CONVERGENCE OF THE ALLEN-CAHN EQUATION WITH NEUMANN BOUNDARY CONDITIONS

MASASHI MIZUNO AND YOSHIHIRO TONEGAWA

ABSTRACT. In this note we describe the results obtained by the paper (titled the same) which studies a singular limit problem of the Allen-Cahn equation with Neumann boundary conditions and general initial data of uniformly bounded energy. In it we prove that the time-parametrized family of limit energy measures is Brakke's mean curvature flow with a generalized right angle condition on the boundary.

1. INTRODUCTION

We consider the following Allen-Cahn equation:

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}, & t>0,\ x\in\Omega, \\ \frac{\partial u^\varepsilon}{\partial\nu}\Big|_{\partial\Omega} = 0, & t>0, \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, $\varepsilon>0$ is a small positive parameter, ν is the outer unit normal vector field on $\partial\Omega$ and W is a bi-stable potential with two equal wells at ±1. $W(u)=\frac{1}{4}(1-u^2)^2$ is a typical example. The equation (1.1) is a gradient flow of

$$E^\varepsilon[u] := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right)dx$$

as one may check easily that $\frac{dE^\varepsilon}{dt}\le 0$. Under the assumption that a given family $\{u_0^\varepsilon\}_{0<\varepsilon<1}$ satisfies

$$\sup_{0<\varepsilon<1} E^\varepsilon[u_0^\varepsilon]<\infty,$$

it is interesting to study the limiting behavior of the solution u^ε of (1.1) as $\varepsilon\to0$. Heuristically, one expects that the finiteness assumption for $E^\varepsilon[u^\varepsilon(\cdot,t)]$ for very small ε implies a 'phase separation', i.e., Ω is mostly divided into two regions where $u^\varepsilon(\cdot,t)$ is close to 1 on one of them and to -1 on the other, with thin 'transition layer' of order ε thickness separating these two regions. With this heuristic picture, one may also expect that the following measures μ_t^ε defined by

$$d\mu_t^\varepsilon := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon(x,t)|^2 + \frac{W(u^\varepsilon(x,t))}{\varepsilon}\right)dx \tag{1.2}$$

behave more or less like surface measures of moving phase boundaries. It is thus interesting and natural to study $\lim_{\varepsilon\to0}\mu_t^\varepsilon$. By the well-known heuristic argument using the signed distance functions to the moving phase boundaries composed with the one-dimensional standing wave

2000 *Mathematics Subject Classification.* 28A75,35K20,53C44.
Key words and phrases. Boundary monotonicity formula, Allen-Cahn equation, mean curvature flow, varifold.
M. Mizuno worked done during a visit to the Institut Mittag-Leffler (Djursholm, Sweden). This work was supported by JSPS KAKENHI Grant Numbers 21224001, 25800084, 25247008.

solution of $\varepsilon^2 u'' = W'(u)$, one may also expect that the motion of the phase boundaries is the mean curvature flow (abbreviated hereafter as MCF). The rigorous proof of this in the most general setting, on the other hand, requires extensive use of tools from geometric measure theory.

The singular limit of (1.1) without boundary is studied by many researchers with different settings and assumptions. The most relevant among them to the present paper is Ilmanen's work [14], which showed that the limit measures of μ_t^ε are the MCF in the sense of Brakke [4] (where $\Omega = \mathbb{R}^n$). There was a technical assumption in [14] on the initial condition, which was removed by Soner [26]. The second author observed that Ilmanen's work can be extended to bounded domains, and showed that the limit measures have integer densities a.e. modulo division by a constant [29]. If the densities are equal to 1 a.e., it has been proved recently that the support of the measures is smooth a.e. as well [4, 16, 30]. By these works, interior behavior of the limit measures has been rigorously characterized as Brakke's MCF. There are numerous earlier and relevant results on (1.1) and we additionally mention [5, 6, 7, 8, 21, 23, 25, 27, 28] which is by no means an exhaustive listing.

Now turning to the attention to the problem with Neumann boundary conditions, one may heuristically expect that the limit phase boundaries intersect $\partial\Omega$ with 90 degree angle. Katsourakis et al. [15] basically proved this connecting the singular limit of (1.1) to the unique viscosity solutions of level set equation of the MCF with right angle boundary conditions studied in [9, 23]. The differences of the present paper from [15] are explained as follows. While one does not know in [15] if the particular individual level set obtained as a singular limit of (1.1) satisfies MCF equation or boundary conditions in some measure-theoretic sense, we show that the limit measure satisfies Brakke's inequality with a generalized right angle condition. If we assume that the limit measure has density 1 a.e., then, it is smooth a.e. in the interior due to [4, 16, 30]. We also obtain a characterization for any finite energy initial data in $W^{1,2}(\Omega)$ and not necessarily for a carefully prepared initial data. Perhaps the most insightful aspect of the present paper is that our study motivates a measure-theoretic formulation of Brakke's MCF up to the boundary (see Section 2.4) for which one may further pursue the establishment of up to the boundary regularity theorem.

More technically speaking, in this paper, we prove that (1) the limit measures μ_t have bounded first variation on $\overline{\Omega}$ for a.e. $t \geq 0$, (2) μ_t is $n-1$-rectifiable on $\overline{\Omega}$ and integral (modulo division by a constant) on Ω for a.e. $t \geq 0$, (3) μ_t satisfies Brakke's inequality of MCF up to the boundary with a suitable modification for the first variation on $\partial\Omega$. If we assume in addition that $\mu_t(\partial\Omega) = 0$, then the right angle condition on the boundary is satisfied in the sense that the first variation of μ_t on $\partial\Omega$ is perpendicular to $\partial\Omega$. We make an assumption that Ω is strictly convex, even though some generalization is possible. The proof uses various ideas developed through [14, 29, 28]. In those paper, the Huisken/Ilmanen monotonicity formula played a central role and the situation is the same in this paper as well. We first prove up to the boundary monotonicity formula by a boundary reflection method, and this leads us to similar estimates as in the interior case. We need to be concerned with measures concentrated on $\partial\Omega$ as well as the limit of 'boundary measures of phase boundary'. All those quantities are incorporated in the final formulation appearing in Theorem 2.6.

2. Preliminaries and main results

2.1. **Basic notation.** Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}^+ := \{x \geq 0\}$. For $0 < r < \infty$ and $a \in \mathbb{R}^k$, define $B_r^k(a) := \{x \in \mathbb{R}^k \,:\, |x - a| < r\}$. When $k = n$, we omit writing k and we write $B_r := B_r^n(0)$. The Lebesgue measure is denoted by $\mathcal{L}^n$ and the k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. Let $\omega_n := \mathcal{L}^n(B_1)$.

For any Radon measure μ on $\mathbb{R}^n$ and $\phi \in C_c(\mathbb{R}^n)$ we often write $\mu(\phi)$ for $\int \phi \, d\mu$. We write $\operatorname{spt}\mu$ for the support of μ. Thus $x \in \operatorname{spt}\mu$ if $\forall r > 0$, $\mu(B_r(x)) > 0$. We use the standard notation for the Sobolev spaces such as $W^{1,p}(\Omega)$ from [10].

For $A, B \in \operatorname{Hom}(\mathbb{R}^n; \mathbb{R}^n)$ which we identify with $n \times n$ matrices, we define

$$A \cdot B := \sum_{i,j} A_{ij} B_{ij}.$$

The identity of $\operatorname{Hom}(\mathbb{R}^n; \mathbb{R}^n)$ is denoted by I. For $k \in \mathbb{N}$ with $k < n$, let $\mathbf{G}(n,k)$ be the space of k-dimensional subspaces of $\mathbb{R}^n$. For $S \in \mathbf{G}(n,k)$, we identify S with the corresponding orthogonal projection of $\mathbb{R}^n$ onto S and its matrix representation. For $a \in \mathbb{R}^n$, $a \otimes a \in \operatorname{Hom}(\mathbb{R}^n; \mathbb{R}^n)$ is the matrix with the entries $a_i a_j$ $(1 \leq i, j \leq n)$. For any unit vector $a \in \mathbb{R}^n$, $I - a \otimes a \in \mathbf{G}(n, n-1)$. For $x, y \in \mathbb{R}^n$ and $t < s$, define

$$\rho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}}. \tag{2.1}$$

2.2. **Varifold.** We recall some definitions related to varifold and refer to [2, 24] for more details. In this paper, for a bounded open set $\Omega \subset \mathbb{R}^n$, we need to consider various objects on $\overline{\Omega}$ instead of Ω. For this reason, let $X \subset \mathbb{R}^n$ be either open or compact in the following. Let $G_k(X) := X \times \mathbf{G}(n,k)$. A general k-varifold in X is a Radon measure on $G_k(X)$. We denote the set of all general k-varifold in X by $\mathbf{V}_k(X)$. For $V \in \mathbf{V}_k(X)$, let $\|V\|$ be the weight measure of V, namely,

$$\|V\|(\phi) := \int_{G_k(X)} \phi(x) \, dV(x,S), \quad \forall \phi \in C_c(X).$$

We say $V \in \mathbf{V}_k(X)$ is rectifiable if there exist a $\mathcal{H}^k$ measurable countably k-rectifiable set $M \subset X$ and a locally $\mathcal{H}^k$ integrable function θ defined on M such that

$$V(\phi) = \int_M \phi(x, \operatorname{Tan}_x M) \theta(x) \, d\mathcal{H}^k \tag{2.2}$$

for $\phi \in C_c(G_k(X))$. Here $\operatorname{Tan}_x M$ is the approximate tangent space of M at x which exists $\mathcal{H}^k$ a.e. on M. Rectifiable k-varifold is uniquely determined by its weight measure through the formula (2.2). For this reason, we naturally say a Radon measure μ on X is rectifiable if there exists a rectifiable varifold such that the weight measure is equal to μ. If in addition that $\theta \in \mathbb{N}$ $\mathcal{H}^k$ a.e. on M, we say V is integral. The set of all rectifiable (resp. integral) k-varifolds in X is denoted by $\mathbf{RV}_k(X)$ (resp. $\mathbf{IV}_k(X)$). If $\theta = 1$ $\mathcal{H}^k$ a.e. on M, we say V is a unit density k-varifold.

For $V \in \mathbf{V}_k(X)$ let δV be the first variation of V, namely,

$$\delta V(g) := \int_{G_k(X)} \nabla g(x) \cdot S \, dV(x,S) \tag{2.3}$$

for $g \in C_c^1(X; \mathbb{R}^n)$. If the total variation $\|\delta V\|$ of δV is locally bounded (note in the case of $X = \overline{\Omega}$, this means $\|\delta V\|(\overline{\Omega}) < \infty$), we may apply the Radon-Nikodym theorem to δV with respect to $\|V\|$. Writing the singular part of $\|\delta V\|$ with respect to $\|V\|$ as $\|\delta V\|_{\text{sing}}$, we have

$\|V\|$ measurable $h(V,\cdot)$, $\|\delta V\|$ measurable ν_{sing} with $|\nu_{\mathrm{sing}}| = 1$ $\|\delta V\|$ a.e., and a Borel set $Z \subset X$ such that $\|V\|(Z) = 0$ with,

$$\delta V(g) = -\int_X h(V,\cdot)\cdot g\, d\|V\| + \int_Z \nu_{\mathrm{sing}}\cdot g\, d\|\delta V\|_{\mathrm{sing}}$$

for all $g \in C^1_c(X;\mathbb{R}^n)$. We say $h(V,\cdot)$ is the generalized mean curvature vector of V, ν_{sing} is the (outer-pointing) generalized co-normal of V and Z is the generalized boundary of V.

2.3. **Setting of the problem.** Suppose that $n \geq 2$ and

(2.4) $\quad \Omega \subset \mathbb{R}^n$ is a bounded, strictly convex domain with smooth boundary $\partial\Omega$.

Here the strict convexity means that the principal curvatures of $\partial\Omega$ are all positive. Suppose that $W : \mathbb{R} \to \mathbb{R}$ is a C^3 function with $W(\pm 1) = 0$, $W(u) \geq 0$ for all $u \in \mathbb{R}$,

(2.5) $\quad$ for some $-1 < \gamma < 1$, $W' < 0$ on $(\gamma, 1)$ and $W' > 0$ on $(-1,\gamma)$,

(2.6) $\quad$ for some $0 < \alpha < 1$ and $\kappa > 0$, $W''(u) \geq \kappa$ for all $\alpha \leq |u| \leq 1$.

A typical example of such W is $(1-u^2)^2/4$, for which we may set $\gamma = 0$, $\alpha = \sqrt{2/3}$ and $\kappa = 1$. For a given sequence of positive numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty}\varepsilon_i = 0$, suppose that $u_0^{\varepsilon_i} \in W^{1,2}(\Omega)$ satisfies

$$\|u_0^{\varepsilon_i}\|_{L^\infty(\Omega)} \leq 1 \tag{2.7}$$

and

$$\sup_i E^{\varepsilon_i}[u_0^{\varepsilon_i}] \leq c_1. \tag{2.8}$$

The condition (2.7) may be dropped if we assume a suitable growth rate upper bound on W which is suitable for the existence of solution for (1.1). A typical example of sequence of $u_0^{\varepsilon_i}$ may be given as in [20]. We include the detail for the convenience of the reader. Let $U \subset \mathbb{R}^n$ be any domain with C^1 boundary $M = \partial U$, and let Φ be a solution of ODE $\Phi'' = W'(\Phi)$ with $\Phi(\pm\infty) = \pm 1$ and $\Phi(0) = 0$. Note that such a solution exists uniquely, and Φ also satisfies $\Phi' = \sqrt{2W(\Phi)}$. Let d be the signed distance function to M so that it is positive inside of U. Define $u_0^{\varepsilon_i}(x) := \Phi(d(x)/\varepsilon_i)$ for $x \in \Omega$. Then one can check that, using $\Phi' = \sqrt{2W(\Phi)}$ and $|\nabla d| = 1$ a.e.,

$$E^{\varepsilon_i}[u_0^{\varepsilon_i}] = \int_\Omega \varepsilon_i^{-1}(\Phi')^2\, dx = \int_\Omega \varepsilon_i^{-1}\Phi'\sqrt{2W(\Phi)}|\nabla d|\, dx. \tag{2.9}$$

By the co-area formula, then,

$$E^{\varepsilon_i}[u_0^{\varepsilon_i}] = \int_{-\infty}^{\infty}\int_{\Omega\cap\{d=\varepsilon_i s\}} \Phi'(s)\sqrt{2W(\Phi(s))}\, d\mathcal{H}^{n-1} ds. \tag{2.10}$$

If M is transverse to $\partial\Omega$, $\mathcal{H}^{n-1}(\Omega\cap\{d=\varepsilon_i s\}) \approx \mathcal{H}^{n-1}(M\cap\Omega)$ for small ε_i and (2.10) shows

$$\lim_{i\to\infty} E^{\varepsilon_i}[u_0^{\varepsilon_i}] = \sigma\mathcal{H}^{n-1}(\Omega\cap M), \quad \sigma := \int_{-1}^{1}\sqrt{2W(u)}\, du. \tag{2.11}$$

Thus in this case, we may take $c_1 = \sigma\mathcal{H}^{n-1}(M\cap\Omega) + 1$, for example.

We next solve the problem (1.1) with ε_i and $u_0^{\varepsilon_i}$ satisfying (2.7) and (2.8). By the standard parabolic existence and regularity theory, for each i, there exists a unique solution u^{ε_i} with

$$u^{\varepsilon_i} \in L^2_{loc}([0,\infty); W^{2,2}(\Omega)) \cap C^\infty(\overline{\Omega}\times(0,\infty)), \quad \partial_t u^{\varepsilon_i} \in L^2([0,\infty); L^2(\Omega)). \tag{2.12}$$

By the maximum principle and (2.7),

$$\sup_{x\in\overline{\Omega},\, t>0} |u^{\varepsilon_i}(x,t)| \leq 1, \tag{2.13}$$

and due to the gradient structure and (2.8), we also have

$$E^{\varepsilon_i}[u^{\varepsilon_i}(\cdot,T)] + \int_0^T \int_\Omega \varepsilon_i \big(\Delta u^{\varepsilon_i} - \frac{W'}{\varepsilon_i^2}\big)^2 \,dxdt = E^{\varepsilon_i}[u^{\varepsilon_i}(\cdot,0)] \leq c_1 \tag{2.14}$$

for any $T > 0$. Thus, for each i through (1.2), we have a family $\{\mu_t^{\varepsilon_i}\}_{t\in[0,\infty)}$ of uniformly bounded Radon measures.

2.4. **Main results.** The following sequence of theorems and definitions constitutes the main results of the present paper.

Theorem 2.1. *Under the assumptions* (2.4)-(2.8), *let u^{ε_i} be the solution of* (1.1). *Define $\mu_t^{\varepsilon_i}$ as in* (1.2). *Then there exists a subsequence (denoted by the same index) and a family of Radon measures $\{\mu_t\}_{t\geq 0}$ on $\overline{\Omega}$ such that for all $t \geq 0$, $\mu_t^{\varepsilon_i} \rightharpoonup \mu_t$ as $i \to \infty$ on $\overline{\Omega}$. Moreover, for a.e. $t \geq 0$, μ_t is rectifiable on $\overline{\Omega}$.*

Due to Theorem 2.1, we may define rectifiable varifolds as follows.

Definition 2.2. For a.e. $t \geq 0$, let $V_t \in \mathbf{RV}_{n-1}(\overline{\Omega})$ be the unique rectifiable varifold such that $\|V_t\| = \mu_t$ on $\overline{\Omega}$. For any t such that μ_t is not rectifiable, define $V_t \in \mathbf{V}_{n-1}(\overline{\Omega})$ to be an arbitrary varifold with $\|V_t\| = \mu_t$ (for example $V_t(\phi) := \int_{\overline{\Omega}} \phi(\cdot, \mathbb{R}^{n-1}\times\{0\})\, d\mu_t$ for $\phi \in C(G_{n-1}(\overline{\Omega}))$).

Theorem 2.3. *Let V_t be defined as above. Then the following property holds.*

(1) *For a.e. $t \geq 0$, $\sigma^{-1}V_t\lfloor_\Omega \in \mathbf{IV}_{n-1}(\Omega)$.*
(2) *For a.e. $t \geq 0$, $\|\delta V_t\|(\overline{\Omega}) < \infty$ and $\int_0^T \|\delta V_t\|(\overline{\Omega})\, dt < \infty$ for all $T > 0$.*

We next define the tangential component of the first variation δV_t on $\partial\Omega$.

Definition 2.4. For a.e. $t \geq 0$ such that $\|\delta V_t\|(\overline{\Omega}) < \infty$, define

$$\delta V_t\lfloor_{\partial\Omega}^{\top}(g) := \delta V_t\lfloor_{\partial\Omega}(g - (g\cdot\nu)\nu) \quad \text{for } g \in C(\partial\Omega;\mathbb{R}^n) \tag{2.15}$$

where ν is the unit outward-pointing normal vector field on $\partial\Omega$.

We have the following absolute continuity result.

Theorem 2.5. *For a.e. $t \geq 0$, we have $\|\delta V_t\lfloor_{\partial\Omega}^{\top} + \delta V_t\lfloor_\Omega\| \ll \|V_t\|$, and there exists $h_b = h_b(t) \in L^2(\|V_t\|)$ such that*

$$\delta V_t\lfloor_{\partial\Omega}^{\top} + \delta V_t\lfloor_\Omega = -h_b(t)\|V_t\|. \tag{2.16}$$

Moreover,

$$\int_0^\infty \int_{\overline{\Omega}} |h_b|^2 \, d\|V_t\| dt \leq c_1. \tag{2.17}$$

Note that $h_b = h(V_t,\cdot)$ in Ω. Finally, using the above quantities, we have

Theorem 2.6. *For $\phi \in C^1(\overline{\Omega}\times[0,\infty)\,;\mathbb{R}^+)$ with $\nabla\phi(\cdot,t)\cdot\nu = 0$ on $\partial\Omega$ and for any $0 \leq t_1 < t_2 < \infty$, we have*

$$\int_{\overline{\Omega}} \phi(\cdot,t)\, d\|V_t\|\Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{\overline{\Omega}} \big(-\phi|h_b|^2 + \nabla\phi\cdot h_b + \partial_t\phi\big)\, d\|V_t\| dt. \tag{2.18}$$

If $\phi(\cdot, t)$ has a compact support in Ω, (2.18) is Brakke's inequality [4] in an integral form. If we have a situation that $\|V_t\|(\partial\Omega) = 0$, then Theorem 2.5 shows $\delta V_t \lfloor_{\partial\Omega}^{\top} = 0$ and $\delta V_t \lfloor_{\partial\Omega}$ is singular with respect to $\|V_t\|$. It is parallel to ν for $\|\delta V_t\|$ a.e. which would, if spt $\|V_t\|$ is smooth up to the boundary, correspond to 90 degree angle of intersection.

2.5. **Comment.** It seems likely that, if $\|V_0\|(\partial\Omega) = 0$, then $\|V_t\|(\partial\Omega) = 0$ holds for all $t > 0$. Intuitively, due to the strict convexity of the domain and the Neumann boundary condition (which should intuitively imply 90 degree angle of intersection), interior of moving hypersurfaces should not touch $\partial\Omega$. Due to the maximum principle, this cannot happen if the hypersurfaces are smooth up to the boundary. But within the general framework of this paper, we do not know how to prove such statement or if it is indeed true.

Though it may first appear counter intuitive in view of the connection to the MCF, if we have $\|V_0\|(\partial\Omega) > 0$, then it is possible to have $\|V_t\|(\partial\Omega) > 0$ for all $t > 0$. An example can be provided by a limit of time-independent solutions of (1.1) where $\mu^\varepsilon \rightharpoonup c\mathcal{H}^{n-1}\lfloor_{\partial\Omega}$ on $\overline{\Omega}$ as $\varepsilon \to 0$, where $c > 0$ is some constant. One can obtain such family of solutions u^ε by considering $\Omega = B_1$ and a mountain path solution connecting two constant functions 1 and -1 within a class of radially symmetric functions. There are uniform positive lower and upper bounds of $E^\varepsilon(u^\varepsilon)$ and the limiting varifold V is non-trivial. On the other hand, if $\|V\|(B_1) > 0$, due to [13], spt $\|V\|$ has to be a minimal surface, which contradicts the radially symmetry. Thus $\|V\|$ is concentrated only on ∂B_1 and is non-trivial. In this particular case, note that $\delta V = -\frac{x}{|x|}\mathcal{H}^{n-1}\lfloor_{\partial B_1}$ and the tangential component $\delta V\lfloor_{\partial B_1}^{\top}$ is 0. Using more explicit and sophisticated method, Malchiodi-Ni-Wei [19] constructed a family of solutions with multiple layers whose energy concentrates on ∂B_1 with $\|V\|(\partial B_1) = N\sigma\mathcal{H}^{n-1}$, $N \in \mathbb{N}$. N may be arbitrarily chosen. Furthermore, for general strictly mean convex domain Ω, Malchiodi-Wei [18] constructed a family of single layered solutions whose limit energy concentrates on $\partial\Omega$. Even though such limit measures are not certainly the MCF in $\mathbb{R}^n$ in the usual sense (it should shrink), such time independent measures satisfies (2.18) trivially since $h_b = 0$. This is the reason that we need to decompose the first variation on $\partial\Omega$ to accommodate such cases in general.

The existence result of the present paper suggests a reasonable setting for proving the boundary regularity of MCF. It is interesting to extend interior regularity theorem (see [4, 16, 30]) to the corresponding boundary regularity theorem. For the time-independent case, interior regularity [1] has been extended to boundary regularity [2, 3, 11].

REFERENCES

[1] Allard, W. K., *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.

[2] Allard, W. K., *On the first variation of a varifold: boundary behavior*, Ann. of Math. (2) **101** (1975), 418–446.

[3] Bourni, T., *Allard type boundary regularity theorem for varifolds with $C^{1,\alpha}$ boundary*, arXiv:1008.4728.

[4] Brakke, K. A., *The motion of a surface by its mean curvature*, Mathematical Notes, vol. 20, Princeton University Press, 1978.

[5] Bronsard, L. and Kohn, R. V., *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations **90** (1991), 211–237.

[6] Chen, X., *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential Equations **96** (1992), 116–141.

[7] de Mottoni, P. and Schatzman, M., *Évolution géométrique d'interfaces*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), 453–458.

[8] Evans, L. C., Soner, H. M. and Souganidis, P. E., *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992), 1097–1123.

[9] Giga, Y. and Sato, M.-H., *Neumann problem for singular degenerate parabolic equations*, Differential Integral Equations **6** (1993), 1217–1230.

[10] Gilbarg, D. and Trudinger, N. S., *Elliptic partial differential equations of second order*, 2nd ed. , Springer, 1983.

[11] Grüter, M. and Jost, J., *Allard type regularity results for varifolds with free boundaries*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **13** (1986), 129–169.

[12] Huisken, G., *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), 285–299.

[13] Hutchinson, J. E. and Tonegawa, Y., *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. PDE **10** (2000), 49–84.

[14] Ilmanen, T., *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. **38** (1993), 417–461.

[15] Katsoulakis, M., Kossioris, G. T. and Reitich, F., *Generalized motion by mean curvature with Neumann conditions and the Allen-Cahn model for phase transitions*, J. Geom. Anal. **5** (1995), 255–279.

[16] Kasai, K. and Tonegawa, Y., *A general regularity theory for weak mean curvature flow*, Calc. Var. Partial Differential Equations. **50** (2014), 1–68.

[17] Liu, C., Sato, N. and Tonegawa, Y., *On the existence of mean curvature flow with transport term*, Interfaces Free Bound. **12** (2010), 251–277.

[18] Malchiodi, A. and Wei, J., *Boundary interface for the Allen-Cahn equation*, J. Fixed Point Theory Appl. **1** (2007), 305–336.

[19] Malchiodi, A., Ni, W.-M. and Wei, J., *Boundary-clustered interfaces for the Allen-Cahn equation*, Pacific J. Math. **229** (2007), 447–468.

[20] Modica, L., *Gradient theory of phase transitions and minimal interface criteria*, Arch. Rational Mech. Anal. **98** (1987), 123–142.

[21] Rubinstein, J., Sternberg, P. and Keller, J. B., *Fast reaction, slow diffusion, and curve shortening*, SIAM J. Appl. Math. **49** (1989), 116–133.

[22] Sato, M.-H., *Interface evolution with Neumann boundary condition*, Adv. Math. Sci. Appl. **4** (1994), 249–264.

[23] Sato, N., *A simple proof of convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, Indiana Univ. Math. J. **57** (2008), 1743–1751.

[24] Simon, L., *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, **3**, Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[25] Soner, H. M., *Convergence of the phase-field equations to the Mullins-Sekerka problem with kinetic undercooling*, Arch. Rational Mech. Anal. **131** (1995), 139–197.

[26] Soner, H. M., *Ginzburg-Landau equation and motion by mean curvature. I. Convergence*, J. Geom. Anal. **7** (1997), 437–475.

[27] Soner, H. M., *Ginzburg-Landau equation and motion by mean curvature. II. Development of the initial interface*, J. Geom. Anal. **7** (1997), 477–491.

[28] Takasao, K. and Tonegawa, Y., *Existence and regularity of mean curvature flow with transport term in higher dimensions*, arXiv:1307.6629v2.

[29] Tonegawa, Y., *Integrality of varifolds in the singular limit of reaction-diffusion equations*, Hiroshima Math. J. **33** (2003), 323–341.

[30] Tonegawa, Y., *A second derivative Hölder estimate for weak mean curvature flow*, Adv. Cal. Var. **7** (2014), 91–138.

(Masashi Mizuno) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE AND TECHNOLOGY, NIHON UNIVERSITY, TOKYO 101-8308 JAPAN
E-mail address: mizuno@math.cst.nihon-u.ac.jp

(Yoshihiro Tonegawa) DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO 060-0810 JAPAN
E-mail address: tonegawa@math.sci.hokudai.ac.jp