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Some New Properties of Lagrange Function

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1 Introduction and Preliminaries

Let us consider the following problem

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) := (f_1(x), \cdots, f_m(x)) \\ & \text{subject to} & g_t(x) \leqq 0,\ t \in T, \\ & & x \in C, \end{array}$$

where the functions $f_i : X \to \mathbb{R}, i \in M := \{1, \cdots, m\}$ and $g_t : X \to \mathbb{R}, t \in T$ are locally Lipschitz on a Banach space X, T is an arbitrary (possibly infinite) index set, and C is a closed convex subset of X. We denote the feasible set by $F =: \{x \in C | g_t(x) \leqq 0, t \in T\}$.

In this paper, due to Chankong-Haimes method, for $j \in M$ and $z \in C$, we associate to (MP) the following scalar problem,

$$\begin{array}{lll} (\mathrm{P_j(z)}) & \text{Minimize} & f_j(x) \\ & \text{subject to} & f_k(x) \leqq f_k(z), k \in M^j := M \setminus \{j\}, \\ & & g_t(x) \leqq 0, t \in T, \\ & & x \in C. \end{array}$$

Relationships between (MP) and $(\mathrm{P_j(z)})$ are established and optimality conditions of the problems are given.

Let us denote by $\mathbb{R}^{(T)}$ a following linear space,

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

For each $\lambda \in \mathbb{R}^{(T)}$, the supporting set corresponding to λ is $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$. It is a finite subset of T. We denote $\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0, t \in T\}$. It is a nonnegative cone of $\mathbb{R}^{(T)}$. For $\lambda \in \mathbb{R}^{(T)}$ and $\{z_t\}_{t \in T} \subset Z$, Z being a real linear space, we understand that

$$\sum_{t \in T} \lambda_t z_t = \begin{cases} \sum_{t \in T(\lambda)} \lambda_t z_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

Throughout this paper X is a Banach space, C is a nonempty closed convex subset of X, T is a compact topological space, $f : X \to \mathbb{R}$ is a locally Lipschitz function, and $g_t : X \to \mathbb{R}, t \in T$, are locally Lipschitz with respect to x uniformly in t, i.e.,

$$\forall x \in X, \exists U(x), \exists K > 0, \ \ |g_t(u) - g_t(v)| \leq K\|u - v\|, \ \ \forall u, v \in U(x), \ \ \forall t \in T.$$

The following concepts can be found in the Clarke's books [1, 2]. Let D be a nonempty closed convex subset of X. The normal cone to D at a point $z \in D$ coincides with the normal cone in the sense of convex analysis and given by $N_D(, z) := \{v \in X^* \mid v(x - z) \leq 0, \forall x \in D\}$.

Let $g : X \to \mathbb{R}$ be a locally Lipschitz function. The directional derivative of g at $z \in X$ in direction $d \in X$, is $g'(z; d) = \lim\limits_{t \to 0+} \frac{g(z+td)-g(z)}{t}$ if the limit exists. The Clarke generalized directional derivative of g at $z \in X$ in direction $d \in X$ is $g^c(z; d) := \limsup\limits_{\substack{y \to z \\ t \to 0^+}} \frac{g(y+td)-g(y)}{t}$. The Clarke subdifferential of g at $z \in X$, denoted by $\partial^c g(z)$, is defined by $\partial^c g(z) := \{v \in X^* \mid v(d) \leq g^c(z; d), \forall d \in X\}$.

A locally Lipschitz function g is said to be regular (in the sense of Clarke) at $z \in X$ if $g'(z; d)$ exists and

$$g^c(z; d) = g'(z; d), \forall d \in X.$$

Definition 1.1 *Let C be a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function.*

(i) The function f is said to be pseudoconvex at $x \in C$ if $f(y) < f(x) \Rightarrow u(y - x) < 0, u \in \partial^c f(x), y \in C$, equivalently, $u(y - x) \geqq 0 \Rightarrow f(y) \geqq f(x), u \in \partial^c f(x), y \in C$. The function f is said to be pseudoconvex on C if it is pseudoconvex at every $x \in C$.

(ii) The function f is said to be quasiconvex at $x \in C$ if $f(y) \leqq f(x) \Rightarrow u(y - x) \leqq 0, u \in \partial^c f(x), y \in C$, equivalently, $u(y - x) > 0 \Rightarrow f(y) > f(x), u \in \partial^c f(x), y \in C$. The function f is said to be quasiconvex on C if it is quasiconvex at every $x \in C$.

We need the following lemmas.

Lemma 1.1 *Let $f : X \to \mathbb{R}$ be a locally Lipschitz pseudoconvex function. If there exits $u \in \partial^c f(x)$ such that $u(y - x) \geq 0,\ y, x \in X,$ then $f(y) \geq f(x)$.*

Lemma 1.2 *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. If f is pseudoconvex then f is quasiconvex.*

Now we give the definition of efficient solution of (MP).

Definition 1.2 *A point $z \in F$ is said to be an efficient solution of* (MP) *if there exists no other $x \in F_M$ such that*

$$f_i(x) \leqq f_i(z), \text{for all } i \in M$$

and

$$f_{i_0}(x) < f_{i_0}(z), \text{for some } i_0 \in M.$$

The criteria of Chankong-Haimes method applied for (MP) is as follows.

Lemma 1.3 *A feasible point z of* (MP) *is an efficient solution if and only if it is a solution of* $(\mathrm{P_j(z)})$ *for each $j \in M$.*

2 Optimality Conditions

First of all, let us consider the following scalar optimization problem in order to recall some concepts of solution for a single objective optimization problem.

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & f(x) \\ & \text{subject to} & g_t(x) \leqq 0, t \in T, \\ & & x \in C \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz function and functions $g_t, t \in T$ and C are as above. Also, the feasible set of (P) is denoted by $F_P := \{x \in C | g_t(x) \leqq 0, t \in T\}$.

Let $x \in \mathbb{R}^n$. We need the following condition,

$(\mathcal{A})$: $\exists d \in T_C(x) : g_t^\circ(x; d) < 0$, for all $t \in I(x) := \{t \in T | g_t(x) = 0\}$.

According to Theorems 4.1 and 4.2 presented in [10] (where the problem (P) is defined on a Banach space), we derive the following theorems for the case of the involved functions are defined on $\mathbb{R}^n$ and the index set T is compact. The proofs can be omitted.

Theorem 2.1 *Let z be an optimal solution for* (P). *Assume that the condition $(\mathcal{A})$ holds for z. Then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial f(z) + \sum_{t \in T} \lambda_t \partial g_t(z) + N_C(z), \quad g_t(z) = 0 \text{ for all } t \in T(\lambda). \tag{2.1}$$

We now establish optimality conditions for $(\mathrm{P_j(z)})$ and (MP). The following condition is associated to the problem $(\mathrm{P_j(z)})$, and the feasible set of $(\mathrm{P_j(z)})$ is denoted by $F_j(z)$.

Let $x \in \mathbb{R}^n$, $I(x) = \{t \in T \mid g_t(x) = 0\}$, $H_j(x) = \{k \in M^j \mid f_k(x) = f_k(z)\}$, and $\bar{T}(x) = I(x) \cup H_j(x)$.

$$(\mathcal{A}_j): \quad \exists d \in T_C(x): \begin{cases} g_t^\circ(x;d) < 0, & \text{for all } t \in I(x), \\ f_k^\circ(x;d) < 0, & \text{for all } k \in H_j(x). \end{cases}$$

Lemma 2.1 *Let z be an optimal solution of* $\mathrm{P_j(z)}$, *assume that the condition* $(\mathcal{A}_j)$ *holds for z, then there exist $\mu_k \geqq 0, k \in M^j$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f_j(z) + \sum_{k \in M^j} \mu_k \partial^c f_k(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N_C(z), g_t(z) = 0, \forall t \in T(\lambda). \tag{2.2}$$

Lemma 2.2 *Let $z \in F_j(z)$. Assume that the function f_j is pseudoconvex, the functions $f_k, k \in M^j$ and $g_t, t \in T$ are quasiconvex. If there exist $\mu_k \geqq 0, k \in M^j$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that* (2.2) *holds. Then z is an optimal solution for* $(\mathrm{P_j(z)})$.

Theorem 2.2 (Necessary Condition) *If z is an efficient solution, then there exist $\tau > 0$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that the following condition holds*

$$0 \in \sum_{i \in M} \tau_i \partial^c f_i(z) + \sum_{t \in T} \lambda_t \partial^c g_t(z) + N(C,z), \quad \lambda_t g_t(z) = 0, \quad \forall t \in T. \tag{2.3}$$

Theorem 2.3 (Sufficient Condition) *Let $z \in F$, assume that $\tau^T f$ is pseudoconvex at z and $\lambda_t g_t, t \in T$ are quasiconvex. If there exist $\tau > 0$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that* (2.3) *holds, then z is an efficient solution of* (MP).

3 Mixed Duality

The dual problem of (MP) in a mixed type of Wolfe type and in Mond-Weir type is formulated by

$$
\begin{array}{lll}
\text{(MD)} & \text{Maximize} & f(y) + \sum_{t\in T} \lambda_t g_t(y)e \\
& \text{subject to} & 0 \in \sum_{i\in M} \tau_i \partial^c f_i(y) + \sum_{t\in T} (\lambda_t + \mu_t)\partial^c g_t(y) + N(C, y), \\
& & \mu_t g_t(y) \geqq 0, t \in T, \\
& & \tau^T e = 1, \tau > 0, \tau \in \mathbb{R}^m, e = (1, \cdots, 1) \in \mathbb{R}^m, \\
& & (y, \tau, \lambda, \mu) \in C \times \mathbb{R}^m \times \mathbb{R}_+^{(T)} \times \mathbb{R}_+^{(T)}.
\end{array}
$$

Let us denote by G the feasible set of (MD). The optimal values of the problems (MP) and (MD) are denoted by V(MP) and V(MD) , respectively.

Theorem 3.1 (Weak Duality) *Let x and (y, τ, λ, μ) be the feasible solution of* (MP) *and* (MD), *respectively. Assume that $(\tau^T f + \sum_{t\in T}(\lambda_t + \mu_t) g_t)$ is pseudoconvex, $f_i, i \in M$ and $g_t, t \in T$ are regular on C. Then the following cannot hold:*

$$f(x) < f(y) + \sum_{t\in T} \lambda_t g_t(y)e.$$

Theorem 3.2 (Strong Duality) *Suppose that y is an efficient solution for* (MP) *and weak duality theorem (Theorem* 3.1) *holds, then there exist $\lambda, \mu \in \mathbb{R}_+^{(T)}$ and $\tau_i, i \in M$ such that (y, τ, λ, μ) is an efficient solution for* (MD).

4 Properties of Lagrange Function

The Lagrange function associated to (MP) is formulated by

$$L(x, \lambda) = \begin{cases} f(x) + \sum_{t\in T} \lambda_t g_t(x)e, & (x, \lambda) \in C \times \mathbb{R}_+^{(T)} \\ +\infty, & \text{otherwise.} \end{cases}$$

For every $\lambda \in \mathbb{R}_+^{(T)}$, the function $L(\cdot, \lambda)$ is locally Lipschitz on X.

From now on, we suppose that the function $L(\cdot,\lambda)$ is pseudoconvex on X for every $\lambda \in \mathbb{R}_+^{(T)}$, and $f, g_t, t \in T$, are regular on X.

Theorem 4.1 *If z is an efficient solution of* (MP) *and there exists $j \in M$ such that the condition $(\mathcal{A}_j)$ holds for z, then $(z, \bar{\tau}, \bar{\lambda}, 0)$ and $(z, \bar{\tau}, 0, \bar{\lambda})$ are solutions of* (MD).

Theorem 4.2 *Suppose that $(y^*, \tau^*, \lambda^*, \mu^*)$ is a weakly efficient solution of* (MD).

i) It holds $L(y, \lambda^*+\mu^*) = V(\mathrm{MD})$ for all $y \in G_1 := \{y \in C \mid (y, \tau^*, \lambda^*, \mu^*) \in G\}$ and $\mu_t^* g_t(y^*) = 0$ for all $t \in T$.

ii) Furthermore, if $V(\mathrm{MD}) = V(\mathrm{MP})$ then $L(y, \lambda^* + \mu^*) = V(\mathrm{MD})$ for all $y \in Sol(\mathrm{MP})$ and $(\lambda_t^* + \mu_t^*) g_t(y) = 0$ for all $t \in T$.

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