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完備測地距離空間上の二写像に対する近似法

An iterative scheme for two mappings defined on a complete geodesic space

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1 Introduction

Let X be a metric space and $T : X \to X$ a nonexpansive mapping, that is, T satisfies that $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. A point $z \in X$ such that $Tz = z$ is called a fixed point of T. Approximation of fixed points of T is one of the central topics in fixed point theory because it includes various types of problems in nonlinear analysis.

In particular, approximation of fixed points of a mapping defined on a complete CAT(κ) space is a trend of this study and there are a large number of researches related to this problem. For example, the following result is a convergence theorem of an iterative scheme called the shrinking projection method on a CAT(1) space.

Theorem 1 (Kimura-Satô [5]). *Let X be a complete* CAT(1) *space such that $d(u, v) < \pi/2$ for every $u, v \in X$ and suppose that the subset $\{z \in X : d(z, u) \leq d(z, v)\}$ of X is convex for every $u, v \in X$. Let $T : X \to X$ be a nonexpansive mapping such that the set of fixed points $F = \{z \in X : Tz = z\}$ is nonempty. For a given initial point $x_0 \in X$ and $C_0 = X$, generate a sequence $\{x_n\}$ as follows:*

$$C_{n+1} = \{z \in X : d(Tx_n, z) \leq d(x_n, z)\} \cap C_n,$$
$$x_{n+1} = P_{C_{n+1}} x_0,$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and converges to $P_F x_0 \in X$, where $P_C : X \to C$ is the metric projection of X onto a nonempty closed convex subset C of X.

The shrinking projection method was first proposed by Takahashi, Takeuchi, and Kubota [10], and it has been generalized to various directions. See, for instance, Takahashi and Zembayashi [11], Plubtieng and Ungchittrakool [8], Inoue, Takahashi, and Zembayashi [2], Qin, Cho, and Kang [9], Wattanawitoon and Kumam [13, 12], Kimura, Nakajo, and Takahashi [4], Kimura and Takahashi [7], Kimura [3], Kimura and Satô [6], and others.

In this paper, we deal with an approximation of common fixed points for two mappings. We attempt to prove the main result without using the notion of Δ-convergence because it is not easy to understand for the beginners of this study. The proof shown in this paper only uses basic notions.

2 Preliminaries

Let X be a metric space. We say that X is a geodesic space if, for any $u, v \in X$, there exists a mapping $c : [0, d(u,v)] \to X$, which is called a geodesic between endpoints u and v, such that $c(0) = u$, $c(d(u,v)) = v$, and $d(c(s), c(t)) = |s - t|$ for every $s, t \in [0, d(u,v)]$.

If a geodesic is unique for each pair of endpoints, X is said to be uniquely geodesic. In what follows, we always assume that X is a complete uniquely geodesic space such that $d(u,v) < \pi/2$ for every $u, v \in X$. On a uniquely geodesic space, the convex combination of two points $u, v \in X$ can be defined in a natural way and we denote it by $\alpha u \oplus (1-\alpha)v$, where $\alpha \in [0,1]$. For $C \subset X$, if every geodesics having the endpoints in C is contained in C, then C is said to be convex.

Let $\mathbb{S}^2$ be a unit sphere of 3-dimensional Euclidean space $\mathbb{R}^3$ and $d_{\mathbb{S}^2}$ be the spherical metric defined on $\mathbb{S}^2$. A geodesic space X is called a CAT(1) space if for each geodesic triangle on X is thinner than or equal to its comparison triangle on $\mathbb{S}^2$. Namely, every $p, q \in \triangle \subset X$ and their comparison points $\overline{p}, \overline{q} \in \overline{\triangle} \subset \mathbb{S}^2$ satisfy the following which is called CAT(1) inequality:

$$d(p,q) \leq d_{\mathbb{S}^2}(\overline{p}, \overline{q}).$$

If X is a CAT(1) space, then for $x, y, z \in X$ and $t \in [0,1]$, the following inequality holds; see [5].

$$\begin{aligned} &\cos d(tx \oplus (1-t)y, z) \sin d(x,y) \\ &\qquad \geq \cos d(x,z) \sin(td(x,y)) + \cos d(y,z) \sin((1-t)d(x,y)). \end{aligned}$$

Let C be a nonempty closed convex subset C of X. Since X satisfies in our setting that $d(u,v) < \pi/2$ for every $u, v \in X$, we know that for every $x \in X$, there exists a unique $y_x \in C$ such that $d(x, y_x) = d(x, C)$, where $d(x, C) = \inf_{y \in C} d(x,y)$. We define a mapping $P_C : X \to C$ by $P_C x = y_x$ for $x \in X$ and we call it the metric projection of X onto C.

For more details of CAT(1) spaces and related notions, see [1].

We say a mapping $T : X \to X$ is quasinonexpansive if the set $F(T) = \{z \in X : Tz = z\}$ of fixed points is nonempty and $d(Tx, z) \leq d(x,z)$ for every $x \in X$ and $z \in F(T)$. We also know that if X is CAT(1) space with $d(u,v) < \pi/2$ for every $u, v \in X$, then $F(T)$ is closed and convex.

3 Approximation of a common fixed point

In this section, we prove a convergence theorem of an iterative sequence generated by the shrinking projection method for two quasinonexpansive mappings defined on a complete CAT(1) space.

Theorem 2. *Let X be a complete* CAT(1) *space such that $d(u,v) < \pi/2$ for every $u,v \in X$ and suppose that the subset $\{z \in X : d(z,u) \le d(z,v)\}$ of X is convex for every $u,v \in X$. Let S and T be continuous quasinonexpansive mappings of X to itself such that the set of common fixed points $F = \{z \in X : Sz = z = Tz\}$ is nonempty. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ such that there exists a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ converging to $\alpha_\infty \in]0,1[$. For a given initial point $x_0 \in C$ and $C_0 = X$, generate a sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
y_n &= \alpha_n S x_n \oplus (1-\alpha_n) T x_n, \\
C_{n+1} &= \{z \in X : d(y_n, z) \le d(x_n, z)\} \cap C_n, \\
x_{n+1} &= P_{C_{n+1}} x_0,
\end{aligned}$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and converges to $P_F x_0 \in X$, where $P_C : X \to C$ is the metric projection of X onto a nonempty closed convex subset C of X.

To prove this type of convergence theorems, one tends to make use of the following theorem.

Theorem 3 (Kimura-Satô [5]). *Let X be a complete* CAT(1) *space and $\{C_n\}$ a sequence of nonempty closed π-convex subsets of X. Let C_∞ be a nonempty closed π-convex subset of X. Then the following are equivalent:*

(i) *$C_\infty = \Delta_1\text{M-}\lim_{n\to\infty} C_n$;*
(ii) *for $x \in X$ and a subsequence $\{C_{n_i}\}$ of $\{C_n\}$, if one of $\limsup_{i\to\infty} d(x, C_{n_i})$ and $d(x, C_\infty)$ is less than $\pi/2$, then the other is also less than $\pi/2$ and $\{P_{C_{n_i}} x\}$ converges to $P_{C_\infty} x$.*

Although this result is useful, one may think that it is rather difficult to understand because it requires the notion of Δ-Mosco convergence of a sequence of subsets in X. We actually do not need to use this concept since we only use the result for the case where a sequence $\{C_n\}$ of subsets of X is decreasing with respect to inclusion. Here, we show the proof of Theorem 2 without using the notion of Δ-Mosco convergence.

Proof of Theorem 2. We first prove the well-definedness of $\{x_n\}$ by showing that every C_n is closed, convex, and it includes $F \ne \emptyset$ by induction. It is trivial that $C_0 = X$ is a closed convex set such that $F \subset C_0$, and a point $x_0 \in X$ is given. Suppose that C_k is defined as a closed convex subset of X which includes F for some $k \in \mathbb{N}$. Then, $x_k = P_{C_k} x_0$ is defined. Since S and T are quasinonexpansive and $\sin t$ is concave on

$t \in [0, \pi/2]$ with $\sin 0 = 0$, for $z \in F$ we have that

$$\begin{aligned}
&\cos d(y_k, z) \sin d(Sx_k, Tx_k) \\
&= \cos d(\alpha_k Sx_k \oplus (1-\alpha_k)Tx_k, z) \sin d(Sx_k, Tx_k) \\
&\geq \cos d(Sx_k, z) \sin(\alpha_k d(Sx_k, Tx_k)) + \cos d(Tx_k, z) \sin((1-\alpha_k) d(Sx_k, Tx_k)) \\
&\geq \cos d(x_k, z)(\sin(\alpha_k d(Sx_k, Tx_k)) + \sin((1-\alpha_k) d(Sx_k, Tx_k))) \\
&\geq \cos d(x_k, z)(\alpha_k \sin d(Sx_k, Tx_k) + (1-\alpha_k) \sin d(Sx_k, Tx_k)) \\
&= \cos d(x_k, z) \sin d(Sx_k, Tx_k),
\end{aligned}$$

and thus $d(y_k, z) \leq d(x_k, z)$. This implies that

$$F \subset \{z \in X : d(y_k, z) \leq d(x_k, z)\} \cap C_k = C_{k+1}.$$

It is obvious from the continuity of the metric and the assumption of the space that C_k is closed and convex. Hence $\{x_n\}$ is well defined and $\{C_n\}$ is a sequence of closed convex subsets of X satisfying that $F \subset C_n$ for every $n \in \mathbb{N}$.

It holds by definition that $\{C_n\}$ is decreasing with respect to inclusion and $C_\infty = \bigcap_{n=1}^\infty C_n$ is nonempty since $C_\infty \supset F$. Since $x_n = P_{C_n} x_0$ for every $n \in \mathbb{N}$, we have that $\{d(x_n, x_0)\}$ is nondecreasing and bounded above. Thus there exists $d = \lim_{n\to\infty} d(x_n, x_0)$.

Let $m, n \in \mathbb{N}$ such that $m \leq n$. Then, both x_m and x_n belong to C_m and since C_m is convex, we have that

$$\begin{aligned}
&\cos d(x_m, x_0) \sin d(x_m, x_n) \\
&\geq \cos d\left(\frac{1}{2}x_m + \frac{1}{2}x_n, x_0\right) \sin d(x_m, x_n) \\
&\geq \cos d(x_m, x_0) \sin\left(\frac{1}{2} d(x_m, x_n)\right) + \cos d(x_n, x_0) \sin\left(\frac{1}{2} d(x_m, x_n)\right).
\end{aligned}$$

Since

$$\cos d(x_m, x_0) \sin d(x_m, x_n) = 2\cos d(x_m, x_0) \sin\left(\frac{1}{2} d(x_m, x_n)\right) \cos\left(\frac{1}{2} d(x_m, x_n)\right),$$

we have that

$$2\cos d(x_m, x_0) \cos\left(\frac{1}{2} d(x_m, x_n)\right) \geq \cos d(x_m, x_0) + \cos d(x_n, x_0)$$

and since $d(x_m, x_0) \leq d(x_n, x_0)$, we get that

$$\begin{aligned}
\cos\left(\frac{1}{2} d(x_m, x_n)\right) &\geq \frac{\cos d(x_m, x_0) + \cos d(x_n, x_0)}{2\cos d(x_m, x_0)} \\
&\geq \frac{\cos d(x_n, x_0)}{\cos d(x_m, x_0)},
\end{aligned}$$

which is equivalent to that

$$-\log\cos\left(\frac{1}{2}d(x_m,x_n)\right) \le \log\cos d(x_m,x_0) - \log\cos d(x_n,x_0).$$

Since $\{\log\cos d(x_n,x_0)\}$ is a convergent sequence to $\log\cos d$, there exists a sequence $\{t_n\}$ converging to 0 such that

$$0 \le \log\cos d(x_m,x_0) - \log\cos d(x_n,x_0) \le t_n$$

for all $m,n \in \mathbb{N}$ with $m \le n$. Then we have that

$$d(x_m,x_n) \le 2\arccos e^{-t_n}$$

for all $m,n \in \mathbb{N}$ with $m \le n$ and $\lim_{n\to\infty} 2\arccos e^{-t_n} = 0$. It shows that $\{x_n\}$ is a Cauchy sequence and therefore it has a limit $x_\infty \in X$.

For fixed $k \in \mathbb{N}$, $\{x_{n+k}\}$ is a sequence in C_k. It follows from the closedness of C_k that x_∞ is a point in C_k and thus we have that

$$d(y_k,x_\infty) \le d(x_k,x_\infty).$$

Tending $k \to \infty$, we obtain that $\{y_k\}$ also converges to x_∞. In addition, we also have that $x_\infty \in \bigcap_{k=1}^{\infty} C_k = C_\infty$. We next show that x_∞ belongs to F. For $z \in F$, we have that $z \in C_\infty$ and

$$\begin{aligned}
&\cos d(y_n,z)\sin d(Sx_n,Tx_n)\\
&= \cos d(\alpha_n Sx_n \oplus (1-\alpha_n)Tx_n, z)\sin d(Sx_n,Tx_n)\\
&\ge \cos d(Sx_n,z)\sin(\alpha_n d(Sx_n,Tx_n)) + \cos d(Tx_n,z)\sin((1-\alpha_n)d(Sx_n,Tx_n))\\
&\ge \cos d(x_n,z)(\sin(\alpha_n d(Sx_n,Tx_n)) + \sin((1-\alpha_n)d(Sx_n,Tx_n)))\\
&= 2\cos d(x_n,z)\sin\left(\frac{1}{2}d(Sx_n,Tx_n)\right)\cos\left(\left(\frac{1}{2}-\alpha_n\right)d(Sx_n,Tx_n)\right).
\end{aligned}$$

Since

$$\sin d(Sx_n,Tx_n) = 2\sin\left(\frac{1}{2}d(Sx_n,Tx_n)\right)\cos\left(\frac{1}{2}d(Sx_n,Tx_n)\right),$$

we have that

$$\begin{aligned}
&\cos d(y_n,z)\cos\left(\frac{1}{2}d(Sx_n,Tx_n)\right)\\
&\ge \cos d(x_n,z)\cos\left(\left(\frac{1}{2}-\alpha_n\right)d(Sx_n,Tx_n)\right).
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, for a subsequence $\{\alpha_{n_i}\}$ of $\{a_n\}$ whose limit is $\alpha_\infty \in]0,1[$,

$$\cos d(x_\infty,z)\cos\left(\frac{1}{2}\limsup_{i\to\infty} d(Sx_{n_i},Tx_{n_i})\right)$$

$$\geq \cos d(x_\infty, z) \cos\left(\left(\frac{1}{2} - \alpha_\infty\right) \limsup_{i\to\infty} d(Sx_{n_i}, Tx_{n_i})\right),$$

which implies that $\lim_{i\to\infty} d(Sx_{n_i}, Tx_{n_i}) = 0$. Hence we have that

$$\begin{aligned} d(x_\infty, Sx_\infty) &= \lim_{i\to\infty} d(y_{n_i}, Sx_{n_i}) \\ &= \lim_{i\to\infty} d(\alpha_{n_i} Sx_{n_i} \oplus (1-\alpha_{n_i})Tx_{n_i}, Sx_{n_i}) \\ &= \lim_{i\to\infty} (1-\alpha_{n_i}) d(Tx_{n_i}, Sx_{n_i}) \\ &= (1-\alpha_\infty) \lim_{i\to\infty} d(Tx_{n_i}, Sx_{n_i}) \\ &= 0, \end{aligned}$$

and, in a similar fashion, we get that $d(x_\infty, Tx_\infty) = 0$. Thus $x_\infty \in F(S) \cap F(T) = F$. Since $F \subset C_\infty$, we have that

$$d(x_0, x_\infty) = \lim_{i\to\infty} d(x_0, P_{C_i} x_0) \leq d(x_0, P_F x_0) \leq d(x_0, x_\infty)$$

and, from the uniqueness of the minimizing point of the distance between x_0 and F, we have $x_\infty = P_F x_0$. This is the desired result. □

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