mathcal{X})'' =: \mathcal{M}$ in the standard form $(\pi^{\overset{\shortmid\shortmid}{\circ\circ}}, c(\pi)\mathfrak{H}_u)$ can be understood as ***unitary implementation*** of a normal subgroup $G_{\mathcal{M}} := Isom(\mathcal{M}_*)^{\mathcal{M}} \lhd Isom(\mathcal{M}_*)$ fixing $\mathcal{M}$ pointwise by unitary group $\mathcal{U}(\mathcal{M}')$ in the commutant $\mathcal{M}'$: namely, for $\gamma \in G_{\mathcal{M}}$, $\exists U'_\gamma \in \mathcal{U}(\mathcal{M}')$ s.t. $\langle \gamma\omega, x\rangle = \langle \omega, \gamma^*(x)\rangle = \langle \omega, U'^*_\gamma x U'_\gamma\rangle$ for $\omega \in \mathcal{M}_*$, and $U'^*_\gamma x U'_\gamma = x \Longleftrightarrow x \in \mathcal{M}$. Through modular conjugation $J_\varphi(-)J_\varphi$, this unitary group can naturally be related to the modular group Δ^{it}_φ.

iv) Quasi-equivalence $\pi_1 \approx \pi_2$ defines sector-classifying groupoid $\Gamma_\approx$ consisting of ***invertible intertwiners*** in $Rep_{\mathcal{X}}$, which reduces on each $\pi \in Rep_{\mathcal{X}}$ to the automorphism group, $\Gamma_\approx(\pi, \pi) = Aut(\pi(\mathcal{X})'')$ $\simeq Isom(\pi(\mathcal{X})''_*)$, isomorphic to the isometry group of predual $\pi(\mathcal{X})''_*$.

6.3 Quasi-equivalence & Galois structure

From the relation $\mathfrak{Z}(\mathcal{M}) = (\mathcal{M}')^{\mathcal{U}(\mathcal{M}')} = (\mathcal{M} \vee \mathcal{M}')^{\mathcal{U}(\mathcal{M})\times\mathcal{U}(\mathcal{M}')}$, sector-classifying space can be viewed as Grassmannian-like symmetric space (or, Hecke algebra): $Sp(\mathfrak{Z}(\mathcal{M})) = \mathcal{U}(\mathcal{M})\backslash[\mathcal{U}(\mathcal{M}\vee\mathcal{M}')]/\mathcal{U}(\mathcal{M}')$. This can be seen as the basis of the connection between symmetry breaking and symmetric space.

For $\mathcal{M}$ of type III, following Galois-type relations hold with crossed product by a coaction of $\mathcal{U}(\mathcal{M}')$ on $\mathcal{M}$:

$$\begin{aligned} \mathfrak{Z}(\mathcal{M})' = \mathcal{M} \vee \mathcal{M}' = \mathcal{M} \rtimes \widehat{\mathcal{U}(\mathcal{M}')}&: \text{Galois extension of } \mathcal{M}, \\ \mathcal{M} = (\mathcal{M} \vee \mathcal{M}')^{\mathcal{U}(\mathcal{M}')}&: \text{fixed-point subalgebra under } \mathcal{U}(\mathcal{M}'), \\ \mathcal{U}(\mathcal{M}') = Gal(\mathfrak{Z}(\mathcal{M})'/\mathcal{M})&: \text{Galois group of } \mathcal{M} \hookrightarrow \mathfrak{Z}(\mathcal{M})', \end{aligned}$$

according to which trivial centre $\mathfrak{Z}(\mathcal{M}) = \mathbb{C}\mathbf{1}$ to characterize a ***sector*** can be reinterpreted as **ergodicity** condition on $\mathcal{M}$ under $Aut(\mathcal{M})$ or $G_{\mathcal{M}}$:

$$\mathbb{C}\mathbf{1} = \mathcal{M} \cap \mathcal{M}' = \mathcal{M}' \cap \mathcal{U}(\mathcal{M}')' = (\mathcal{M}')^{\mathcal{U}(\mathcal{M}')} \supset (\mathcal{M}')^{Aut(\mathcal{M})}.$$

Through the above consideration, symmetric-space completion $(G/H)^{\overset{\shortmid\shortmid}{\circ\circ}}$ can now be identified with the completion of G/H in the factor spectrum with respect to the disjoint completion $\pi \to \pi^{\overset{\shortmid\shortmid}{\circ\circ}}$.

7 Sector Bundle & Holonomy along Goldstone Condensates

In use of sector bundle $\widehat{H} \hookrightarrow G \underset{H}{\times} \hat{H} \twoheadrightarrow G/H$, physical origin of space-time concept can be seen in its ***physical emergence process*** [8].

For simplicity, we assume here that a group G of broken internal symmetry be extended by a group $\mathcal{R}$ of space-time symmetry (typically translations) into a larger group $\Gamma = \mathcal{R} \dot{\times} G$ defined by a semi-direct product of $\mathcal{R}$ & G with $\Gamma/G = \mathcal{R}$. In this case, the sector bundles have a double fibration structure:

$$\begin{array}{ccccc} \hat{H} & \hookrightarrow & G \underset{H}{\times} \hat{H} & \hookrightarrow & \Gamma \underset{G}{\times} (G \underset{H}{\times} \hat{H}) = \Gamma \underset{H}{\times} \hat{H} \\ & & \downarrow & & \downarrow \\ & & G/H & & \Gamma/G = \mathcal{R} \end{array} .$$

Thus we have three different axes on different levels in Spec:

a) sectors $\hat{H}$ of *unbroken* symmetry H,

b) deg. vacua $G/H = M$ due to *broken internal* symmetry [1, 2],

c) $\Gamma/G = \mathcal{R}$ as emergent *space-time* [8] in broken external symmetry.

These axes arise in a series of structure-group contractions $H \leftarrow G \leftarrow \Gamma$ of principal bundles $P_H \hookrightarrow P_G \hookrightarrow P_\Gamma$ over $\mathcal{R}$, specified by ***solderings*** as bundle sections, $\mathcal{R} \overset{\rho}{\hookrightarrow} P_G/H = P_H \underset{H}{\times} (G/H)$, $\mathcal{R} \overset{\tau}{\hookrightarrow} P_\Gamma/G = P_G \underset{G}{\times} (\Gamma/G) = P_G \underset{G}{\times} \mathcal{R}$, corresponding physically to ***Goldstone modes***:

$$\begin{array}{ccccc} P_H & \hookrightarrow & P_G & \hookrightarrow & P_\Gamma \\ H\downarrow & \circlearrowleft & \downarrow H & \circlearrowleft & \downarrow H \\ \mathcal{R} & \overset{\rho}{\hookrightarrow} & P_G/H & \overset{\sigma}{\hookrightarrow} & P_\Gamma/H \\ & \searrow\searrow \circlearrowleft & \downarrow G/H & \circlearrowleft & \downarrow G/H \\ & & \mathcal{R} & \overset{\tau}{\hookrightarrow} & P_\Gamma/G \\ & & & \searrow\searrow \circlearrowleft & \downarrow \mathcal{R} \\ & & & & \mathcal{R} \end{array} .$$

8 Augmented Algebra as algebraic dual of Helgason Duality

From the algebraic viewpoint (dual to ***Helgason duality*** $K\backslash G \leftrightarrow G/H$:

$$\begin{array}{ccccc} & \nearrow & K\backslash G/H & \nwarrow & \\ K\backslash G & & \leftrightarrow & & G/H \\ & \nwarrow & G & \nearrow & \end{array}$$

, with Radon transforms & ***Hecke algebra*** $K\backslash G/H$), the essence of the relevant structures can be viewed as the "***stereo-graphic extension***" of such *planar* diagrams as controlling "augmented algebras" [1] of crossed products to describe symmetry breaking:

$$\begin{array}{ccccc} {}_{G/H}\swarrow & \mathcal{X}^H = \tilde{\mathcal{X}}^G & \searrow_H & & \\ \tilde{\mathcal{X}}^H & \Downarrow & \mathcal{X} & & \\ \downarrow_H \searrow\searrow & \tilde{\mathcal{X}} & \swarrow_{G/H} \downarrow & & \\ \downarrow \;\swarrow & \Downarrow & \searrow\searrow \;\downarrow & & \\ \widehat{H\backslash G} \hookrightarrow & \hat{G} & \twoheadrightarrow \hat{H} & & \end{array} \rightleftarrows \begin{array}{ccc} {}_{\mathcal{R}}\swarrow & \mathcal{O}_\rho = \mathcal{O}_d^H & \searrow_H \\ \mathcal{A}(\mathcal{R}) & \Downarrow & \mathcal{O}_d \\ \downarrow_H \searrow\searrow & \mathcal{X}(\mathcal{R}) & \swarrow_{\mathcal{R}} \downarrow \\ \downarrow \;\swarrow & \Downarrow & \searrow\searrow \;\downarrow \\ \hat{\mathcal{R}} \hookrightarrow & \hat{\Gamma} & \twoheadrightarrow \hat{H} \end{array} :$$

[same sort of lines are in the same exact seq]

Note that push-out diagram shows up here (right) in DR reconstruction [9] of field algebra $\mathcal{X}(\mathcal{R})$ with its internal symmetry unbroken.

9 Symmetric Space Structure & Maxwell Equation

Symmetric space structures of $G/H = M$ & $\Gamma/G = \mathcal{R}$ due to symmetry breaking is characterized by the equation of type $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, which connects holonomy $[\mathfrak{m}, \mathfrak{m}]$ (in terms of curvature) with generators $\mathfrak{h}$ of unbroken subgroup.

Note that this feature is shared in common by Maxwell & Einstein equations of electromagnetism and of gravity, respectively:

LHS: (curvature $F_{\mu\nu}$ or $R_{\mu\nu}$) = (source current J_μ or $T_{\mu\nu}$) : RHS.

According to the second Noether theorem (developed in the theory of invariants), Maxwell equation is an identity following from the invariance of action integral under space-time dependent transformations. In contrast, however, *no such classical quantities as action integrals nor Lagrangian densities* are available in our algebraic & categorical formulation of quantum fields.

9.1 Spectral functor in Doplicher-Roberts reconstruction of symmetry

The expected roles of action integral are to determine representation contents of a theory. In Doplicher & Roberts (DR) reconstruction [9], this can be substituted by categorical data concerning Galois group in terms of DR category $\mathcal{T}$ of modules of local excitations:

Obj($\mathcal{T}$) : local endomorphisms $\rho \in End(\mathcal{A})$ of observable algebra $\mathcal{A}$, selected by DHR localization criterion $\pi_0 \circ \rho \restriction_{\mathcal{A}(\mathcal{O}')} \cong \pi_0 \restriction_{\mathcal{A}(\mathcal{O}')}$,

Mor($\mathcal{T}$): $T \in \mathcal{T}(\rho \leftarrow \sigma) \subset \mathcal{A}$ intertwining $\rho, \sigma \in \mathcal{T}$: $\rho(A)T = T\sigma(A)$.

In this context, the group H of unbroken internal symmetry is identified with the group $H = End_\otimes(V)$ of unitary tensorial (=monoidal) natural transformations $u : V \dot{\leftarrow} V$ with the spectral functor $V : \mathcal{T} \hookrightarrow Hilb$ to embed $\mathcal{T}$ into category $Hilb$ of Hilbert spaces with morphisms as bounded linear maps.

9.2 Spectral functor in category & its local gauge invariance

Noting the commutativity diagram, $v_\rho W(T) = V(T)v_\sigma$: $\begin{array}{ccc} V(\rho) & \overset{v_\rho}{\leftarrow} & W(\rho) \\ V(T)\uparrow & \circlearrowleft & \uparrow W(T) \\ V(\sigma) & \underset{v_\sigma}{\leftarrow} & W(\sigma) \end{array}$,

to define a natural transformation $v : V \dot{\leftarrow} W$ from a functor W to another V with $T \in \mathcal{T}(\rho \leftarrow \sigma)$ [10], we re-interpret it as a categorical definition of a *local gauge transformation* $W \overset{\tau_v}{\to} \tau_v(W) = V$ of a functor W to V on the basis of definition:

$$\tau_v(W)(T) := v_\rho W(T) v_\sigma^{-1} \quad \text{for } T \in \mathcal{T}(\rho \leftarrow \sigma).$$

Note that similar formulae appear for gauge links in lattice gauge theory.

Then, the commutativity, $u_\rho V(T) = V(T) u_\sigma$ for $u \in End_\otimes(V)$, can be interpreted as *local gauge invariance* $\tau_u(V) = V$ of the functor V under *local gauge transformation* $V \to \tau_u(V)$ induced by a natural transformation $u \in H = End_\otimes(V)$.

9.3 Local gauge invariance & Maxwell equation

In the original DR theory, local endomorphisms $\rho \in \mathcal{T} \subset End(\mathcal{A})$ have, unfortunately, been regarded as *global* constant objects, owing to the emphasis on space-time transportability[1], and hence, the left-right difference of u_ρ and u_σ in $\tau_u(V)(T) := u_\rho V(T) u_\sigma^{-1}$ has not been recognized as important signal of local gauge structures.

From the viewpoint of forcing method, however, the essential features of logical extension ***from constants to variables*** [6] naturally lead to the interpretation of $\tau_u(V)(T) = u_\rho V(T) u_\sigma^{-1} = V(T)$ as the characterization of local gauge invariance of the functor V under local gauge transformaion $u : \mathcal{T} \ni \rho \longmapsto u_\rho$. This is in harmony also with the alternative formulation of principal bundles in terms of group-valued Čech cohomologies.

9.4 Spectral functors in *-Categories

In the usual definition, Galois group $G = Gal(\mathcal{X}/\mathcal{A}) = G(\mathcal{X}, \mathcal{A})$ is a group simply determined by two such arguments as algebra $\mathcal{X}$ and its subalgebra $\mathcal{A}$, with "quotient" $\mathcal{X}/\mathcal{A}$ having no actual meaning.

With symbol $/\mathcal{A}$ interpreted as $\mathcal{A}$ reduced to scalar, we can regard $\mathcal{X}/\mathcal{A}$ as a G-module with $Gal(\mathcal{X}/\mathcal{A})$ as its inverse Fourier transform.

In terms of natural transformations, this re-interpretation can be extended categorically, according to which we obtain functors to extract groups or algebras from *-categories of modules as follows:

1) $End_\otimes(\mathcal{T} \hookrightarrow Hilb) = G$: internal symmetry group derived from DR category $\mathcal{T}(\subset End(\mathcal{A}))$ of modules of local excitations

2) $Nat(Mod_B \hookrightarrow Hilb) = B''$: Rieffel's extraction of universal enveloping von Neumann algebra B'' from a category of B-modules

3) Takesaki-Bichteler's admissible operator fields on $Rep(B \to \mathfrak{H})$ in a

[1]This has led to the mathematical definition of "sectors" of $\mathcal{A}$ by $End(\mathcal{A})/Inn(\mathcal{A})$.

sufficiently big Hilbert space $\mathfrak{H}$ to reproduce von Neumann algebra B (Third example, focused up in Dr. Okamura's PhD thesis as a non-commutative extension of Gel'fand-Naimark theorem, can be viewed as a full subcategory of the second one according to Rieffel.)

10 Second Noether Theorem & Maxwell Equation

To adapt the roles of DR category $\mathcal{T} \subset End(\mathcal{A}) = End(\mathcal{X}^H)$ in determining the factor spectrum $Sp(\mathfrak{Z}(\mathcal{X}^H)) = \hat{H}$ to our present purpose, we need to replace $\mathcal{T}$ by $\tilde{\tilde{\mathcal{T}}} = End(\tilde{\tilde{\mathcal{X}}}^H)$ with $\tilde{\tilde{\mathcal{X}}} = \mathcal{X}^H \rtimes \widehat{\mathcal{R}}$ and with $\Gamma/G = \mathcal{R}$(: space-time) in the two-step construction of augmented algebras associated with the series of group extensions: unbroken $H \hookrightarrow$ broken internal $G \hookrightarrow$broken external Γ.

By repeating the categorical formulation of $End_{\otimes}(V : \mathcal{T} \hookrightarrow Hilb)$ with $\mathcal{T}$ and V replaced by $\tilde{\tilde{\mathcal{T}}}$ and $\tilde{\tilde{V}}$, respectively, we can reproduce the essence of the second Noether theorem to connect the local gauge invariance and Maxwell equation. In this context, the second Noether theorem can be generalized into a form with three type arguments, $x \in \mathcal{R}, \xi \in G/H, a \in \hat{H}$.

For simplicity, we reproduce its standard form with infinitesimal local gauge transformation $\delta_\Lambda \varphi^a(x) = G^a(x) \cdot \Lambda(x) + T^{a\mu}(x) \cdot \partial_\mu \Lambda(x)$ of fields $\varphi^a(x)$ specified by an "inifinitesimal parameter" $\Lambda = \Lambda(x)$ of a natural transformation depending on sector parameter $x \in \mathcal{R}$. Then Maxwell-type equation holds identically,

$$\partial_\nu K^{\nu\mu} + J^\mu = 0,$$

with $K^{\nu\mu}$ and J^μ defined in relation with the "infinitesimal transforms" of spectral functor V:

$$K^{\nu\mu} := T^{a\mu} \frac{\partial}{\partial(\partial_\nu \varphi^a)} V,$$
$$J^\mu := T^{a\mu} (\frac{\partial}{\partial \varphi^a} - \frac{\partial}{\partial(\partial_\nu \varphi^a)}) V + G^a \frac{\partial}{\partial(\partial_\mu \varphi^a)} V.$$

Choosing $\xi \in G/H$ as the parameter-dependence of local gauge transformations, we can incorporate the low-energy theorem (with "soft pions") due to symmetry breaking in the present context.

In the case with $a \in \hat{H}$, we note that the recovered group H of unbroken symmetry is compact in DR theory [9] which implies that the group dual $\hat{H}$ of sector parameters is discrete. While it seems difficult to adapt this case to the standard formulation of the second Noether theorem in terms of differential operations, we expect some interesting lessons to be learned from the attempt to unify it in the present context.

With the aid of this machinery, such a perspective (which has long been advocated by Dr. Saigo and also emphasized recently by Dr. Okamura) can now be envisaged that all the contents of QFT in quadrality scheme are unified into a C*-tensor category of physical quantities (work in progress).

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