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On Triality and Pre-structurable Algebras

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abstract We discuss a concept of pre-structurable Algebras based upon triality relations and study its relation to structurable algebra introduced by Allison, as well as to Lie algebras satisfying triality. And this note is an announcement of new results.

I Introduction, preliminary and pre-structurable algebra

The structurable algebra [A], which is a class of nonassociative algebras has many interesting properties. First, it satisfies a triality relation [A-F]. Moreover, we can construct a Lie algebra L (see [A-F]) that any simple classical Lie algebra can be constructed from some appropriate structurable algebra. Further any such Lie algebra L is invariant under the symmetric group S_4 and is a BC_1-graded Lie algebra of type B_1 (see [E-O.2] and [E-K-O]).

Although the structurable algebra has been originally defined in term of Kantor's triple system [Kan], or equivalently of $(-1,1)$ Freudenhthal-Kantor triple system [Y-O], it can be also defined without any reference upon the triple system [A-F]. Here, following [O.1] we base its definition in terms of triality relations as follows:

Let $(A,-)$ be an algebra over a field F with bi-linear product denoted by juxtaposition $ab\in A$ for $a,b\in A$ with the unit element e and with the involution map $a\rightarrow\bar{a}$ and $\overline{ab}=\bar{b}\bar{a}$.

Let

$$t_j : A\otimes A\rightarrow \text{ End } A, (j=0,1,2) \tag{1.1}$$

be given by (see [A-F])

$$t_1(a,b):=l(\bar{b})l(a)-l(\bar{a})l(b) \tag{1.2a}$$

$$t_2(a,b):=r(\bar{b})r(a)-r(\bar{a})r(b) \tag{1.2b}$$

$$t_0(a,b):=r(\bar{a}b-\bar{b}a)+l(b)l(\bar{a})-l(a)l(\bar{b}) \tag{1.2c}$$

where $l(a)$ and $r(a)$ are standard multiplication operaters given by

$$l(a)b=ab \tag{1.3a}$$

$$r(a)b=ba. \tag{1.3b}$$

We then see that $t_0(a,b)$ satisfies automatically

$$t_0(a,b)c+t_0(b,c)a+t_0(c,a)b=0. \tag{1.4}$$

Suppose now that $t_j(a,b)$ satisfy the triality relation

$$\overline{t_j(a,b)}(cd)=(t_{j+1}(a,b)c)d+c(t_{j+2}(a,b)d) \tag{1.5}$$

for any $a,b,c,d\in A$ and for any $j=0,1,2$, where the indices are defined modulo 3, i.e.

$$t_{j\pm 3}(a,b)=t_j(a,b) \tag{1.6}$$

for $j=0,1,2$. Here, $\bar{Q}\in$ End A for any $Q\in$ End A is defined by

$$\overline{Qa}=\bar{Q}\bar{a}. \tag{1.7}$$

We call the unital involutive algebra A satisfying Eq.(1.5) to be a pre-structurable algebra. Moreover, let

$$Q(a,b,c):=t_0(a,\bar{b}\bar{c})+t_1(b,\bar{c}\bar{a})+t_2(c,\bar{a}\bar{b}) \tag{1.8}$$

to satisfy $Eq(X)$ of [A-F], which is rewritten as (see [O.1])

$$Q(a,b,c)=0, \tag{X}$$

then we call the pre-structurable algebra A to be structurable.

We note that these concept are a generalization of well known *"the principle of triality"*, because the octonion algebra is a structurable algebra.

The main purpose of this note is to study properties of pre-structurable algebra as well as those satisfied by $Q(a,b,c)$, and its relation to Lie algebras. First, we note the following Theorem (see Theorem 3.1 and Lemma 3.6 of [A-F])

Theorem 1.1 *Let A be any unital algebra with involution, and introduce $A(a,b,c)$ and $B(a,b,c)\in$ EndA by*

$$A(a,b,c)d := ((da)\bar{b})c - d(a(\bar{b}c)) \tag{1.9a}$$

$$B(a,b,c)d := ((da)\bar{b})c - d((a\bar{b})c). \tag{1.9b}$$

Then, a necessary and sufficient condition that the unital involutive algebra $(A,-)$ is pre-structurable is to have

$$A(a,b,c)-A(b,a,c)=A(c,a,b)-A(c,b,a). \tag{A}$$

Moreover, we have

(i) Eq.(A) implies the validity of

$$B(a,b,c)-B(b,a,c)=B(c,a,b)-B(c,b,a) \tag{B}$$

and

$$[a,\bar{b},c]-[b,\bar{a},c]=[c,\bar{a},b]-[c,\bar{b},a] \tag{A1}$$

(ii) Eq.(B) implies

$$[a-\bar{a},b,c]+[b,a-\bar{a},c]=0 \tag{sk}$$

(iii) Eq.(sk) implies

$$[c,\bar{a},b]-[c,\bar{b},a]=[c,a,\bar{b}]-[c,b,\bar{a}] \tag{sk 1}$$

where $[a,b,c]$ is the associator of A defined by

$$[a,b,c] := (ab)c-a(bc). \tag{1.10}$$

Remark 1.2 Taking the involution of Eq.(sk), it yields

$$[a-\bar{a},b,c]=-[b,a-\bar{a},c]=[b,c,a-\bar{a}]. \tag{sk$'$}$$

Also, Eqs.(A1) and (sk1) can be combined to give

$$[a,\bar{b},c]-[b,\bar{a},c]=[c,\bar{a},b]-[c,\bar{b},a]=[c,a,\bar{b}]-[c,b,\bar{a}]. \tag{A.1$'$}$$

Hereafter in what follows in this note, A is always designated to be a pre-structurable algebra over a field F, unless it is stated otherwise.

Proposition 1.3 *Let A be a pre-structurable algebra. Then we have*

(i)

$$\overline{t_j(a,b)}=t_{3-j}(\bar{a},\bar{b}) \tag{1.11a}$$

(ii)

$$[t_j(a,b),t_k(c,d)]=t_k(t_{j-k}(a,b)c,d)+t_k(c,t_{j-k}(a,b)d), \tag{1.11b}$$

for any $a,b,c,d\in A$ and for any $j,k=0,1,2$.

Corollary.1.4 *Under the assumpsion as in above, let us introduce a triple product in A by*

$$abc := t_0(a,b)c = c(\bar{a}b-\bar{b}a)+b(\bar{a}c)-a(\bar{b}c) \tag{1.12}.$$

Then, it defines a Lie triple system, i.e. it satisfies

(i)

$$abc=-bac \tag{1.13a}$$

(ii)

$$abc + bca + cab = 0 \tag{1.13b}$$

(iii)

$$ab(cdf) = (abc)df + c(abd)f + cd(abf) \tag{1.13c}$$

for any $a, b, c, d, f \in A$.

Remark 1.5 If we set

$$L(a,b) := t_0(a,b) + t_2(\bar{a},\bar{b})$$

then it also satisfies

$$[L(a,b), L(c,d)] = L(L(a,b)c, d) + L(c, L(a,b)d)$$

although we will not go to its detail.

We next set

$$D(a,b) := t_0(a,b) + t_1(a,b) + t_2(a,b). \tag{1.14}$$

Proposition 1.6. *Under the asumpsion as in above, we have*

(i)

$$\overline{D(a,b)} = D(\bar{a},\bar{b}) = D(a,b) \tag{1.15a}$$

(ii)

$$D(a,b)(cd) = (D(a,b)c)d + c(D(a,b)d) \tag{1.15b}$$

i.e. $D(a,b)$ *is a derivation of* A.

(iii)

$$[D(a,b), t_k(c,d)] = t_k(D(a,b)c, d) + t_k(c, D(a,b)d). \tag{1.15c}$$

Next, we assume that underlying field F to be of charachteristic $\neq 2$, and set

$$S = \{a|\bar{a} = a,\ a \in A\} \tag{1.16a}$$

$$H = \{a|\bar{a} = -a,\ a \in A\} \tag{1.16b}$$

Then H is an algebra with respect to the anti-commutative product $[a,b] = ab - ba$. Since Eq.(sk) implies then H to be a generalized alternative nucleus of A, H is a Malcev algebra with respect to the product $[a,b]$. (see [P-S])

Proposition 1.7 *Let* A *be a unital involutive algebra over the field* F *of charachteristic* $\neq 2$, *satisfying Eq.(sk). If Dim* $S = 1$, *then* A *is alternative and hence structurable.Moreover it is quadratic, satisfying*

$$a\bar{a} = <a|a> e, \quad \bar{a} = 2 <a|e> e - a, \tag{1.17}$$

for a symmetric bi-linear form $< \cdot | \cdot >$. *Especially,* A *is a composition algebra satisfying*

$$<ab|ab> = <a|a><b|b>,$$

$$<\bar{a}|bc> = <\bar{b}|ca> = <\bar{c}|ab>$$

.for $a, b, c \in A$. *However Dim* A *needs not be limited to the canonical value ([S]) of 1,2,4, or 8, here since* $< \cdot | \cdot >$ *may be degenerate.*

Remark 1.8 Let us consider the case of *Dim* $A = 3$ with $S = Fe$ and $H = <f, g>_{span}$. Then a general solution satisfying Eq. (sk) is obtained as

$$f^2 = \alpha^2 e, \quad g^2 = \beta^2 e,$$

$$fg = -\alpha\beta e + \beta f + \alpha g, \quad gf = -\alpha\beta e - \beta f - \alpha g \tag{1.18}$$

for $\alpha,\ \beta \in F$, satisfying

$$\alpha^2 = - <f|f>, \quad \beta^2 = - <g|g>, \quad \alpha\beta = <f|g> = <g|f>,$$

together with $<e|f> = <e|g> = 0$ and $<e|e> = 1$. These give a composition algebra satisfying $<ab|ab> = <a|a><b|b>$ with *Dim* $A = 3$, e.g. we have

$$<f^2|f^2> = <f|f><f|f>, \text{ and } <fg|fg> = <f|f><g|g> \text{ etc.}$$

We note that $< \cdot|\cdot >$ is degenerate, since we have

$$< (\beta f + \alpha g)|x >= 0 \; for \; any \; x \; \in A.$$

This algebra also satisfies a linear composition law below.

Let $\phi : A \to F$ be a linear form defined by

$$\phi(e) = 1, \; \phi(f) = \alpha, \; \phi(g) = -\beta.$$

Then it satisfies the linear composition law of

$$\phi(xy) = \phi(x)\phi(y) \; for \; any \; x, y \; \in A.$$

II The properties of $Q(a, b, c)$

Here, we assume A to be a pre-structurable algebra and we will discuss some properties of $Q(a, b, c)$.

Theorem 2.1 *Let A be a pre-structurable algebra. Then, we have*

(i) $Q(a, b, c)d$ is totally symmetric in $a, b, c, d \in A$.

(ii) $Q(a, b, c)d$ is identically zero, if at least one of a, b, c, d is the identity element e.

(iii) Suppose the underlying field F to be of characteristic $\neq 2$, then $Q(a, b, c)d$ is identically zero, if at least one of a, b, c and d is a element of H.

(iv) $\overline{Q(a, b, c)} = Q(\bar{a}, \bar{b}, \bar{c}) = Q(a, b, c)$ is a derivation of A.

(v) $3Q(a, b, c) = D(a, \bar{b}\bar{c}) + D(b, \bar{c}\,\bar{a}) + D(c, \bar{a}\bar{b})$.

For a proof of this theorem, we start with the following Lemmas:

Lemma 2.2 *Under the asumpsion as in above,*

$$Q(a, b, c)d = Q(d, b, c)a \tag{2.1}$$

is symmetric in a and d.

Lemma 2.3 *Under the assumpsion as in above, we have*

$$Q(a, b, c)e = Q(e, b, c)a = 0$$

for any $a, b, c \in A$, where e is the unit element of A.

Lemma 2.4 *Under the assumpsion as in above, we have*

(i)

$$\overline{Q(a, b, c)} = Q(\bar{a}, \bar{c}, \bar{b}) \tag{2.2}$$

(ii)

$$\overline{Q(a, b, c)}(df) = \{Q(c, a, b)d\}f + d\{Q(b, c, a)f\}. \tag{2.3}$$

We are now in position to prove Theorem 2.1. We first set $d = e$ or $f = e$ and then letting $f \to d$ in Eq.(2.3).

We find

$$\overline{Q(a, b, c)}d = Q(c, a, b)d = Q(b, c, a)d \tag{2.4}$$

which shows the $Q(c, a, b)$ is cyclic invariant under $a \to b \to c \to a$, and hence, it also gives

$$\overline{Q(a, b, c)}d = Q(a, b, c)d., \; i.e \; .\overline{Q(a, b, c)} = Q(a, b, c). \tag{2.5}$$

Further $Q(a, b, c)d = Q(d, b, c)a$ by Lemma 2.2, so that

$$Q(a, b, c)d = Q(d, b, c)a = Q(b, c, d)a = Q(a, c, d)b = Q(d, a, c)b = Q(b, a, c)d.$$

Therefore, we have also $Q(a, b, c) = Q(b, a, c)$ so that $Q(a, b, c)$ is totally symmetric in $a, b, c \in A$. Especially, this implies $Q(a, b, c)d$ to be totally symmetric in a, b, c and $d.$, proving (i) of Theorem 2.1. Then together with Lemma 2.2, it also proves (ii) of Theorem 2.1. In order to show (iii), we first note the validity of

$$\overline{Q(a, b, c)} = Q(\bar{a}, \bar{b}, \bar{c}) = Q(a, b, c) \tag{2.6}$$

because of Eqs.(2.2) and (2.5). This gives (iv) of the Theorem.

In order to avoid possible confusion, we label elements of S and H, respectively as a_0 and a_1, or b_0 and b_1 etc. so that

$$\bar{a}_0 = a_0, \; \bar{a}_1 = -a_1 \text{ etc.}$$

Then, Eq.(2.6) implies immediately

$$Q(a_0, b_0, c_1) = 0 = Q(a_1, b_1, c_1).$$

Note that we have

$$Q(a_0, b_1, c_1)d_1 = Q(d_1, b_1, c_1)a_0 = 0, \ i.e., \ Q(a_0, b_1, c_1)d_1 = 0.$$

However, we have also

$$Q(a_0, b_1, c_1)d_0 = Q(a_0, d_0, c_1)b_1 = 0$$

so that $Q(a_0, b_1, c_1) = 0$. This proves the statement (iii) of Theorem 2.1. Finally,

$$Q(a, b, c) = t_0(c, \bar{b}\bar{c}) + t_1(b, \bar{c}\bar{a}) + t_2(c, \bar{a}\bar{b}).$$

Letting $a \to b \to c \to a$, and adding all of the resulting relation we obtain

$$Q(a, b, c) + Q(b, c, a) + Q(c, a, b) = D(a, \bar{b}\bar{c}) + D(b, \bar{c}\bar{a}) + D(c, \bar{a}\bar{b})$$

so that

$$3Q(a, b, c) = D(a, \bar{b}\bar{c}) + D(b, \bar{c}\bar{a}) + D(c, \bar{a}\bar{b}). \tag{2.7}$$

This completes the proof of Theorem 2.1.

Proposition 2.5 *Under the assumpsion as in above, if $a = \bar{a} \in S$, then we have*

$$Q(a, a, a)a = [a, aa^2] + 3\{a^2a^2 - a(a^2a)\} = [a^2a, a] + 3\{a^2a^2 - (aa^2)a\}.$$

Remark 2.6 (a) If A is a pre-structurable algebra over the field F of charachtristic $\neq 2$, and $\neq 3$, then Theorem 2.1 and Proposition 2.5 imply that A is structurable, provided that we have $aa^2 = a^2a(:= a^3$ and $a^2a^2 = aa^3(= a^3a)$ for $a \in S$, since then $Q(a, b, c) = 0$ for any $a, b, c \in A$.

Thus if A is power-associative and pre-structurable algebra, then A is structurable.

Remark 2.6 (b) By means of relations (2.3) and(2.6), we note that

$$Q(a, b, c) \ is \ a \ \ derivation \ of \ pre-structurable \ algebra. \ A.$$

Proposition 2.7 *Let A be a pre-structurable algebra and set $A_0 = \{x | x \in A$, and $Q(a, b, c)x = 0$ for any $a, b, c \in A\}$. Then,A_0 is a structurable algebra.*

By Theorem 2.1, A_0 always contains a structurable subalgebra generated by e and elements of H. It is plausible that we may have $A_0 = A$ if A is a simple algebra. However, we could neither prove nor disprove such a conjecture.

We will now prove the converse statement of Theorem 2.1.

Theorem 2.8 *Let A be a unital involutive algebra satisfying*

(i) $\underline{Q(a, b, c)}d$ is totally symmetric in a, b, c, d,

(ii) $\overline{Q(a, b, c)} = Q(a, b, c)$ (2.8),

(iii) $Q(a, b, c) = 0$ whenever at least one of $a, b, c \in A$ is a element of H,

(iv) the validity of Eq. (sk).

Then A is a pre-structurable algebra.

Alternatively any unital involutive algebra A is pre-structurable, if Eq. (sk) holds valid and we have

$$Q(a, b, c) = B(b, a, c) - C(a, b, c) - C(c, b, a) - C'(c, a, b) \tag{2.9}$$

being totally symmetric in $a, b, c \in A$.

Here $C(a, b, c)$ and $C'(a, b, c) \in End\ A$ are defined by (see [A-F])

$$C(a, b, c)d = \{a(\bar{b}\ \bar{d})\}c - (a\bar{b})(\bar{d}c) \tag{2.10}$$

$$C'(a, b, c)d = C(a, b, c)\bar{d} = \{a(\bar{b}\ d)\}c - (a\bar{b})(dc) \tag{2.11}$$

The special case of $Q(a, b, c) = 0$ in Theorem 2.8 reproduces (iii) of Theorem 5.5 of [A-F]:

Corollary 2.9 *A necessary and sufficient condition for a unital involutive algebra to be structurable is the validity of Eq.(sk) and Eq.(X).*

We also note the following Proposition (see Corollary 3.6 of [O.1]):

Proposition 2.10 *Let A be a unital involutive algebra possessting a symmetric bi-linear non-degenerate form $< \cdot | \cdot >$ satisfying*

$$< \bar{a}|bc >=< \bar{b}|ca >=< \bar{c}|ab > . \tag{2.12}$$

Then, Eq.(A) is equivalent to Eq.(X). Especially, we have

(i) Any pre-structurable algebra is automatically structurable.

(ii) If $Q(a, b, c) = 0$, then A is structurable.

Remark 2.11 Many interesting unital involutive algebras containing Jordan and alternative algebras are structurable (see [A-F]). It is rather hard to find example of a pre-structurable but not structurable algebras.

Let A be a commutative algebra with $Dim\ A = Dim\ S = 3$ so that $\bar{a} = a$ for any $a \in A$ and Eq.(sk) is trivially satisfied. Let $A =< e, f, g >_{span}$ with e being the unit element. Suppose hat we have

$$ff = fg = gf = 0, \ (2.13) \quad gg = \alpha e + \beta f, \ (2.14)$$

for $\alpha, \ \beta \ \in F$. Then we can readily verify that we have $Q(a, b, c)d = 0$ for any a, b, c, d assuming values of e, f, and g except for the case of

$$Q(g, g, g)g = (3\alpha\beta)f. \tag{2.15}$$

Therefore, A is pre-structurable but not structurable by Theorem 2.8, provided that $3\alpha\beta \neq 0$. Note that the case of $\alpha\beta = 0$ (or more strongly $\alpha = 0$) corresponds to A being a Jordan (or associative) algebra which is structurable. Note also that the present algebra is not simple since $B = Ff$ is a ideal of A.

Remark 2.12 If A is a structurable algebra, then $D(a, b)$ given by Eq.(1.14) is a derivation of A satisfying

$$D(a, \bar{b}\bar{c}) + D(b, \bar{c}\ \bar{a}) + D(c, \bar{a}\bar{b}) = 0 \tag{2.16}$$

by Eq.(2.7). If we set

$$D_0(a, b) = D(a, \bar{b}) \tag{2.17}$$

then it satisfies

(i)

$$D_0(a, b) = -D_0(b, a) = D_0(\bar{a}, \bar{b}) = \overline{D_0(a, b)} \tag{2.18}$$

(ii)

$$D_0(a, b) \text{ is a derivation of} A \tag{2.19a}$$

(iii)

$$D_0(a, bc) + D_0(b, ca) + D_0(c, ab) = 0. \tag{2.19b}$$

Any algebra A possessing a non-zero $D_0(a, b)$ satisfying

$D_0(a, b) = -D_0(b, a)$ as well as Eq.(2.19a) and (2.19b) has been called in [Kam.2] to be a generalized structurable algebra. Therefore, any structurable algebra is also a generalized structurable algebra, provided that $D(a, b) \neq 0$. Note that there exists a structurable algebra such that we have $D(a, b) = 0$ identically as in Example 25.3 of [O.2].

III Lie Algebras with Triality

In Corollary 1.4, we have seen that we can introduce a Lie triple system for any pre-structurable algebra and hence we can construct a Lie algebra in a canonial way as follows.

Let

$$L_0 = \rho_0(A) \oplus T_0(A, A) \tag{3.1}$$

where $\rho_0(A)$ is a copy of A itself and $T_0(a, b)$, for $a, b \in A$ is an analogue (or generalization) of $t_0(a, b)$. If we wish, we may identify $T_0(a, b)$ as $t_0(a, b)$. Then, supposing commutation relations;

(i)

$$[T_0(a, b), T_0(c, d)] = T_0(t_0(a, b)c, d) + T_0(c, t_0(a, b)d) \tag{3.2a}$$

(ii)

$$[T_0(a, b), \rho_0(c)] = \rho_0(t_0(a, b)c) \tag{3.2b}$$

(iii)

$$[\rho_0(a), \rho_0(b)] = T_0(a, b) \tag{3.2c}$$

for $a, b, c, d \in A$, L_0 becomes a Lie algebra as we may easily verify. Note that Eq (3.2a) is an analogue of Eq.(1.13d).

A extra advantage of A being structurable is that we can further enlarge the Lie algebra by utilizing Eq.(1.11b) for any $j, k = 0, 1, 2$ as follows:

Let $\rho_j(A)$ for $j = 0, 1, 2$ be 3 copies of A. Moreover, we introduce three unspecified symbols $T_j(a, b)$ for $j = 0, 1, 2$ and for $a, b \in A$, which may be regarded as a generalization of $t_j(a, b)$. If we wish, we may identify $T_j(a, b)$ to be $t_j(a, b)$ itself. Now, consider

$$L = \rho_0(A) \oplus \rho_1(A) \oplus \rho_2(A) \oplus T(A, A) \tag{3.3}$$

where $T(A, A)$ is a vector space spanned by $T_j(a, b)$ for any $j = 0, 1, 2$ and for any $a, b \in A$. We first assume the commutation relation of

$$\begin{aligned} [T_l(a, b), T_m(c, d)] &= -[T_m(c, d), T_l(a, b)] \\ &= T_m(t_{l-m}(a, b)c, d) + T_m(c, t_{l-m}(a, b)d)) \end{aligned} \tag{3.4}$$

for any $l, m = 0, 1, 2$ and for any $a, b, c, d \in A$. Then, it is easy to verify that it defines a Lie algebra in view of Eq.(1.11b). Of course, Eq.(3.4) is also automatically satisfied if we identify $T_l(a, b) = t_l(a, b)$.

In order to enlarge this Lie algebra, let us assume (i, j, k) to be any cyclic permutation of $(0, 1, 2)$, and let γ_j $(j = 0, 1, 2)$ to be any non-zero constants . We now assume

(i)

$$[\rho_i(a), \rho_i(b)] = \gamma_j \gamma_k^{-1} T_{3-i}(a, b) \tag{3.5a}$$

(ii)

$$[\rho_i(a), \rho_j(b)] = -[\rho_j(b), \rho_i(a)] = -\gamma_j \gamma_i^{-1} \rho_k(\overline{ab}) \tag{3.5b}$$

(iii)

$$[T_l(a, b), \rho_j(c)] = -[\rho_j(c), T_l(a, b)] = \rho_j(t_{l+j}(a, b)c). \tag{3.5c}$$

Assuming $\rho_j(a)$ to be F-linear in $a \in A$, then

$$L_j = T_{3-j}(A, A) \oplus \rho_j(A) \tag{3.6}$$

yields Lie algebras for each value of $j = 0, 1, 2$. This generalizes Eqs.(3.1) and (3.2). Here, the indices l and j for $T_l(a, b)$ and for $\rho_j(a)$ are defined modulo 3.

Introducing the Jacobian in L by

$$J(X, Y, Z) = [[X, Y], Z] + [[Y, Z,], X] + [[Z, X], Y] \tag{3.7}$$

for $X, Y, Z \in L$, we can show (see Theorem 3.1. of [O.1])

Proposition 3.1 *Let A be a pre-structurable algebra. Then, the Jacobian $J(X, Y, Z)$ in L are identically zero except for the case of*

$$J(a, b, c) := J(\rho_0(a), \rho_1(b), \rho_2(c)) \tag{3.8a}$$

which is given by

$$J(a, b, c) = T_0(a, \overline{bc}) + T_1(c, \overline{ab}) + T_2(b, \overline{ca}). \tag{3.8b}$$

We next note that in view of Eq.(1.11b) and Theorem 2.1, we have

$$[J(a, b, c), \rho_j(d)] = \rho_j(Q(a, b, c)d), \tag{3.9a}$$

$$[J(a, b, c), T_j(d, f)] = T_j(Q(a, b, c)d, f) + T_j(d, Q(a, b, c)f). \tag{3.9b}$$

Suppose now that A is structurable. We then have $Q(a, b, c) = 0$ identically, and Eqs.(3.9) yields

$$[J(a, b, c), \rho_j(d)] = 0 = [J(a, b, c), T_i(d, f)] \tag{3.10}$$

so that $J(a, b, c)$ is a center element of L. Let J be a vector space spanned by all $J(a, b, c)$, $(a, b, c \in A)$. Then, the quotient algebra $\tilde{L} = L/J$ is a Lie algebra. Therefore, we can effectively set

$$J(a, b, c) = T_0(a, \overline{bc}) + T_1(c, \overline{ab}) + T_2(b, \overline{ca}) = 0. \tag{3.11}$$

Then, it is more economical to identify $T_j(a,b)$ with $t_j(a,b)$ itself or with a triple of

$$T(t_j(a,b), t_{j+1}(a,b), t_{j+2}(a,b)) \tag{3.12}$$

as in [A-F], [E.1] and [E.2]. In that case, Eqs.(3.4) and (3.11) are automatically satisfied by Eqs.(1.11b) and Eq.(X). See also the construction given in[K-O].

In what follows, we suppose now that A is structurable so that $L = L_0 + L_1 + L_2$ is a Lie algebra assuming the valitity of Eq. (3.11). A special case of $\gamma_0 = \gamma_1 = \gamma_2 = 1$ in Eq.(3.5) is of particular interest then, since L is invariant under a cyclic permutation group Z_3 given by

$$\rho_0(a) \to \rho_1(a) \to \rho_2(a) \to \rho_0(a), \tag{3.13}$$

$$T_0(a,b) \to T_2(a,b) \to T_1(a,b) \to T_0(a,b).$$

Actually, L is known to be invariant under a larger symmetric group S_4 (see [E-O,2], [K-O]), although we will not go into its detail.

We can visualize the structure of the Lie algebra L given by Eqs.(3.5) as in Fig.1, which exhibits a triality:

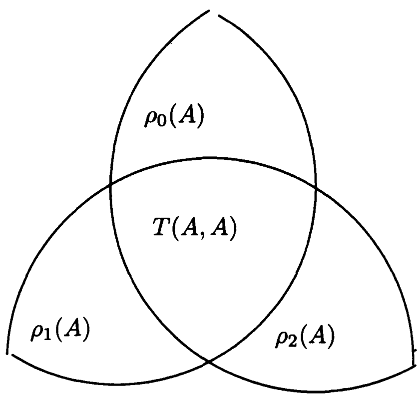


Fig.1 *Graphical Representation* of the Lie Algebra L.

In each branch of the tri-foglio in Fig.1,

$$\hat{L}_0 = \rho_0(A) \oplus T(A,A)$$
$$\hat{L}_1 = \rho_1(A) \oplus T(A,A)$$
$$\hat{L}_2 = \rho_2(A) \oplus T(A,A)$$

yields these sub-Lie algebra of L as in Eq.(3.6). They are isomorphic to each other and interchanges under the Z_3-group Eq.(3.13) as in

$$L_0 \to L_1 \to L_2 \to L_0.$$

Also, $T(A,A)$ is a sub-Lie algebra of L, which transforms among themselves under Z_3 as in Eq.(3.13).

We will give some examples below, assuming underlying field F to be algebraically closed of characterictic zero.

First let A be an octonion algebra which is structurable ([A-F]). In that case, these Lie algebras are $L = F_4$, $L_j = B_4$ for $j = 0,1,2$ and $T(A,A) = D_4$, corresponding to the case of the classical triality relation, as we may see from works of ([B-S], [E.1] and [E.2]).

If we choose A to be the Zorn's vector matrix algebra with Eq.(1.20) and (1.21), corresponding to B beging the cubic admissible algebra associated with the 27-dimentional exceptional Jordan algebra (i.e.Albert algebra), then A is structurable and the resulting Lie algebra L is of type E_8 (see e.g. [Kan] and [Kam.1]). Moreover, $L_j (j = 0,1,2)$ is a Lie algebra $E_7 \oplus A_1$ and $T(A,A)$ is realized to be isomorhic to $E_6 \oplus gl(1) \oplus gl(1)$. Another way of obtaining E_8 is to consider a structurable algebra $A = O_1 \otimes O_2$ of two octonion algebra O_1 and O_2 (see [A-F]). In that case, it is known (see [B-S],[E.1] and [E.2]) that it yields also the Lie algebra E_8. However, the sub-Lie algebra $L_j (j = 0, 1, 2)$ in this case are Lie algebra D_8, while $T(A, A)$ is $D_4 \oplus D_4$.

It may be worth-while to make the following comment here. The Lie algebra constructed by Eqs.(3.3)-(3.6) manifests the explicit Z_3 -symmetry (i.e. the triality), but not a 5-graded structure. On the other side, the standard construction of the Lie algebra on the basis of the (-1,1) Freudenthal-Kantor triple system[Y-O] is on the contrary explicitly 5-graded but not manifestly Z_3-invariant . The relationship between these two approaches has been studied in [K-O].

In ending of this note, from a geometrical point of view, we remark

Remark 3.2 Any simple structurable algebra A may be identified with some symmetric space as follows. First, if A is structurable, then we can construct Lie algebras by Eqs.(3.5) with Eqs.(3.4) and (3.13) from the standard construction based upon $(-1, 1)$Freudenthal-Kantor triple system [K-O].Then by Eqs.(3.5), we have

$$[\rho_0(a), \rho_0(b)] = (\gamma_2/\gamma_0)T_0(a, b)$$
$$[T_0(a, b), \rho_0(c)] = \rho_0(t_0(a, b)c)$$
$$[T_0(a, b), T_0(c, d)] = T_0(t_0(a, b)c, d) + T_0(c, t_0(a, b)d)$$

so that $\rho_0(A)$ may be identified with the symmetric space $L_0/T_0(A, A)$. because the tangent space of a symmetric space has a structure of Lie triple system. Further, the mapping $A \to \rho_0(A)$ is one-to-one if A is simple. To prove it, let

$$B = \{a | \rho_0(a) = 0, \ a \in A\}$$

and calculate

$$0 = [\rho_0(a), \rho_1(x)] = -(\gamma_1/\gamma_0)\rho_2(\overline{ax})$$
$$0 = [\rho_2(x), \rho_0(a)] = -(\gamma_0/\gamma_2)\rho_1(\overline{xa})$$

for any $a \in B$ and any $x \in A$, so that $\rho_1(\overline{xa}) = \rho_2(\overline{ax}) = 0$. Moreover, we compute

$$0 = [\rho_1(e), \rho_2(\overline{ax})] = -(\gamma_2/\gamma_1)\rho_0(ax)$$
$$0 = [\rho_1(\overline{xa}), \rho_2(e)] = -(\gamma_1/\gamma_2)\rho_0(xa)$$

which yields $\rho_0(ax) = \rho_0(xa) = 0$. Therefore B is a ideal of A and hence $A \to \rho_0(A)$ is one-to-one, if A is simple with $B = 0$. Note that the other possibility of $B = A$ leads to the trivial case of $\rho_0(A) = T_0(A, A) = 0$ identically.

Appendix

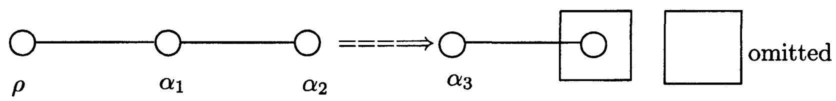
In this section, we discuss a characterization of the corresponding about the Lie algebras

$$L = \rho_1(A) \oplus \rho_2(A) \oplus \rho_3(A) \oplus T(A, A)$$

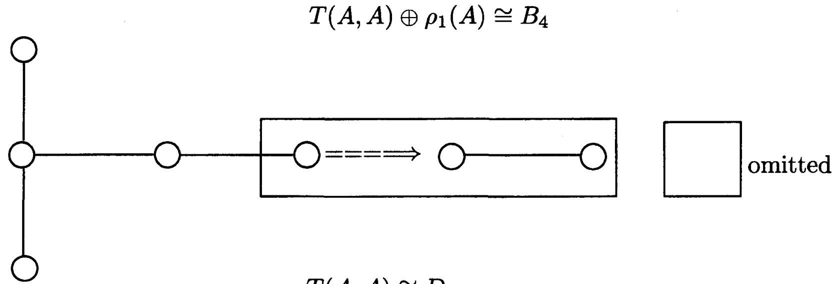
obtained from the structurable algebras A and the Dynkin diagrams associated with L. That is, in particular, we will study the cases of

$$A = \mathbf{O}, \ \mathbf{O} \otimes \mathbf{C}, \ \mathbf{O} \otimes \mathbf{H} \text{ and } \mathbf{O} \otimes \mathbf{O}.$$

(i) $A = \mathbf{O}$, $\rho = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ (F_4type)

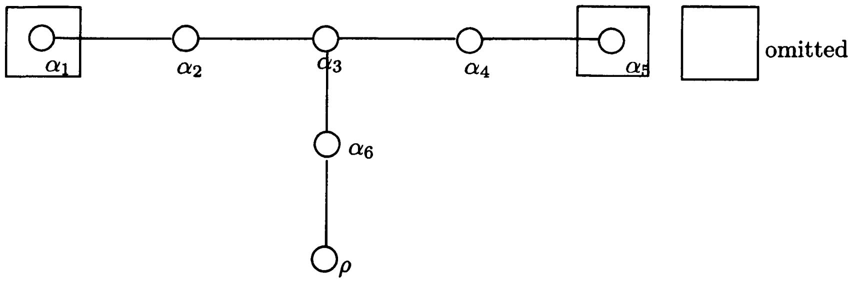


$T(A, A) \oplus \rho_1(A) \cong B_4$

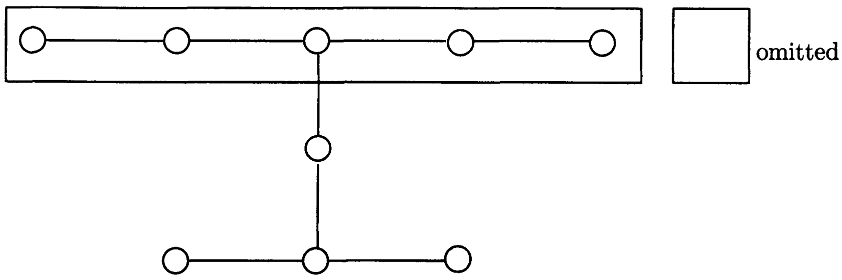


$T(A, A) \cong D_4$

(ii) $A = \mathbf{O} \otimes \mathbf{C}$, $\rho = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_4 + \alpha_5 + 2\alpha_6$ (E_6 type)

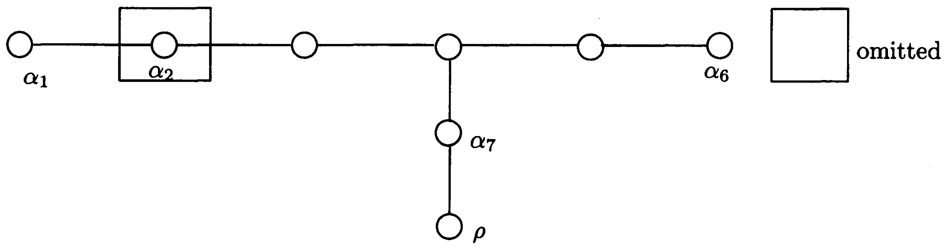


$T(A, A) \oplus \rho_1(A) \cong D_5 \oplus \mathbf{C}$

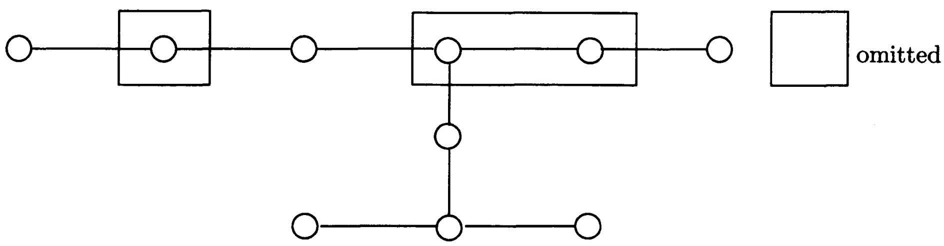


$T(A, A) \cong D_4 \oplus \mathbf{C} \oplus \mathbf{C}$

(iii) $A = \mathbf{O} \otimes \mathbf{H}$, $\rho = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_7$ (E_7 type)

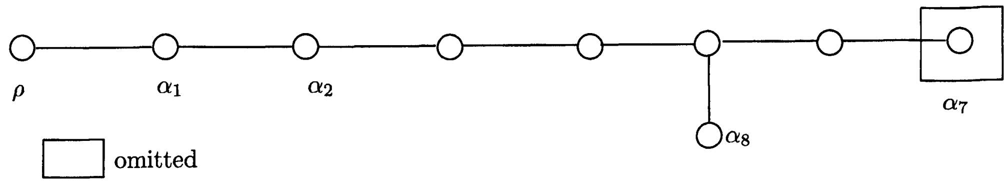


$T(A, A) \oplus \rho_1(A) \cong D_6 \oplus A_1$

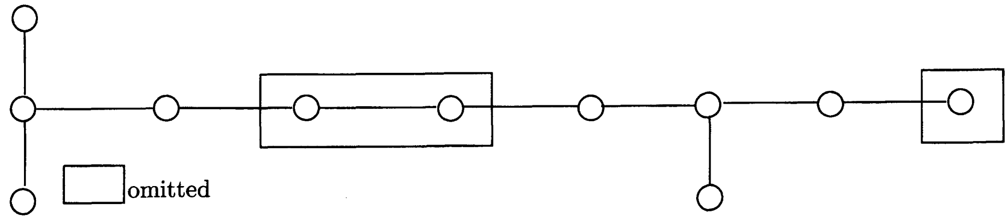


$T(A, A) \cong D_4 \oplus A_1 \oplus A_1 \oplus A_1$

(iv) $A = \mathbf{O} \otimes \mathbf{O}$. $\rho = 2\alpha_1 + 3\alpha_2 + \cdots + 2\alpha_7 + 3\alpha_8$ (E_8 type)



$$T(A,A) \oplus \rho_1(A) \cong D_8$$



$$T(A,A) \cong D_4 \oplus D_4$$

where

$$\{xyz\} = (x \cdot \bar{y}) \cdot z + (z \cdot \bar{y}) \cdot x - (z \cdot \bar{x}) \cdot y, \quad and \quad \overline{x \cdot y} = \bar{y} \cdot \bar{x}, \quad x, y, z \in A.$$

Concluding Remark

In final note of this paper, to make a gaidance of our concept, we give two propositions as follows.

Proposition A ([S]) (well known principle of triality) *Let A be a Cayley algebra of characteristic* $\neq 2, 3$ *with norm n(x) and o(8,n) be the orthogonal Lie algebra of all U in End A which are skew relative to n(x). For every U in o(8,n) there unique* U', U'' *in o(8,n) satisfying*

$$U(xy) = (U'x)y + x(U''y)$$

for all $x,\ y \in A$.

Proposition B ([K-O]) *Let L be the Lie algebra induced from* $\rho(A)$ *as in (3.3) and* $M = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ *be the standard embeding 5 graded Lie algebra associated with a structurable algebra A, such that* $L_{-1} = A$ *as in ([Kam.1], [K-O]). Then we have*

$$L \simeq M \ (as\ Lie\ algebra).$$

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