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Some aspects of vanishing properties of solutions to nonlinear elliptic equations

By

SEPPO GRANLUND* and NIKO MAROLA**

Abstract

We discuss some aspects of vanishing properties of sign changing solutions to certain nonlinear elliptic partial differential equations.

§ 1. Introduction

We discuss some aspects of vanishing properties of sign changing solutions to certain second order quasilinear elliptic differential equations of the form

$$-\nabla\cdot\mathcal{A}(x,u,\nabla u)+\mathcal{B}(x,u,\nabla u)=0. \tag{1.1}$$

We shall specify the class of equations considered in this paper in (2.1)–(2.2) in Section 2. For solutions of linear equations with Lipschitz leading coefficients it is well-known that analyzing an Almgren type frequency function leads to monotonicity formulas and doubling inequalities. The monotonicity formulas and doubling inequalities in turn imply that if a sign changing solution vanishes in some proper open subset of a given domain, then it must vanish identically in the whole domain. We refer the reader to the celebrated papers [15, 16] by Garofalo and Lin. In this note we are interested in such vanishing properties of solutions.

In the nonlinear case on the other hand, it is known that there exists a second order nonlinear elliptic operator of divergence form ($\mathcal{B}=0$ in (1.1) and $p=n$, where

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*University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FI-00014, Finland. e-mail: seppo.granlund@pp.inet.fi
**University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FI-00014, Finland. e-mail: niko.marola@helsinki.fi

$n \geq 3$, in (2.1)–(2.2)) such that a solution to this equation that vanishes in the lower half space $x_n < 0$ of $\mathbb{R}^n$ does not vanish identically in the whole space [23].

In the present note, we investigate a nonlinear frequency function related to a solution of (1.1). The main goal of this paper is to obtain some results on vanishing properties of sign changing solutions to the equation (1.1) by way of such frequency function. Our main result is stated in Theorem 3.14.

We mention a recent paper [7], where the study of certain other generalizations of Almgren's frequency function give new results and insight on the critical set of the solutions to linear elliptic equations.

Finally, let us point out that one of the main estimates in the note, Proposition 3.3, can be considered as a generalized Poincaré-type inequality. Proposition 3.3 covers every $1 < p < \infty$, and although it is an easy generalization of a similar inequality proved for $p = 2$ in a forthcoming monograph by Han and Lin [19], it might be of independent interest to the reader.

Notation Throughout the paper a domain is a proper open connected subset of $\mathbb{R}^n$, $n \geq 2$, and $1 < p < \infty$. We use the notation $B_r = B(x,r)$ for concentric open balls of radii r centered at x. Unless otherwise stated, the letter C denotes various positive and finite constants whose exact values are unimportant and may vary from line to line. Moreover, dx denotes the Lebesgue volume element in $\mathbb{R}^n$, whereas dS denotes the surface element. The characteristic function of a set E is written as χ_E.

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§ 2. Nonlinear equations

Let G be a bounded domain in $\mathbb{R}^n$. We consider the equation (1.1) in weak form, i.e. for any $\eta \in W_0^{1,p}(G)$

$$\int_G \mathcal{A}(x,u,\nabla u) \cdot \nabla \eta \, dx + \int_G \mathcal{B}(x,u,\nabla u)\eta \, dx = 0$$

holds, where $\mathcal{A}\colon G \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{B}\colon G \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are assumed to satisfy the Carathéodory conditions. For the results in this paper it is essential that a weak solution is in $C^1(G)$, and therefore we shall assume this. It is well known, however, that by assuming more on the structure of $\mathcal{A}$ and $\mathcal{B}$ every weak solution is in $C^1(G)$, see

[21] and also [9, 25]. In addition, we shall assume that there are constants $1 < p < \infty$, $0 < a_0 \le a_1 < \infty$, and $0 < b_1 < \infty$ such that for all (t, h) in $\mathbb{R} \times \mathbb{R}^n$ and for almost every $x \in G$ the following structural assumptions hold:

$$\mathcal{A}(x,t,h) \cdot h \ge a_0|h|^p, \quad \mathcal{A}(x,t,h)| \le a_1|h|^{p-1}, \tag{2.1}$$

$$|\mathcal{B}(x,t,h)| \le b_1|h|^{p-1}. \tag{2.2}$$

We also consider the second order nonlinear elliptic equation

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, \tag{2.3}$$

where $1 < p < \infty$, $\lambda > 0$ is a parameter, and $u = 0$ on the boundary of a bounded domain $G \subset \mathbb{R}^n$ with smooth boundary ∂G. In fact, (2.3) is the p-Laplace generalization of the classical eigenvalue problem for the Laplace equation which can be recovered from (2.3) by setting $p = 2$. A good introduction to this nonlinear eigenvalue problem is [22], the references given there, and in particular [14]. For the results in this paper no regularity assumptions are needed about the boundary of G.

We interpret equation (2.3) in the weak sense; A function $u \in W_0^{1,p}(G)$, u not identically zero, is a weak solution to (2.3) if there exists $\lambda > 0$ such that

$$\int_G |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx = \lambda \int_G |u|^{p-2} u \eta \, dx, \tag{2.4}$$

where η is a test-function in $W_0^{1,p}(G)$. Standard elliptic regularity theory implies that u is locally in $C^{1,\alpha}(G)$, where the Hölder exponent α depends only on n and p. For this regularity result see [9] or [25]. For other properties we refer the reader to [22].

§ 3. Frequency function and vanishing of solutions

Let us consider the following frequency function for solutions to (1.1) or (2.3)

$$F_p(r) = \frac{r^{p-1} \int_{B_r} |\nabla u|^p \, dx}{\int_{\partial B_r} |u|^p \, dS}, \tag{3.1}$$

where $\overline{B}_r \subset G$. When it is necessary to stress also the function for which the frequency function is defined we write $F_p(r; u)$. We set

$$I(r) := \int_{\partial B_r} |u|^p \, dS.$$

Observe that $F_p(r)$ is not defined for such radii r for which $I(r) = 0$. We remark that $F_p(r)$ is a generalization of the well known Almgren frequency function $F_2(r)$ for harmonic functions in $\mathbb{R}^n$, see [1]. For harmonic functions the frequency function $F_2(r)$ is known to be non-decreasing as a function of r. This is not at all clear for $F_p(r)$.

It is straigthforward to check that for each positive real number τ the frequency function $F_p(r)$ satisfies the following scaling property $F_p(r;v) = F_p(\tau r;u)$, where we write $v(x) = u(\tau x)$.

Theorem 3.2. *Suppose $u \in C^1(G)$. Assume further that there exist two concentric balls $B_{r_b} \subset \overline{B}_{R_b} \subset G$ such that the frequency function $F_p(r)$ is defined, i.e. $I(r) > 0$ for every $r \in (r_b, R_b]$, and moreover, $\|F_p\|_{L^\infty((r_b,R_b])} < \infty$. Then there exists some $r^\star \in (r_b, R_b]$ such that*

$$\int_{\partial B_{r_1}} |u|^p \, dS \le 4 \int_{\partial B_{r_2}} |u|^p \, dS,$$

for every r_1, $r_2 \in (r_b, r^\star]$. In particular, the following weak doubling property is valid

$$\int_{\partial B_{r^\star}} |u|^p \, dS \le 4 \int_{\partial B_r} |u|^p \, dS,$$

for every $r \in (r_b, r^\star]$.

Proof. The proof can be found in [17, Section 4]; see also Section 5 in [18]. However, a minor modification in use of Young's inequality is needed due to the factor r^{p-1} instead of r in the numerator in (3.1). □

The next proposition can be considered as a generalization of a Poincaré inequality and it is interesting as such. Inequality (3.4) below is usually covered in the case in which $p = 2$; we refer the reader to [19] and [13]. It might be known for general $1 < p < \infty$, as the proof is rather straightforward, but due to a lack of a proper reference we provide a proof.

Proposition 3.3. *For any $u \in W^{1,p}(B_r) \cap C^1(B_r)$ with $r > 0$, there holds*

$$\int_{B_r} |u|^p \, dx \le \frac{2r}{n} \int_{\partial B_r} |u|^p \, dS + Cr^p \int_{B_r} |\nabla u|^p \, dx, \tag{3.4}$$

where C depends only on n and p.

Proof. We introduce radial and angular coordinates ρ and $\omega \in \partial B_1$, and define the sets $P = \{\omega \in \partial B_1 : u(\rho\omega) > 0\}$ and $N = \{\omega \in \partial B_1 : u(\rho\omega) < 0\}$. Let us calculate as follows

$$\begin{aligned}\int_{B_r} |u|^p\,dx &= \int_0^r \left(\int_{\partial B_\rho} |u(\rho\omega)|^p\,d\omega\right) d\rho = \int_0^r \left(\int_{\partial B_1} |u(\rho\omega)|^p\,d\omega\right) \rho^{n-1}\,d\rho \\ &= \frac{r^n}{n}\int_{\partial B_1} |u(r\omega)|^p\,d\omega \\ &\quad - \frac{p}{n}\int_0^r \left(\int_{\partial B_1} \left(u(\rho\omega)^{p-1}\chi_P u_\rho(\rho\omega) - (-u(\rho\omega))^{p-1}\chi_N u_\rho(\rho\omega)\right) d\omega\right)\rho^n\,d\rho \\ &\le \frac{r}{n}\int_{\partial B_r} |u|^p\,dS + \frac{p}{n}\int_{B_r} |x||u|^{p-1}|u_\rho|\,dx,\end{aligned}$$

where $u_\rho = \nabla u\cdot(x/\rho)$, $\rho = |x|$. Applying Young's inequality we have for any $\varepsilon > 0$

$$\int_{B_r} |u|^p\,dx \le \frac{r}{n}\int_{\partial B_r} |u|^p\,dS + \frac{p-1}{n(\varepsilon p)^{q/p}}\int_{B_r} |u|^p\,dS + \frac{p\varepsilon}{n}\int_{B_r} |x|^p|\nabla u|^p\,dx,$$

where $p = q(p-1)$. We obtain (3.4) by taking $\varepsilon = (2(p-1)/n)^{p-1}p^{-1}$. □

Remark 3.5. It seems obvious that one could assume less regularity on u in Proposition 3.3. However, we do not consider it here.

The use of Proposition 3.3 results in the estimate (3.7) in Lemma 3.6 stated next. A stronger version of the estimate was obtained in [17] for solutions to the p-Laplace equation in the form of an identity. An analogous estimate for solutions to (1.1) holds as well; we shall treat it separately in Lemma 3.11

Lemma 3.6. *Suppose u is a solution to* (2.3) *in G. Then there exists a radius r_0, depending on n, p, and λ, such that*

$$\int_{B_r} |\nabla u|^p\,dx \le C_1\int_{\partial B_r} |u||\nabla u|^{p-1}\,dS + C_2 r\int_{\partial B_r} |u|^p\,dS \tag{3.7}$$

is valid for every $\overline{B}_r \subset G$, where $r \le r_0$. Positive constants C_1 and C_2 depend on n, p, and λ only.

Proof. Let $B_r \subset B_\rho$ be concentric balls so that $\overline{B}_\rho \subset G$. We interpret equation (2.3) in the weak sense and plug in a test-function $\eta = u\xi^p$, where $\xi \in C_0^\infty(G)$, $0 \le \xi \le 1$, with $\xi = 1$ on B_r, $\xi = 0$ on $G\setminus B_\rho$, and $|\nabla\xi| \le C/(\rho - r)$; we hence obtain

$$\begin{aligned}\int_{B_r} |\nabla u|^p\,dx &\le p\int_{B_\rho} |u|\xi^{p-1}|\nabla u|^{p-1}|\nabla\xi|\,dx + \lambda\int_{B_\rho} |u|^p\xi^p\,dx \\ &\le \frac{Cp}{\rho - r}\int_{B_\rho\setminus B_r} |u||\nabla u|^{p-1}\,dx + \lambda\int_{B_\rho} |u|^p\,dx.\end{aligned} \tag{3.8}$$

Letting ρ tend to r in (3.8) we have

$$\int_{B_r} |\nabla u|^p \, dx \le Cp \int_{\partial B_r} |u||\nabla u|^{p-1} \, dS + \lambda \int_{B_r} |u|^p \, dx. \tag{3.9}$$

Using (3.4) for the second integral on the right-hand side in (3.9) we obtain

$$\begin{aligned} \int_{B_r} |\nabla u|^p \, dx \le C_1 p \int_{\partial B_r} |u||\nabla u|^{p-1} \, dS + \frac{2\lambda r}{n} \int_{\partial B_r} |u|^p \, dS \\ + C_2 \lambda r^p \int_{B_r} |\nabla u|^p \, dx. \end{aligned} \tag{3.10}$$

For small enough radii $r \le r_0$, where r_0 is chosen so that $C_2 \lambda r_0^2 = 1/2$, we obtain (3.7) from (3.10). □

Lemma 3.11. *Suppose u is a solution to* (1.1) *in G. Then there exists a radius r_0, depending on n, p, a_0, a_1, and b_1, such that*

$$\int_{B_r} |\nabla u|^p \, dx \le C_1 \int_{\partial B_r} |u||\nabla u|^{p-1} \, dS + C_2 r \int_{\partial B_r} |u|^p \, dS \tag{3.12}$$

is valid for every $\overline{B}_r \subset G$, where $r \le r_0$. Positive constants C_1 and C_2 depend on n, p, a_0, a_1, and b_1.

Proof. Let $B_r \subset B_\rho$ be concentric balls so that $\overline{B}_\rho \subset G$. Similarly as in the proof of Lemma 3.6, after plugging the test-function $\eta = u\xi^p$ into the weak formulation of the equation (1.1) and applying the structural conditions (2.1)–(2.2), we obtain by letting ρ tend to r

$$\int_{B_r} |\nabla u|^p \, dx \le \frac{Cpa_1}{a_0} \int_{\partial B_r} |u||\nabla u|^{p-1} \, dS + \frac{b_1}{a_0} \int_{B_r} |u||\nabla u|^{p-1} \, dx. \tag{3.13}$$

We treat the second integral on the right-hand side in (3.13) by applying first Young's inequality with $\varepsilon > 0$. Then we apply estimate (3.4) in Proposition 3.3 and obtain the desired estimate for sufficiently small radii. We leave the details for the reader. □

The following is our main theorem.

Theorem 3.14. *Suppose u is a solution to* (1.1) *or* (2.3) *in G. Consider arbitrary concentric balls $B_{r_b} \subset \overline{B}_{R_b} \subset G$. Assume that*

$$\|F_p\|_{L^\infty((r_b, R_b])} < \infty,$$

whenever $I(r) > 0$ for every $r \in (r_b, R_b]$. If u vanishes on some open non-empty subset of G, then u is identically zero in G.

Proof. The proof is by contradiction: Suppose that the function u, a non-trivial solution to (1.1) (or to (2.3)), vanishes identically in an open non-empty proper subset D of G, but u is not identically zero in G. It is possible to pick arbitrary small concentric neighborhoods B_{r_1} and B_{r_2}, $r_1 < r_2$, where $\overline{B}_{r_2} \subset G$, such that u vanishes identically in $\overline{B}_{r_1}$ but u is not identically zero in B_{r_2}. Due to this we may assume that $r_2 < r_0$ where r_0 is the radius in Lemma 3.11 (or Lemma 3.6).

Let $t > 0$ and consider an open ball B_t which is concentric with B_{r_1} and B_{r_2}. Define $s = \sup\{t > 0 : u|_{\partial B_t} \equiv 0\}$. The preceding assumptions imply that s must be in the interval $[r_1, r_2)$. We note, in addition, that due to Lemma 3.11 (or Lemma 3.6) we may conclude that $u|_{\partial B_\rho}$ does not vanish identically for any radii $\rho \in (s, r_2]$, hence $I(\rho) \neq 0$. We note that it is not known whether $I(r)$ is monotone on $(s, r_s]$.

The frequency function $F_p(r)$ is defined on $(s, r_2]$ and by the hypothesis of the theorem $F_p(r)$ is bounded on $(s, r_2]$. Theorem 3.2 implies the existence of a radius $r^\star \in (s, r_2]$ such that $I(r^\star) \leq 4I(r)$ holds for every $r \in (s, r^\star]$. Since $I(r) \searrow 0$ as $r \searrow s$ we have reached a contradiction. □

§ 4. Infinity harmonic equation

Let us close this note by discussing briefly the infinity Laplacian operator

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{4.1}$$

which leads to the infinity harmonic equation

$$\Delta_\infty u = 0.$$

The infinity harmonic equation arises as the Euler-Lagrange equation for the problem of finding absolute minimizers for the L^∞-energy $\|\nabla u\|_{L^\infty}$. We refer the reader to [2, 3, 6, 8, 20], and the references therein, for detailed discussion on this equation, applications, and for the properties of its solutions.

We mention in passing that the equation is highly nonlinear and degenerate as it degenerates on the hyperplane $\{\xi \in \mathbb{R}^n : \xi \perp \nabla u(x)\}$. The equation is not in divergence form, in particular, it does not have a weak formulation. The appropriate notion is that of viscosity solution.

It is an interesting open problem whether an infinity harmonic function in a domain G can vanish in an open subset of G without being identically zero in G. By a result due to Yu [26], for a C^2 solution of the infinity harmonic equation in a domain G it is known that if $\nabla u(x_0) = 0$ for some $x_0 \in G$, then $u \equiv \nabla u(x_0)$, i.e. a nonconstant C^2

solution cannot have interior critical points. This phenomenon was first observed by Aronsson [2] in the plane. For a C^4 solution in every dimensions Evans [10] established a Harnack estimate for $|\nabla u|$, and hence the fact that nonconstant C^4 solutions have no interior critical points. Yu's method in [26] follows Evan's work.

Solutions to the infinity harmonic functions need not be C^2 smooth as Aronsson's example

$$u(x,y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}, \qquad (x,y) \in \mathbb{R}^2,$$

indicates. Indeed, it is a $C^{1,\frac{4}{3}}$ smooth infinity harmonic function. Smooth C^2 solutions to the infinity harmonic equation possess some special properties, such as Yu's result discussed above, which general viscosity solutions do not have; Yu's theorem does not hold for the aforementioned $C^{1,\frac{4}{3}}$ solution since $(0,0)$ is clearly its critical point.

Optimal regularity of viscosity solutions is the primary open problem and very challenging one in higher dimensions. In the plane $C^{1,\alpha}$ regularity was recently proved in [11], see also the seminal paper [24]. In higher dimensions everywhere differentiability of viscosity solutions to the infinity harmonic equation is known thanks to [12].

Another open problem, or a conjecture, is to show that a global Lipschitz solution must be linear. We refer the reader to [10], [4, 5].

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