数理解析研究所講究録
第 1962 巻 2015 年 45-49

Dependency of polarity on the drift of Brownian motion of a compact manifold

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CONTENTS

1. INTRODUCTION

It is well-known that the Brownian motion on a Riemannian manifold M will not hit a subset Σ of M if and only if the capacity related to the Brownian motion of Σ is zero [2]. However, the situation is not clear for a Brownian motion with a drift; in particular, it would be interesting to know if the capacity of Σ associated to the Brownian motion with a drift being zero is independent of the drift. In this note, we will study this problem. A lower bounded non-symmetric semi Dirichlet form generates a non-symmetric Markov process [3, 5], and this relationship will be the foundation for our study. The main aim of this note is to answer the following two questions:

- Does the operator $\Delta+\langle F,\nabla\cdot\rangle+V$, where Δ is the sub-Laplacian, F is a one-form, and V is a non-negative continuous function, generate a lower bounded semi Dirichlet form?
- Find a characterisation of the capacity for a lower bounded semi Dirichlet form in terms of that for the Dirichlet integral.

The structure of the note is the following. Section 2 will be devoted for the first question and the second question will be studied in Section 3.

2. CLOSED FORMS

Let (M, g) be a compact smooth Riemannian manifold without boundary. Let $\sigma>0$ be a positive continuous function on M. We consider the weighted space, $L^2=L^2(M, dm)$, where $dm=\sigma dv_g$ and v_g is the Riemannian volume associated with the metric g. Let $F\in\Gamma(TM^*)$ be a smooth 1-form and $V\in C(M)$, the space of continuous functions on M, with $V\geq 0$. Suppose that TM admits a system of Hörmander vector fields $\{X_i\}$ and the $X_x\subset T_xM$ is the subspace spanned by $\{X_i\}$ at point $x\in M$. Let π be the orthogonal projection $T_xM\to X_x$. The sub-gradient ∇ is then defined pointwise as $\nabla u=\pi\circ\mathrm{grad}(u)$, where grad is the gradient operator associated to g. The energy form $\mathcal{E}$ is

$$\mathcal{E}(u,v)=\int_M(g(\nabla u,\nabla v)+\langle F,\nabla u\rangle v+Vuv)\,dm,\quad u,v\in C^\infty(M)$$

where $\langle\cdot,\cdot\rangle$ is the pairing between cotangent and tangent vector spaces. We will denote $\mathcal{E}(u)=\mathcal{E}(u,u)$ and $\mathcal{E}_\alpha(u,v)=\mathcal{E}(u,v)+\alpha(u,v)$ for some $\alpha>0$, where $(u,v)=\int_M uv\,dm$ and $\|u\|=(u,u)^{1/2}$, for short. The symbol $|\cdot|$ stands for the pointwise norm. The weighted divergence, which is the negative of the formal joint of ∇, will be denoted by div. We will employ $W^{1,2}=$

$\{u \in L^2 \mid \nabla u \in L^2(TM, dm)\}$. Let us recall some basic definitions regarding with semi-Dirichlet forms stated in the current setting.

Definition 1 (closed forms). A quadratic form Q defined on a dense subspace $D(Q) \subset L^2$ will be called *closed* on L^2 provided the following three conditions:

($\mathcal{E}$.1) Q is lower bounded: There exists $\alpha_0 \geq 0$ such that

$$Q_{\alpha_0}(u) \geq 0, \quad \forall u \in D(Q).$$

($\mathcal{E}$.2) Sector condition: There exists $K \geq 1$ such that

$$|Q(u,v)|^2 \leq K Q_{\alpha_0}(u)\mathcal{E}_{\alpha_0}(v), \quad \forall u, v \in D(Q).$$

($\mathcal{E}$.3) $D(Q)$ is a Hilbert space with respect to the inner product

$$Q_\alpha^{(s)}(u,v) = \frac{1}{2}\left(Q_{\alpha_0}(u,v) + Q_{\alpha_0}(v,u)\right), \quad \forall \alpha \geq \alpha_0.$$

Theorem 1. *The form* $(\mathcal{E}, W^{1,2})$ *is a closed form.*

Proof. The proof follows from Propositions 1 and 2. □

Proposition 1. *The energy* $(\mathcal{E}_\alpha, C^\infty(M))$ *is closable in* L^2 *whenever*

$$\alpha > \sup\left(\frac{1}{2}(divF) - V\right). \tag{1}$$

Proof. We must show:

$$\lim_{m,n\to\infty} \mathcal{E}_\alpha(u_n - u_m) = 0, \quad \lim_{n\to\infty} \|u_n\| = 0 \implies \lim_{n\to\infty} \mathcal{E}_\alpha(u_n) = 0. \tag{2}$$

Let use denote the sub-Dirichlet integral by $\mathcal{D}(u) = \|\nabla u\|^2$. By Green's formula,

$$\mathcal{E}_\alpha(u) = \mathcal{D}(u) + \int_M \frac{1}{2}\langle F, \nabla(u^2)\rangle + (\alpha + V)u^2\, dm = \mathcal{D}(u) + \int_M \left(-\frac{1}{2}(\text{div}F) + \alpha + V\right) u^2\, dm.$$

Letting α so that $0 < \lambda_1 = \inf(-\frac{1}{2}(\text{div}F) + \alpha + V)$, we get

$$\lambda_1 \mathcal{D}_1(u) \leq \mathcal{E}_\alpha(u) \leq \lambda_2 \mathcal{D}_1(u), \tag{3}$$

where $\lambda_2 = \sup(-\frac{1}{2}(\text{div}F) + \alpha + V)$. The assertion will follow from the fact that $\mathcal{D}$ is closable, which is well known and proved for the sake of completeness: As (∇u_n) is a Cauchy sequence in $L^2(TM, dm)$, we denote its limit by X. For any smooth vector field Y,

$$\int_M g(X,Y)\, dm = \lim_{n\to\infty} \int_M g(\nabla u_n, Y)\, dm = -\lim_{n\to\infty} \int_M u_n \text{div}Y\, dm = 0.$$

□

Proposition 2. *The energy* $(\mathcal{E}_\alpha, C^\infty(M))$ *satisfies the sector condition, that is, there exists a constant* $K \geq 1$ *such that*

$$|\mathcal{E}(u,v)|^2 \leq K\mathcal{E}_\alpha(u)\mathcal{E}_\alpha(v), \quad \forall u, v \in C^\infty(M). \tag{4}$$

Proof. Let $u, v \in C^\infty(M)$. Denoting $C = \sup(|F| + |V|)$ and $C' = 2(1+2C^2)$,

$$\begin{aligned}
&|\mathcal{E}(u,v)|^2 \\
=&\left|\int_M (g(\nabla u, \nabla v) + (\langle F, \nabla u\rangle + Vu)v)\,dm\right|^2 \\
\leq&\left|\int_M (|\nabla u||\nabla v| + (|F||\nabla u| + |Vu|)|v|)\,dm\right|^2 \\
\leq&\left|\int_M (|\nabla u||\nabla v| + C(|\nabla u| + |u|)|v|)\,dm\right|^2 \\
\leq&2\left(\left(\int_M |\nabla u||\nabla v|\,dm\right)^2 + \left(C\int_M (|\nabla u| + |u|)|v| dm\right)^2\right) \\
\leq&2\left(\left(\int_M |\nabla u||\nabla v|\,dm\right)^2 + 2\left(C\int_M |\nabla u||v|\,dm\right)^2 + 2\left(C\int_M |u||v|\,dm\right)^2\right) \\
\leq&C'\left(\left(\int_M |\nabla u||\nabla v|\,dm\right)^2 + \left(\int_M |\nabla u||v|\,dm\right)^2 + \left(\int_M |u||v|\,dm\right)^2\right).
\end{aligned}$$

By the Cauchy Schwarz inequality,

$$|\mathcal{E}(u,v)|^2 \leq C'\left(\|\nabla u\|^2\|\nabla v\|^2 + \left(\|\nabla u\|^2 + \|u\|^2\right)\|v\|^2\right) \tag{5}$$

On the other hand, for any $a > 0$,

$$\begin{aligned}
\mathcal{E}_\alpha(u) &= \|\nabla u\|^2 + \int_M \frac{1}{2}\langle F, \nabla(u^2)\rangle + (\alpha + V)u^2\,dm \\
&\geq \|\nabla u\|^2 - \int_M |F||u||\nabla u| + (\alpha+V)u^2\,dm \\
&\geq \|\nabla u\|^2 - 2\left(\frac{1}{a}\int_M |F|^2|u|^2\,dm + a\int_M |\nabla u|^2\,dm\right) + \int_M (\alpha+V)u^2\,dm \\
&= (1-2a)\|\nabla u\|^2 + \int_M \left(-\frac{2}{a}|F|^2 + \alpha + V\right)u^2\,dm \\
&= \frac{1}{2}\|\nabla u\|^2 + \int_M \left(-8|F|^2 + \alpha + V\right)u^2\,dm
\end{aligned}$$

by letting $a = 1/4$. Setting $\beta \leq \sup(-8|F|^2 + \alpha + V)$, we have

$$\begin{aligned}
\mathcal{E}_\alpha(u)\mathcal{E}_\alpha(v) &\geq \left(\frac{1}{2}\|\nabla u\|^2 + \beta\int_M u^2\,dm\right)\left(\frac{1}{2}\|\nabla v\|^2 + \beta\int_M v^2\,dm\right) \\
&\geq \frac{1}{4}\|\nabla u\|^2\|\nabla v\|^2 + \beta\left(\frac{1}{2}\|\nabla u\|^2 + \beta\|u\|^2\,dm\right)\|v\|^2.
\end{aligned}$$

This together with (5), and by the fact that we may take β arbitrary large, we get the desired conclusion. □

By a standard semigroup theory, Theorem 1 yields

Corollary 1. *There exists a strongly semigroup $\{T_t\}_{t\geq 0}$ on L^2 such that $\|T_t\| \leq e^{\alpha_0}$ whose resolvent $G_\alpha u = \int_0^\infty e^{-\alpha t}T_t u\,dt$ with $\alpha > \alpha_0$ satisfying*

$$\mathcal{E}_\alpha(G_\alpha u, v) = (u, v), \quad \forall u \in L^2,\ v \in \mathcal{F}.$$

Definition 2 (Dirichlet forms). A closed form $(Q, D(Q))$ is called a lower-bounded semi-Dirichlet form if it satisfies

$$u \in D(Q),\ a \geq 0 \implies v = u \wedge a \in D(Q),\ Q(v, u-v) \geq 0. \tag{6}$$

Theorem 2. *The form $(\mathcal{E}, \mathcal{F})$ is a lower-bounded semi-Dirichlet form.*

Proof. We need to prove (6). The fact that $u \wedge a \in W^{1,2}$ whenever $u \in W^{1,2}$ and $a \in \mathbb{R}$ can be proved as in the Euclidean case (see, e.g., [2]). It suffices to prove the second statement only for $u \in C^\infty(M)$ by the density argument. Setting $D_+ = \{u > a\}$ and $D_- = \{u < a\}$, we note: $u - u\wedge a = 0$ on D_- and $u \wedge a = a$ on D_+. Taking into account that the measures of the boundaries of these sets are 0,

$$\begin{aligned}&\mathcal{E}(u \wedge a, u - u \wedge a)\\ &= \int_M g(\nabla(u \wedge a), \nabla(u - u \wedge a))\, dm\\ &+ \int_M \langle F, \nabla(u \wedge a)\rangle (u - u \wedge a)\, dm + \int_M V(u \wedge a)(u - u \wedge a)\, dm = \int_{D_+} Va(a-u)\, dm \geq 0.\end{aligned}$$

□

An important consequence of Theorem 2 is

Corollary 2 (see, e.g., Theorem 3.3.4 [5]). *There exists a Hunt process whose resolvent is a q.e. modification of G_α in L^∞.*

Remark 1. *I. Shigekawa [6] obtained a condition for F so that the operator $\Delta + \langle F, \nabla\cdot\rangle$ without the sector condition generates a Markovian semigroup on a complete Riemannian manifold. We will need the sector condition for the existence of equilibrium potential in the next section.*

3. Capacity associated to $\mathcal{E}_\alpha$

Hereafter, $\alpha_0 > 0$ is the constant which was specified in the previous section and $\alpha > \alpha_0$. For an open set $A \subset M$, set a subset $\mathcal{L}_A \subset \mathcal{F}$ by

$$\mathcal{L}_A = \{u \in \mathcal{F} \mid u|_A \geq 1 \ m\text{-a.e.}\}.$$

Clearly, $\mathcal{L}_A$ is a non-empty closed convex set. For arbitrary fixed $u \in \mathcal{F}$, set:

$$J(w) = \mathcal{E}_\alpha(u, w), \quad w \in \mathcal{F}.$$

Since J is a continuous linear functional on $\mathcal{F}$, we may apply Stampaccia's theorem and find a unique $v \in \mathcal{F}$ such that

$$\mathcal{E}_\alpha(v, w - v) \geq J(w - v), \quad \forall w \in \mathcal{F}.$$

This determines a projection $\pi : \mathcal{F} \to \mathcal{L}_A$ by $\pi(u) = v$. The element $\pi(0)$ is called the equilibrium potential of A denoted by e_A. It follows that

$$\mathcal{E}_\alpha(e_A) \leq \mathcal{E}_\alpha(e_A, w) \leq K\mathcal{E}_\alpha(e_A)^{1/2}\mathcal{E}_\alpha(w)^{1/2}, \quad \forall w \in \mathcal{F}. \tag{7}$$

Changing J to $\hat{J}$, where $\hat{J}(w) = \mathcal{E}_\alpha(w, u)$, we find the co-equilibrium potential of A in $\mathcal{L}_A$ denoted by $\hat{e}_A$ and satisfying

$$\mathcal{E}_\alpha(\hat{e}_A) \leq K^2\mathcal{E}_\alpha(w), \quad \forall w \in \mathcal{F}.$$

Moreover, (see, e.g., Lemma 2.1.1 in [5]),

$$e_A|_A = 1, \ m\text{-a.e.}$$

and for $u \in \mathcal{F}$ such that $u|_A = 1$ m-a.e.,

$$\mathcal{E}_\alpha(e_A, u) = \mathcal{E}_\alpha(e_A), \quad \mathcal{E}_\alpha(u, \hat{e}_A) = \mathcal{E}_\alpha(e_A, \hat{e}_A)$$

The (α-)capacity of A is defined as

$$\mathrm{Cap}(A) = \mathcal{E}_\alpha(e_A, \hat{e}_A).$$

By (3) and (7),

$$\lambda_1\mathcal{D}(e_A) \leq \mathcal{E}_\alpha(e_A) \leq \mathrm{Cap}(A) \leq K^2\mathcal{E}_\alpha(e_A) \leq K^2\lambda_2\mathcal{D}(e_A). \tag{8}$$

The capacity of an arbitrary set $B \subset M$ is defined as

$$\mathrm{Cap}(B) = \inf_{B \subset A}\{\mathrm{Cap}(A) \mid A \text{ is open an set in } M\}.$$

Now we answer the second question in

Theorem 3. *For any set $B \subset M$,*

$$\mathrm{Cap}(B) = 0 \iff \mathrm{Cap}_{\mathcal{D}}(B) = 0,$$

where $\mathrm{Cap}_{\mathcal{D}}(B)$ is the capacity of B associated to $\mathcal{D}$.

Proof. First, let us suppose that $\mathrm{Cap}(B) = 0$. Then (8) implies that

$$0 \le \mathrm{Cap}_{\mathcal{D}}(B) \le \liminf_{n\to\infty} \mathcal{D}(e_{A_n}) \le \lambda_1^{-1} \liminf_{n\to\infty} \mathcal{E}_\alpha(e_{A_n}) \le \lambda_1^{-1} \lim_{n\to\infty} \mathrm{Cap}(A_n) = 0,$$

where (A_n) is a sequence of open sets in M approximating $\mathrm{Cap}(B)$.

Next, let us suppose that $\mathrm{Cap}_{\mathcal{D}}(B) = 0$ and let (A_n) be its approximation sequence. Denoting by $\eta_n \in \mathcal{L}_{A_n}$ the equilibrium potential of A_n associated with $\mathcal{D}$,

$$\begin{aligned} 0 &\le \mathrm{Cap}(B) \le \liminf_{n\to\infty} \mathrm{Cap}(A_n) \\ &= \liminf_{n\to\infty} \mathcal{E}_\alpha(\hat{e}_{A_n}) \le \liminf_{n\to\infty} \mathcal{E}_\alpha(\eta_n) \le \lambda_2 \lim_{n\to\infty} \mathcal{D}(\eta_n) = \lambda_2 \mathrm{Cap}_{\mathcal{D}}(B) = 0. \end{aligned}$$

Therefore, we have the assertion. □

Remark 2. *In closing this note, let us mention two related questions to our study.*

- *As we have studied in this note, it turned out that the capacity of a closed set of a* compact *manifold being 0 is independent of drifts. Clearly, the situation will be different for a non-compact Riemannian manifold. In particular, it would be interesting to extend the theory of Cauchy boundary and polity of a singular set of a singular manifold (see, e.g.,* [4]*) to non-symmetric case.*
- *It is known that a capacity of a symmetric Dirichlet form is related with a quantum mechanical tunnelling phenomena* [1]. *Can one formulate a non-symmetric quantum mechanical tunnelling, and if yes, how is it related with the capacity of a non-symmetric Dirichlet form?*

Acknowledgements. I wish to show my gratitude toward Professor Kawashita for having invited the author to his stimulating workshop at Research Institute for Mathematical Sciences, Kyoto.

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