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GLOBAL SMOOTHING EFFECT OF INFINITE ENERGY SOLUTIONS TO THE HOMOGENEOUS BOLTZMANN EQUATION OF MAXWELLIAN MOLECULES

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ABSTRACT. The purpose of this note is to announce recent development of researches about the smoothing effect and the asymptotic behavior of measure valued solutions with infinite energy for the homogeneous non-cutoff Boltzmann equation of Maxwellian molecules. The contents are based on joint works [15, 16, 17, 18] with Tong Yang, Shuaikun Wang, City University of Hong Kong, and Huijiang Zhao, Wuhan University.

1. INTRODUCTION

We consider the spatially homogeneous Boltzmann equation,

$$\partial_t f = Q(f,f), \tag{1.1}$$

where $f = f(t,v)$ is the density distribution function of particles with velocity $v \in \mathbb{R}^3$ at time t. The right hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\left\{g(v_*')f(v') - g(v_*)f(v)\right\} d\sigma dv_*\,,$$

where the conservation of momentum and energy implies that for $\sigma \in \mathbb{S}^2$

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\ \ v_*' = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma\,.$$

The equation (1.1) is supplemented with an intial datum

$$f(0,v) = dF_0 \ge 0, \tag{1.2}$$

where F_0 is a probability measure.

The non-negative cross section B usually takes the form

$$B = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma\,,\ \ 0 \le \theta \le \frac{\pi}{2}, \tag{1.3}$$

where

$$\begin{aligned} &\Phi(|z|) = \Phi_\gamma(|z|) = |z|^\gamma, \ \text{ for some } \gamma > -3, \\ &b(\cos\theta)\theta^{2+2s} \ \to K_0 \ \text{ when } \ \theta \to 0+, \text{for } 0 < s < 1 \text{ and } K_0 > 0. \end{aligned} \tag{1.4}$$

In fact, for the physical model, if the inter-molecule potential satisfies the inverse power law potential $U(\rho) = \rho^{-(q-1)}, q > 2$(, where ρ denotes the distance between two interacting molecules), then s and γ are given by

$$0 < s = 1/(q-1) < 1 \ , \quad 1 > \gamma = 1 - 4s = (q-5)/(q-1) > -3 \ .$$

As usual, the hard ($\gamma > 0$) and soft ($\gamma < 0$) potentials correspond to $q > 5$ and $2 < q < 5$, respectively, and the Maxwellian potential ($\gamma = 0$) corresponds to $q = 5$.

Motivated by this physical model, throughout this note we assume that the non-negative cross section B takes the form (1.3) with the angular factor b satisfying (1.4), and moreover, except for this introduction, we will only consider the case when

$$\Phi(|v - v_*|) = 1,$$

that is, the Maxwellian molecule type cross section.. In this case, the analysis relies on the good structure of the equation after taking Fourier transform in v variable by means of the Bobylev formula (see (2.8)), together with the precise characterization of the Fourier image of probability measures with α-order moment (see (2.7)).

We remark that the angle θ is the deviation angle, that is, the angle between post- and pre-collisional velocities. The range of θ can be restricted to $[0, \pi/2]$, by

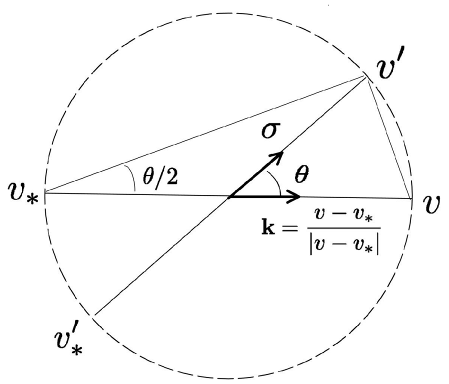

FIGURE 1. post- and pre-collisional velocities

replacing $b(\cos\theta)$ by its "symmetrized" version

$$[b(\cos\theta) + b(\cos(\pi - \theta))]\mathbf{1}_{0 \le \theta \le \pi/2},$$

which is possible due to the invariance of the product $f(v')f(v'_*)$ in the collision operator $Q(f, f)$ under the change of variables $\sigma \to -\sigma$.

Before stating main results in Section 2, we recall the special feature of the non-cutoff Boltzmann collision term $Q(f, f)$ which behaves the fractional power of $-\Delta$. Indeed, we have the following lower and upper estimates for the collision integral operator:

Theorem 1.1 ([1, 3], Coercivity Estimate). *Assume that $f \ge 0, \not\equiv 0$ and $f \in L^1_{\max\{2,|\gamma|\}} \cap L \log L$. If $\gamma > -2s$ then there exists a $C_f > 0$, depending on $\|f\|_{L^1_{\max\{2,|\gamma|\}} \cap L\log L}$ and $1/\|f\|_{L^1}$ at a monotone increase, such that for any smooth function h we have*

$$\frac{1}{C_f} \|\langle D_v \rangle^s h\|^2_{L^2_{\gamma/2}} \le -(Q(f, h), h)_{L^2} + C_f \|h\|^2_{L^2_{\gamma/2}},$$

where for $p \geq 1$ *and* $\beta \in \mathbb{R}$

$$\|f\|_{L^p_\beta} = \Big(\int_{\mathbb{R}^3} |\langle v\rangle^\beta f(v)|^p dv \Big)^{1/p},$$

$$\|f\|_{L\log L} = \int_{\mathbb{R}^3} |f(v)| \log(1+|f(v)|) dv.$$

If $-3 < \gamma \leq -2s$ *then*

$$\frac{1}{C_f} \|\langle D_v\rangle^s h\|^2_{L^2_{\gamma/2}} \leq -\left(Q(f,h),h\right)_{L^2} + C_f \|h\|^2_{L^2_{\gamma/2}} + C_1 \|f\|_{L^{3/(3+\gamma+2s')}_{-\gamma}} \|\langle D_v\rangle^{s'} h\|^2_{L^2_{\gamma/2}},$$

provided that f *belongs to* $L^{3/(3+\gamma+2s')}_{-\gamma}$ *for* $s' \in (0,s)$.

Theorem 1.2 ([2]). *Assume* $\gamma > \max\{-3, -3/2-2s\}$. *Then for any* $m \in (s-1, 2s]$ *and* $\beta \in \mathbb{R}$ *we have*

$$\Big|\Big(Q(f,g),h\Big)_{L^2}\Big| \lesssim \Big(\|f\|_{L^1_{\beta^+ + (\gamma+2s)^+}} + \|f\|_{L^2}\Big) \|g\|_{H^{s+m}_{(\beta+\gamma+2s)^+}} \|h\|_{H^{s-m}_{-\beta}}.$$

If $\gamma + 2s > 0$ *then for any* $m \in [-s, s]$ *and* $\beta \in [-\gamma-2s, 0]$

$$\Big|\Big(Q(f,g),h\Big)_{L^2}\Big| \lesssim \|f\|_{L^1_{\gamma+2s}} \|g\|_{H^{s+m}_{(\beta+\gamma+2s)}} \|h\|_{H^{s-m}_{-\beta}},$$

where $a^+ = \max(a,0)$ *for* $a \in \mathbb{R}$. *Here* $A \lesssim B$ *means that* $A \leq CB$ *for a constant* $C > 0$.

It follows from both theorems that, when $\gamma > 2s$,

$$\frac{1}{C_f} \|h\|^2_{H^s_{\gamma/2}} \leq -\left(Q(f,h),h\right)_{L^2} + C_f \|h\|^2_{L^2_{\gamma/2}},$$

$$\Big|\Big(Q(f,h),h\Big)_{L^2}\Big| \leq C \|f\|_{L^1_{\gamma+2s}} \|h\|^2_{H^s_{\gamma/2+s}}.$$

Therefore, neglecting the weight $\langle v\rangle$, we may say

$$-Q(f,f) \approx C'_f(-\Delta)^s f + \text{lower order terms},$$

and the spatially homogeneous Boltzmann equation (1.1) might behave like the heat equation $(\partial_t - \Delta_v)f = 0$ having the smoothing effect. After a pioneer work [9] by Desvillettes-Wennberg, the smoothing effect of L^1-weak solutions for the spatially homogeneous Boltzmann equation has been almost completely solved as follows:

Theorem 1.3 ([3]). *Let* B *be of the form* (1.3) *with* b *satisfying* (1.4).
1) *Suppose that* $\gamma > \max\{-2s, -1\}$. *Let* f *be a* L^1*-weak solution of the Cauchy problem* (1.1)-(1.2) *with a density initial data* $f_0 \in L^1$. *For* $0 \leq T_0 < T_1$, *if* f *satisfies*

$$|v|^\ell f \in L^\infty([T_0, T_1];\, L^1(\mathbb{R}^3)) \quad \textit{for any} \quad \ell \in \mathbb{N}, \tag{1.5}$$

then $f \in L^\infty([t_0, T_1];\, \mathcal{S}(\mathbb{R}^3))$, *for any* $t_0 \in (T_0, T_1)$.

2) *When* $-1 \geq \gamma > -2s$, *the same conclusion as above holds if we have the following entropy dissipation estimate*

$$\int_{T_0}^{T_1} D(f(t), f(t))dt < \infty, \tag{1.6}$$

where $D(f,f) = -\int Q(f,f)\log f dv \geq 0$.

It is known by Villani[22] that we have L^1-weak solution in the following sense:

Definition 1.4. *Let* $f_0 \geq 0$ *be a function defined on* $\mathbb{R}^3$ *with finite mass, energy and entropy, that is,*

$$\int_{\mathbb{R}^3} f_0(v)[1+|v|^2+\log(1+f_0(v))]dv < +\infty.$$

f *is a weak solution of the Cauchy problem* (1.1)-(1.2), *if it satisfies the following conditions:*

$$\begin{aligned}
&f \geq 0,\ f \in C(\mathbb{R}^+;\mathcal{D}'(\mathbb{R}^3)) \cap L^1([0,T]; L^1_{2+\gamma^+}(\mathbb{R}^3)),\\
&f(0,\cdot) = f_0(\cdot),\\
&\int_{\mathbb{R}^3} f(t,v)\psi(v)dv = \int_{\mathbb{R}^3} f_0(v)\psi(v)dv \ \ for\ \psi = 1, v_1, v_2, v_3, |v|^2;\\
&f(t,\cdot) \in L\log L,\ \ \int_{\mathbb{R}^3} f(t,v)\log f(t,v)dv \leq \int_{\mathbb{R}^3} f_0\log f_0 dv,\ \ \forall t \geq 0,\\
&\int_{\mathbb{R}^3} f(t,v)\varphi(t,v)dv - \int_{\mathbb{R}^3} f_0(v)\varphi(0,v)dv\\
&-\int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau\varphi(\tau,v)dv = \int_0^t d\tau\int_{\mathbb{R}^3} Q(f,f)(\tau,v)\varphi(\tau,v)dv
\end{aligned}$$

where $\varphi \in C^1(\mathbb{R}^+; C_0^\infty(\mathbb{R}^3))$.

It should be noted that the weak solutions constructed by [22] satisfy the moment gain property (1.5) for any $T_0 > 0$ if $\gamma > 0$, though it is known by [12] that the uniqueness does not holds in the case $\gamma > 0$ if the energy conservation law is removed from the definition of the weak solution. His weak solutions satisfy (1.6) if $\gamma \geq -2$. Without the condition (1.5) we have another smoothing effect of weak solutions

$$f \in L^\infty([t_0,\infty);\ H^\infty(\mathbb{R}^3)),$$

for any $t_0 > 0$, if either $\gamma = 0$ or $\gamma > 0$ and $0 < s < 1/2$ (see [14] and Theorem 5.2 of [3]).

By Theorem 1.3, the smoothing effect for L^1-weak solutions holds. However, if one considers the measure-valued solutions, the smoothing effect does not always occur, since the single Dirac mass is its stationary singular solution. The Boltzmann equation is non-linear and different from the heat equation. To end this introduction, I would like to mention the following conjecture posed personally by Cedric Villani [23], which is true for the Maxwellian molecule type cross section (see Theorem 2.1 below).

Conjecture 1.5. *Any weak solution to the Cauchy problem* (1.1)-(1.2) *with measure initial datum except a single Dirac mass acquires* C^∞ *regularity in the velocity variable in any positive time.*

2. Main results

We start with the review on a historical work given by the probabilist H. Tanaka[20]. Denote by $P_\alpha(\mathbb{R}^d), 0 < \alpha \le 2$, the set of all probability measures F on $\mathbb{R}^d$ such that

$$(2.7) \qquad \int_{\mathbb{R}^d} |v|^\alpha dF(v) < \infty\,, \quad \int_{\mathbb{R}^d} v_j dF(v) = 0, j = 1, \cdots, d, \ \text{ if } \alpha > 1.$$

In $P_2(\mathbb{R}^d)$ we define the Wasserstein distance as follows: For $F, G \in P_2(\mathbb{R}^d)$,

$$W_2(F, G) = \Big(\inf_{L \in \Pi(F,G)} \int |v - w|^2 dL(v, w) \Big)^{1/2} \,,$$

where $\Pi(F, G)$ denotes the set of all probability distributions L in $P_2(\mathbb{R}^d \times \mathbb{R}^d)$ having F and G as marginal distributions.

Tanaka Theorem [Existence, Uniqueness and Asymptotic Behavior]. *If an initial datum $F_0 \in P_2(\mathbb{R}^3)$ then there exists a unique solution $F_t \in P_2(\mathbb{R}^3)$ to the Cauchy problem (1.1)-(1.2) and we have*

$$W_2(F_t, \omega) \to 0\,, \ \ t \to \infty,$$

where $\omega(v) = (2\pi E)^{-3/2} e^{-|v|^2/2E}$ for $3E = \int |v|^2 dF_0$,

This result has been analytically treated by Toscani and his coauthors[8, 10, 19, 21], based on the Toscani metric

$$\|\varphi - \tilde\varphi\|_2 = \sup_{0 \ne \xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \tilde\varphi(\xi)|}{|\xi|^2}, \quad \varphi, \tilde\varphi \text{ are Fourier images of } F, G \in P_2(\mathbb{R}^d),$$

and the Fourier transform representation of the Boltzmann equation given by Bobylev [4];

Bobylev formula. If $\psi(t, \xi)$ and $\psi_0(\xi)$ are Fourier transforms of $f(t, v)$ and $f_0(v)$, respectively, then the Cauchy problem (1.1)-(1.2) is reduced to

$$(2.8) \qquad \begin{cases} \partial_t \psi(t, \xi) = \displaystyle\int_{\mathbb{S}^2} b\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \Big(\psi(t, \xi^+)\psi(t, \xi^-) - \psi(t, \xi)\psi(t, 0)\Big) d\sigma, \\[2ex] \psi(0, \xi) = \psi_0(\xi), \ \text{ where } \xi^\pm = \dfrac{\xi}{2} \pm \dfrac{|\xi|}{2}\sigma\,. \end{cases}$$

It is known (see Theorem 1 of [21]) that Wasserstein distance and Toscani metric are equivalent on the subset of $P_2(\mathbb{R}^d)$ with a fixed energy ($\int |v|^2 dF(v) = Ed$). Recently Cannone-Karch [6] extended the existence and uniqueness of solution for the initial datum in P_α with $0 < \alpha < 2$, motivated by the self-similar solution (with infinite energy) given by Bobylev-Cercignani[5]. Following [11, 6], we call the Fourier transform of a probability measure $F \in P_0(\mathbb{R}^d)$, that is,

$$\varphi(\xi) = \hat f(\xi) = \mathcal{F}(F)(\xi) = \int_{\mathbb{R}^d} e^{-iv\cdot\xi} dF(v)\,,$$

a characteristic function. Denote the set of all characcteristic functions by $\mathcal{K}$. Inspired by a series of works by Toscani and his co-authors, Cannone-Karch[6] defined a subspace $\mathcal{K}^\alpha$ for $\alpha \ge 0$ as follows:

$$(2.9) \qquad \mathcal{K}^\alpha = \{\varphi \in \mathcal{K};\ \|\varphi - 1\|_\alpha < \infty\}\,,$$

where

$$\|\varphi - 1\|_\alpha = \sup_{\xi\in\mathbb{R}^d} \frac{|\varphi(\xi)-1|}{|\xi|^\alpha}. \tag{2.10}$$

The space $\mathcal{K}^\alpha$ endowed with the distance

$$\|\varphi - \tilde\varphi\|_\alpha = \sup_{\xi\in\mathbb{R}^d} \frac{|\varphi(\xi)-\tilde\varphi(\xi)|}{|\xi|^\alpha} \tag{2.11}$$

is a complete metric space (see Proposition 3.10 of [6]). It follows that $\mathcal{K}^\alpha = \{1\}$ for all $\alpha > 2$ and the following embeddings (Lemma 3.12 of [6]) hold

$$\{1\} \subset \mathcal{K}^\alpha \subset \mathcal{K}^\beta \subset \mathcal{K}^0 = \mathcal{K} \quad \text{for all } 2 \ge \alpha \ge \beta \ge 0.$$

However, even though the inclusion $\mathcal{F}(P_\alpha(\mathbb{R}^d)) \subset \mathcal{K}^\alpha$ holds (see Lemma 3.15 of [6]), the space $\mathcal{K}^\alpha$ is strictly larger than $\mathcal{F}(P_\alpha(\mathbb{R}^d))$ for $\alpha \in (0,2)$, in other word, $\mathcal{F}^{-1}(\mathcal{K}^\alpha) \supsetneq P_\alpha(\mathbb{R}^d)$. Indeed, for each $\alpha \in (0,2)$, $\varphi_\alpha(\xi) = e^{-|\xi|^\alpha}$ belongs to $\mathcal{K}^\alpha$, which is the Fourier transform of the probability density $P_\alpha(v)$ of α-stable Lévy process. It is known (see Remark 3.16 of [6]) that that $0 < P_\alpha(v) \le C(1+|v|)^{-(\alpha+d)}$ and moreover

$$\frac{P_\alpha(v)}{|v|^{\alpha+d}} \to c_0 \ \text{ when } |v| \to \infty,$$

where $c_0 = \alpha 2^{\alpha-1}\pi^{-(d+2)/2}\sin\left(\frac{\alpha\pi}{2}\right)\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)$.

On the other hand, we remark that $\mathcal{F}(P_2(\mathbb{R}^d)) = \mathcal{K}^2$. Indeed, this can be proved by contradiction. If there exists a $\varphi(\xi) \in \mathcal{K}^2$ such that $F = \mathcal{F}^{-1}(\varphi) \notin P_2$, then we may assume there exist $\omega_0 \in \mathbb{S}^{d-1}$and $A > 0$ such that

$$\int_{\{|\frac{v}{|v|}-\omega_0|<10^{-10}\}\cap\{|v|\le A\}} |v|^2 dF(v) \ge 100\|1-\varphi\|_2\,,$$

from which we have a contradiction because

$$\begin{aligned}
\|1-\varphi\|_2 &\ge \sup_\xi \frac{\mathrm{Re}(1-\varphi(\xi))}{|\xi|^2} \\
&\ge 2\int_{\{|\frac{v}{|v|}-\omega_0|<10^{-10}\}\cap\{|v|\le A\}} \frac{\sin^2\left\{\frac{|v||\xi|}{2}\left(\frac{v}{|v|}\cdot\frac{\xi}{|\xi|}\right)\right\}}{|v|^2|\xi|^2}|v|^2 dF(v) \ \text{ for } \frac{\xi}{|\xi|} = \omega_0,\ |\xi| = \frac{\pi}{A} \\
&\ge \frac{2}{\pi^2}\int_{\{|\frac{v}{|v|}-\omega_0|<10^{-10}\}\cap\{|v|\le A\}} \left(\frac{v}{|v|}\cdot\omega_0\right)^2|v|^2 dF(v) > 50\|1-\varphi\|_2,
\end{aligned}$$

by using

$$\sin z \ge \frac{2z}{\pi} \ \text{ when } 0 \le z \le \frac{\pi}{2}.$$

Since $\mathcal{K}^\alpha \supsetneq \mathcal{F}(P_\alpha(\mathbb{R}^3))$ for $\alpha \in (0,2)$, we introduce $\tilde P_\alpha = \mathcal{F}^{-1}(\mathcal{K}^\alpha)$ endowed also with distance (2.11). We are now ready to state our first result.

Theorem 2.1. *Assume that $b(\cos\theta)$ satisfies (1.4) with $0 < s < 1$ and let $\alpha \in (2s, 2]$. If an initial datum $F_0 \in P_\alpha(\mathbb{R}^3)$ is not a single Dirac mass then there exists a unique solution $f(t,v)$ in $C([0,\infty), \tilde P_\alpha(\mathbb{R}^3))$ to the Cauchy problem (1.1)-(1.2) such that*

$$f(t,v) \in C((0,\infty), L^1_\alpha(\mathbb{R}^3)\cap H^\infty(\mathbb{R}^3)).$$

Remark 2.2. *The existence and uniqueness of solution for an initial datum in $\tilde{P}_\alpha(\mathbb{R}^3)$ was studied by* [6], *in mild singularity case* $0 < s < 1/2$, *and extended by* [13] *to the strong singularity case* $1/2 \le s < 1$. *The* H^∞ *smoothing effect of the solution was proved in* [17, 15]. *It was shown in* [15] *that* $f(t,v) \in C((0,\infty), L^1_\alpha(\mathbb{R}^3))$ *in the case where* $\alpha \in (0,1) \cup (1,2)$. *The restriction of* $\alpha = 1$ *has been removed recently in* [16].

The existence and smoothness are discussed for the Cauchy problem (2.8) in the Fourier space $\mathbb{R}^3_\xi$. To go back the base space $\mathbb{R}^3_v$, we need to fill the gap, $\mathcal{F}^{-1}(\mathcal{K}^\alpha) \supsetneq P_\alpha(\mathbb{R}^3)$. To this end, we introduce a new classification on the characteristic functions as follows. Set

$$\mathcal{M}^\alpha = \{\varphi \in \mathcal{K};\ \|\varphi - 1\|_{\mathcal{M}^\alpha} < \infty\}, \quad \alpha \in (0,2),$$

where

$$\|\varphi - 1\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^d} \frac{|\varphi(\xi) - 1|}{|\xi|^{d+\alpha}} d\xi .$$

For $\varphi, \tilde{\varphi} \in \mathcal{M}^\alpha$, put

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^d} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^{d+\alpha}} d\xi ,$$

and we introduce the distance

$$dis_\alpha(\varphi, \tilde{\varphi}) = \|\varphi - \tilde{\varphi}\|_{\mathcal{M}^\alpha} + \|\varphi - \tilde{\varphi}\|_\alpha . \tag{2.12}$$

Then we have the following complete characterization except for $\alpha = 1$;

Proposition 2.3 ([15]). *If* $0 < \alpha < 2$, *then* $\mathcal{M}^\alpha$ *is a complete metric space endowed with the distance* $dis_\alpha(\varphi, \tilde{\varphi})$. *Moreover, we have*

$$\begin{aligned} &\mathcal{K}^\beta \subset \mathcal{M}^\alpha \ \textit{ if } \alpha < \beta \textit{ and } \alpha \in (0,2), \\ &\mathcal{M}^\alpha \subset \mathcal{F}(P_\alpha(\mathbb{R}^d)) \,(\ \subsetneq \mathcal{K}^\alpha\ )\ \textit{ for } \alpha \in (0,2), \\ &\mathcal{M}^\alpha = \mathcal{F}(P_\alpha(\mathbb{R}^d)),\ \textit{furthermore if } \alpha \neq 1 . \end{aligned}$$

Proposition 2.4 (Existence and Stability in Fourier space). *Assume*

$$\exists \alpha_0 \in (0,2] \quad s.t. \quad \theta^{\alpha_0} b(\cos\theta)\sin\theta \in L^1((0,\pi/2]). \tag{2.13}$$

If the initial datum φ_0 *belongs to* $\mathcal{M}^\alpha$ ($\alpha \in [\alpha_0, 2)$), *then there exists a unique classical solution* $\varphi(t,\xi) \in C([0,\infty), \mathcal{M}^\alpha)$ *to the Cauchy problem* (2.8) *such that*

$$dis_\alpha(\varphi(t,\cdot), \varphi(s,\cdot)) \lesssim |t-s| e^{\lambda_\alpha \max\{t,s\}} dis_\alpha(\varphi_0, 1) .$$

Here

$$\lambda_\alpha = 2\pi \int_0^{\pi/2} b(\cos\theta) \left(\cos^\alpha \frac{\theta}{2} + \sin^\alpha \frac{\theta}{2} - 1\right) \sin\theta d\theta > 0 . \tag{2.14}$$

Furthermore, if $\psi(t,\xi), \varphi(t,\xi) \in C([0,\infty), \mathcal{M}^\alpha)$ *are two solutions to the Cauchy problem* (2.8) *with initial data* $\psi_0, \varphi_0 \in \mathcal{M}^\alpha$, *respectively, then for any* $t > 0$, *the following two stability estimates hold*

$$\begin{aligned} &||\psi(t) - \varphi(t)||_{\mathcal{M}^\alpha} \le e^{\lambda_\alpha t} ||\psi_0 - \varphi_0||_{\mathcal{M}^\alpha} , \\ &||\psi(t) - \varphi(t)||_\alpha \le e^{\lambda_\alpha t} ||\psi_0 - \varphi_0||_\alpha . \end{aligned} \tag{2.15}$$

The above assumption (2.13) holds for the function b satisfying (1.4) if $\alpha_0 > 2s$. The existence and uniqueness parts in the proof of Theorem 2.1 can be done by means of above two propositions except for the case $\alpha = 1$. In order to overcome the exceptional case $\alpha = 1$ we need to introduce another space

$$\widetilde{\mathcal{M}}^\alpha = \{\varphi \in \mathcal{K};\ \|Re\varphi - 1\|_{\mathcal{M}^\alpha} + \|\varphi - 1\|_\alpha < \infty\}, \quad \alpha \in (0,2), \tag{2.16}$$

where $Re\varphi$ stands for the real part of $\varphi(\xi)$. For $\varphi, \tilde{\varphi} \in \widetilde{\mathcal{M}}^\alpha$, put

$$\|\varphi - \tilde{\varphi}\|_{\widetilde{\mathcal{M}}^\alpha} = \int_{\mathbb{R}^d} \frac{|Re\varphi(\xi) - Re\tilde{\varphi}(\xi)|}{|\xi|^{d+\alpha}} d\xi,$$

and, for any $0 < \beta < \alpha < 2, 0 < \varepsilon < 1$, we introduce the distance in $\widetilde{\mathcal{M}}^\alpha$ as

$$dis_{\alpha,\beta,\varepsilon}(\varphi, \tilde{\varphi}) = \|\varphi - \tilde{\varphi}\|_{\widetilde{\mathcal{M}}^\alpha} + \|\varphi - \tilde{\varphi}\|_\beta + \|\varphi - \tilde{\varphi}\|_\beta^\varepsilon. \tag{2.17}$$

Then $\widetilde{\mathcal{M}}^\alpha$ is a complete metric space endowed with this distance and we have $\mathcal{M}^1 \subsetneq \widetilde{\mathcal{M}}^1 = \mathcal{F}(P_1(\mathbb{R}^d))$ and $\widetilde{\mathcal{M}}^\alpha = \mathcal{M}^\alpha$ if $\alpha \in (0,1) \cup (1,2)$ (see the detail in [16]). The smoothing effect part in the proof of Theorem 2.1 essentially relies on the following time degenerate coercivity estimate, instead of Theorem 1.1;

Lemma 2.5 (Time degenerate coercivity). *If $\psi(t,\xi)$ is the Fourier transform of the solution $f(t,v)$ then there exist $T > 0$ and $C > 0$ such that for $t \in [0,T]$*

$$t\int_{\mathbb{R}^3} \langle\xi\rangle^s |h(\xi)|^2 d\xi \le C\Big(\int_{\mathbb{R}^3}\Big(\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(1 - |\psi(t,\xi^-)|)d\sigma\Big)|h(\xi)|^2 d\xi \\ + \int_{\mathbb{R}^3} |h(\xi)|^2 d\xi\Big), \quad \textit{for } \forall h \in L^2_s.$$

The key of its proof is to show the following; $\exists R > 1$ and $\exists C > 0$ such that

$$t|\xi|^{2s} \le C\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(1 - |\psi(t,\xi^-)|)d\sigma \text{ if } |\xi| \ge R,$$

which is derived, in the most crucial case, again from the fact that $\psi(t,\xi^-)$ is the solution to (2.8) (see Section 2.2 of [17]).

In order to state our second result, we recall the self-similar solution given by Bobylev-Cercignani [5]. For $\alpha \in (2s,2)$, denote $\mu_\alpha = \frac{\lambda_\alpha}{\alpha}$. For each $K > 0$, there exists a radially symmetric nonnegative function

$$\Psi_{\alpha,K} \in L^1_\beta(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3), \ \ (\forall\beta < \alpha), \tag{2.18}$$

satisfying

$$\hat{\Psi}_{\alpha,K} \in \mathcal{K}^\alpha, \quad \lim_{|\xi|\to 0} \frac{1 - \hat{\Psi}_{\alpha,K}(\xi)}{|\xi|^\alpha} = K,$$

such that

$$f_{\alpha,K}(t,v) = e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})$$

is a solution of the Boltzmann equation. As pointed out in Cannone-Karch[7], Bobylev-Cercignani constructed their self-similar solution as a power series

$$\hat{\Psi}_{\alpha,K}(\xi) = 1 - K|\xi|^\alpha + a_2|\xi|^{2\alpha} + \cdots$$

in the Fourier space, and conjectured that it belongs to L^1 function of the base space. Namely, (2.18) is a direct consequence of Theorem 2.1 under the non-cutoff assumption (1.4).

As stated in Tanaka Theorem, it is a common sense in the kinetic people that the solution to the Boltzmann equation tends to an equilibrium, that is, a Maxwellian when time tends to infinity. This has been proved in various settings when the initial energy is finite. However, when the initial energy is infinite, the time asymptotic state is no longer described by a Maxwellian, but a self-simiar solution $\Psi_{\alpha,K}$ under suitable conditions.

Theorem 2.6 ([18]). *For $\alpha \in (\max\{1,2s\},2)$, if an initial datum $f_0 \in \tilde{P}_\alpha(\mathbb{R}^3)$ satisfies*

$$\int \Big(f_0(v) - \Psi_{\alpha,K}(v)\Big)|v|^2 dv = 0\,,$$
$$\exists\delta > 0\,; \int \Big|f_0(v) - \Psi_{\alpha,K}(v)\Big||v|^{2+\delta} dv < \infty\,,$$

then there exists a $c_0 > 0$ such that for any $\beta \in \mathbb{Z}^3_+$

$$\sup_v \Big|\partial_v^\beta \Big(f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})\Big)\Big| \lesssim e^{-c_0 t}\,,$$
$$\int |f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})|dv \lesssim e^{-c_0 t}\,,$$

provided that

$$c_0 > 3\mu_\alpha. \tag{2.19}$$

Remark 2.7. *Since $\mu_\alpha \to 0+$ as $\alpha \to 2-0$, the condition (2.19) holds when α is close to 2.*

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