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Note on Cauchy problems for α order fractional differential equations with $1<\alpha\leq 2$

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Abstract

In this paper we consider the Cauchy problem in a class of fractional differential equations. Let $1<\alpha\leq 2$. We consider the Cauchy problem

$$\begin{cases} D_{0+}^{\alpha}u(t)=p(t)t^{a}u(t)^{\sigma},\\ \lim_{t\to 0+}u(t)=0,\ \lim_{t\to 0+}t^{2-\alpha}u'(t)=(\alpha-1)\lambda, \end{cases}$$

where p is continuous, $a, \sigma, \lambda\in \mathbb{R}$ with $\sigma<0$, $\lambda>0$ and D_{0+}^{α} is the Riemann-Liouville fractional derivative. If $\alpha=2$, then this problem is the problem in [6].

1 Introduction

In [6], Knežević-Miljanović considered the Cauchy problem

$$\begin{cases} u''(t)=p(t)t^{a}u(t)^{\sigma},\\ \lim_{t\to 0+}u(t)=0,\ u'(0)=\lambda, \end{cases} \tag{1.1}$$

where p is continuous, $a, \sigma, \lambda\in \mathbb{R}$ with $\sigma<0$ and $\lambda>0$. She proved that if p satisfies

$$\int_{0}^{1}|p(t)|t^{a+\sigma}dt<\infty,$$

then the problem has a solution.

On the other hand, fractional differential equations have been studied by many mathematicians. For example, in [1] and [7], the authors considered the differential equation of fractional order

$$D_{0+}^{\alpha}u(t)+f(t, u(t))=0,$$

where $1<\alpha\leq 2$ and D_{0+}^{α} is the Riemann-Liouville fractional derivative. The Riemann-Liouville fractional derivative of order α of u is given by

$$D_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{0}^{t}(t-s)^{n-\alpha-1}u(s)ds,$$

where $n = [\alpha] + 1$ and Γ is the gamma function. If $\alpha = 2$, then $n = 3$ and

$$D^2_{0+}u(t) = \frac{1}{\Gamma(1)} \frac{d^3}{dt^3} \int_0^t u(s)ds = u''(t).$$

In this paper we consider the Cauchy problem (1.1) in a class of fractional differential equations. Let $1 < \alpha \leq 2$. We consider the Cauchy problem

$$\begin{cases} D^\alpha_{0+}u(t) = p(t)t^a u(t)^\sigma, \\ \lim_{t\to 0+} u(t) = 0, \ \lim_{t\to 0+} t^{2-\alpha}u'(t) = (\alpha - 1)\lambda, \end{cases} \tag{1.2}$$

where p is continuous, $a, \sigma, \lambda \in \mathbb{R}$ with $\sigma < 0$ and $\lambda > 0$. If $\alpha = 2$, then the Cauchy problem (1.2) is the problem (1.1).

2 Main result

In this section we derive first the integral equation which is equivalent to the problem (1.2) (Lemma 2.3). Next, by using the Banach fixed point theorem, we obtain the existence and uniqueness result of solutions of the problem (1.2) (Theorem 2.1).

Let u be a continuous function from $(0, \infty)$ into $\mathbb{R}$ and α be a positive real number. The Riemman-Liouville fractional integral of order α of u is defined by

$$I^\alpha_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds.$$

The following lemmas can be found in [5] and [1].

Lemma 2.1. *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then the fractional differential equation $D^\alpha_{0+}u(t) = 0$ has a unique solution*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$ $(i = 1, \dots n)$ and $n = [\alpha] + 1$.

Lemma 2.2. *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$ satisfying $D^\alpha_{0+}u \in C(0,1) \cap L^1(0,1)$. Then*

$$I^\alpha_{0+}D^\alpha_{0+}u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for some $C_1, C_2, \dots, C_n \in \mathbb{R}$ and $n = [\alpha] + 1$.

Next we derive the integral equation which is equivalent to the problem (1.2).

Lemma 2.3. *Let p be a continuous function, $a \in \mathbb{R}$, $\sigma < 0$ and $\lambda > 0$. Then the solution of the Cauchy problem (1.2) is*

$$u(t) = \lambda t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) s^a u(s)^\sigma ds.$$

Proof. By Lemma 2.2, the equation $D^{\alpha}_{0+}u(t) = p(t)t^a u(t)^{\sigma}$ is equivalent to the integral equation

$$u(t) = I^{\alpha}_{0+}p(t)t^a u(t)^{\sigma} + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$$

for some C_1 and C_2. By the definition of the Riemman-Liouville fractional integral I^{α}_{0+}, we have

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)s^a u(s)^{\sigma}ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}.$$

The condition $\lim_{t\to 0} u(t) = 0$ implies $C_2 = 0$. Thus

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)s^a u(s)^{\sigma}ds + C_1 t^{\alpha-1}.$$

Since

$$\lim_{t\to 0} t^{2-\alpha}u'(t) = (\alpha-1)C_1,$$

we obtain that $C_1 = \lambda$. □

The following is our main result.

Theorem 2.1. *Let p be a continuous function from $[0,1]$ into $\mathbb{R}$ such that*

$$\int_0^1 |p(t)|t^{a+(\alpha-1)\sigma}dt < \infty,$$

where $1 < \alpha \leq 2$, $a \in \mathbb{R}$, $\sigma < 0$ and $\lambda > 0$. Then there exists a unique solution $u : (0,h] \longrightarrow \mathbb{R}$ of the Cauchy problem (1.2) such that $\frac{\lambda}{2}t^{\alpha-1} \leq u(t)$ for any $t \in (0,h]$.

Proof. By Lemma 2.3, instead of the Cauchy problem (1.2) we consider the integral equation

$$u(t) = \lambda t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)s^a u(s)^{\sigma}ds.$$

Choose $0 < h < 1$ satisfying

$$\int_0^h |p(s)|s^{a+\sigma}ds \leq \Gamma(\alpha)\left(\frac{\lambda}{2}\right)^{1-\sigma}$$

and

$$\int_0^h |p(s)|s^{a+(\alpha-1)\sigma}ds < \frac{\Gamma(\alpha)}{|\sigma|}\left(\frac{\lambda}{2}\right)^{1-\sigma}$$

We denote by $C[0,h]$ the space of all continuous functions from $[0,h]$ into $\mathbb{R}$ with the maximum norm given by $\|u\| = \max_{0\le t\le h} |u(t)|$ for any $u \in C[0,h]$. Let X be a subset of $C[0,h]$ defined by

$$X = \left\{ u \in C[0,h] \,\middle|\, u(0) = 0,\ \lim_{t\to 0+} t^{2-\alpha}u'(t) = (\alpha-1)\lambda,\ \frac{\lambda}{2}t^{\alpha-1} \le u(t),\ \forall t \in [0,h] \right\}.$$

Since a mapping $t \longmapsto \lambda t^{\alpha-1}$ belongs to X, we obtain that $X \neq \emptyset$. Let A be an operator from X into $C[0,h]$ defined by

$$Au(t) = \lambda t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) s^a u(s)^\sigma ds.$$

Then $A(X) \subset X$. Indeed, let $u \in X$. We have $Au(0) = 0$ and

$$\lim_{t\to 0} t^{2-\alpha}(Au)'(t) = (\alpha-1)\lambda.$$

Moreover we obtain that

$$\begin{aligned} Au(t) &\ge \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |p(s)| s^a u(s)^\sigma ds \\ &\ge \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |p(s)| s^a \left(\frac{\lambda}{2}s\right)^\sigma ds \\ &= \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^\sigma \int_0^t (t-s)^{\alpha-1} |p(s)| s^{a+\sigma} ds. \end{aligned}$$

Since $(t-s)^{\alpha-1} \le t^{\alpha-1}$ for $0 \le s \le t \le 1$ and

$$\int_0^h |p(s)| s^{a+\sigma} ds \le \Gamma(\alpha) \left(\frac{\lambda}{2}\right)^{1-\sigma},$$

we have

$$\begin{aligned} Au(t) &\ge \lambda t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^\sigma t^{\alpha-1} \int_0^t |p(s)| s^{a+\sigma} ds \\ &\ge \lambda t^{\alpha-1} - \frac{\lambda}{2} t^{\alpha-1} \\ &= \frac{\lambda}{2} t^{\alpha-1}. \end{aligned}$$

Hence we have $Au \in X$. We will find a fixed point of A. Let φ be an operator from X into $C[0,h]$ defined by

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{t^{\alpha-1}}, & t \neq 0, \\ \lambda, & t = 0. \end{cases}$$

Then we obtain that

$$\varphi[X] = \left\{ z \in C[0,h] \;\middle|\; z(0) = \lambda,\ \frac{\lambda}{2} \le z(t),\ \forall t \in [0,h] \right\}$$

and $\varphi[X]$ is a closed subset of $C[0,h]$. Hence it is a complete metric space. Let Φ_A be an operator from $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi_A \varphi[u] = \varphi[Au].$$

By the mean value theorem for any $u_1, u_2 \in X$ there exists a mapping ξ such that

$$\frac{u_1^\sigma(t) - u_2^\sigma(t)}{u_1(t) - u_2(t)} = \sigma \xi(t)^{\sigma-1},$$

where

$$\min\{u_1(t), u_2(t)\} \le \xi(t) \le \max\{u_1(t), u_2(t)\}$$

for almost every $t \in [0,h]$. For $t \ne 0$, we have

$$\begin{aligned} |\Phi_A\varphi[u_1](t) - \Phi_A\varphi[u_2](t)| &= |\varphi[Au_1](t) - \varphi[Au_2](t)| \\ &= \left| \frac{1}{t^{\alpha-1}\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) s^a (u_1(s)^\sigma - u_2(s)^\sigma) ds \right|. \end{aligned}$$

Since $(t-s)^{\alpha-1} \le t^{\alpha-1}$ and

$$\begin{aligned} |u_1(s)^\sigma - u_2(s)^\sigma| &= |\sigma||\xi(s)|^{\sigma-1}|u_1(s) - u_2(s)| \\ &\le |\sigma| \left| \frac{\lambda}{2} s^{\alpha-1} \right|^{\sigma-1} |u_1(s) - u_2(s)| \end{aligned}$$

for $0 \le s \le t \le 1$, we have

$$\begin{aligned} &|\Phi_A\varphi[u_1](t) - \Phi_A\varphi[u_2](t)| \\ &\le \left| \frac{1}{\Gamma(\alpha)} \int_0^t p(s) s^a (u_1(s)^\sigma - u_2(s)^\sigma) ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^{\sigma-1} |\sigma| \int_0^t |p(s)| s^{a+(\alpha-1)\sigma} \left| \frac{u_1(s)}{s^{\alpha-1}} - \frac{u_2(s)}{s^{\alpha-1}} \right| ds \\ &\le \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^{\sigma-1} |\sigma| \int_0^t |p(s)| s^{a+(\alpha-1)\sigma} ds \|\varphi[u_1] - \varphi[u_2]\| \end{aligned}$$

for $0 \le t \le h$. Therefore we have

$$\|\Phi_A\varphi[u_1] - \Phi_A\varphi[u_2]\| \le \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{2}\right)^{\sigma-1} |\sigma| \int_0^t |p(s)| s^{a+(\alpha-1)\sigma} ds \|\varphi[u_1] - \varphi[u_2]\|.$$

Since

$$\int_0^h |p(s)|s^{a+(\alpha-1)\sigma} ds < \frac{\Gamma(\alpha)}{|\sigma|}\left(\frac{\lambda}{2}\right)^{1-\sigma},$$

we have

$$\frac{1}{\Gamma(\alpha)}\left(\frac{\lambda}{2}\right)^{\sigma-1} |\sigma| \int_0^t |p(s)|s^{a+(\alpha-1)\sigma} ds < 1.$$

Hence Φ_A is contractive. By the Banach fixed point theorem, there exists a unique fixed point $\varphi[u] \in \varphi[X]$ of Φ_A. Since $\Phi_A \varphi[u] = \varphi[u]$, we have $Au = u$. Therefore u is a unique solution of of the Cauchy problem (1.2). □

Remark 2.1. If $\alpha = 2$, then Theorem 2.1 is the result of [6]. See also [3]. In [4], we considered the Cauchy problem

$$\begin{cases} u''(t) = f(t, u(t)), \\ u(0) = 0, \ u'(0) = \lambda, \end{cases} \tag{2.1}$$

which is a generalization of the problem (1.1). Theorem 2.1 will be generalized to the case of the problem (2.1). This is a further topic. In [4], we considered the Cauchy problem

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), \\ u(0) = 0, \ u'(0) = \lambda. \end{cases} \tag{2.2}$$

Theorem 2.1 will be generalized to the case of the problem (2.2). This is also a further topic.

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