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Centralizing Monoids on a Three-Element Set Related to Majority Functions

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Abstract

We consider commutation between multi-variable functions defined over a fixed non-empty set A. A centralizing monoid M is a set of unary functions which commute with all members of some given set F of functions, where F is called a witness of M. In this paper, we focus on the case where A is a three-element set and determine all centralizing monoids which have ternary majority functions as their witnesses.

Keywords: clone; centralizing monoid; majority function

1 Introduction

For a fixed non-empty set A we consider multi-variable functions defined on A. A centralizer is the set of functions which commute with all members of some set F of functions and a centralizing monoid M is the unary part of some centralizer, that is, the set of unary functions of some centralizer. (Precise definitions will appear in Section 2.) The set F is called a witness of the centralizing monoid M. Centralizers and centralizing monoids have been studied for many years, sometimes under different names, by various authors (e.g., [Da79], [Sza85], etc.).

As a continuation of our previous work ([MR11] and [MR12]), we restrict our attention to the case where A is a three-element set. We choose the class of ternary majority functions and determine all centralizing monoids which have sets of ternary majority functions as their witnesses. It turns out that there exist considerably small number of centralizing monoids having majority functions as their witnesses. Note that the preliminary version of this paper appeared in [GMR15].

2 Definitions and Basic Facts

Let A be a finite set with $|A| > 1$. By an n-variable function defined over A we mean a map from A^n into A. As usual, we denote by $\mathcal{O}_A^{(n)}$, for $n \geq 1$, the set of n-variable functions defined over A, and by $\mathcal{O}_A$ the set of functions defined over A, i.e., $\mathcal{O}_A = \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}$. A function $e_i^n \in \mathcal{O}_A^{(n)}$, for $1 \leq i \leq n$, is called n-variable i-th *projection* if it satisfies $f(x_1, \ldots, x_i, \ldots, x_n) = x_i$ for any $(x_1, \ldots, x_n) \in A^n$. We denote by $\mathcal{J}_A$ the set of projections over A.

Given $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$, for $m, n \geq 1$, f *commutes* with g, or f and g *commute*, if the following equation holds for all $x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn} \in A$.

$$f(g(x_{11}, \ldots, x_{m1}), \ldots, g(x_{1n}, \ldots, x_{mn})) = g(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{m1}, \ldots, x_{mn}))$$

We write $f \perp g$ when f commutes with g. The relation $\perp$ on $\mathcal{O}_A$ is obviously symmetric.

In this paper we are concerned with commutation where one of the functions is unary, e.g., $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(1)}$. In such case, f and g commute if $f(g(x_1), \ldots, g(x_n)) = g(f(x_1, \ldots, x_n))$ holds for all $x_1, \ldots, x_n \in A$.

For a subset F of $\mathcal{O}_A$ the set of functions in $\mathcal{O}_A$ which commute with all members of F is denoted by F^*, i.e.,

$$F^* = \{ g \in \mathcal{O}_A \mid g \perp f \text{ for all } f \in F \}.$$

When $F = \{f\}$ we often write f^* instead of F^*. We also write F^{**} for $(F^*)^*$. A subset C of $\mathcal{O}_A$ is a *centralizer* if there is a subset $F \subseteq \mathcal{O}_A$ satisfying $C = F^*$. Evidently, for any $F \subseteq \mathcal{O}_A$ the centralizer F^* is a clone.

A non-empty subset M of $\mathcal{O}_A^{(1)}$ is a *monoid* if it is closed under composition and contains the identity. The unary part of any centralizer F^*, i.e., $F^* \cap \mathcal{O}_A^{(1)}$, is a monoid.

For a subset M of $\mathcal{O}_A^{(1)}$, the following conditions (i) and (ii) are equivalent to each other.

(i) $M = M^{**} \cap \mathcal{O}_A^{(1)}$

(ii) There exists a subset F of $\mathcal{O}_A$ such that $M = F^* \cap \mathcal{O}_A^{(1)}$

Remark For an algebra $\mathcal{A} = (A; F)$, a unary function $g \in \mathcal{O}_A^{(1)}$ is called an *endomorphism* of $\mathcal{A}$ if $f \perp g$ holds for all $f \in F$. Denote by $\mathrm{End}(\mathcal{A})$ the set of endomorphisms of $\mathcal{A}$. The conditions above are equivalent to the condition that there exists an algebra $\mathcal{A} = (A; F)$ such that $M = \mathrm{End}(\mathcal{A})$.

A subset M of $\mathcal{O}_A^{(1)}$ is a *centralizing monoid* if M satisfies the above conditions. (In this case, M is obviously a monoid.) By Item (ii) above, a centralizing monoid is the unary part, i.e., the subset consisting of all unary functions, of some centralizer. Henceforth, F in Item (ii) is called a *witness* of a centralizing monoid M. The centralizing monoid M is denoted by $M(F)$ if F is a witness of M. When F is $\{f\}$ we simply write $M(f)$ for $M(\{f\})$.

A centralizing monoid M is *maximal* if M is not $\mathcal{O}_A^{(1)}$ and there is no centralizing monoid M' satisfying $M \subset M' \subset \mathcal{O}_A^{(1)}$. (The symbol $\subset$ denotes proper inclusion.)

Some basic facts on centralizing monoids and witnesses follow. Recall that a function f in $\mathcal{O}_A$ is called a *minimal function* if (i) f generates a minimal clone C and (ii) f has the minimum arity among functions generating C.

Proposition 2.1 *([MR11])*

(1) *Every centralizing monoid has a finite subset F of $\mathcal{O}_A$ as its witness.*

(2) *Every maximal centralizing monoid has a singleton set as its witness. Moreover, this singleton set can be a set consisting of a minimal function.*

Now we review some known results on E_3. From now on, we concentrate on the case where the base set A is $E_3 = \{0, 1, 2\}$. We write $\mathcal{O}_3^{(n)}$ and $\mathcal{O}_3$ for $\mathcal{O}_A^{(n)}$ and $\mathcal{O}_A$, respectively.

The set of all minimal clones on E_3 was described by B. Csákány [Cs83]. There are 84 such clones. Among them, 7 minimal clones are generated by ternary majority functions. As usual, a majority function is defined as follows: For a ternary function f, f is a *majority function* if $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$ holds for all $x, y \in E_3$.

We have determined all maximal centralizing monoids on E_3 in [MR11]. There are 10 maximal centralizing monoids on E_3. More precisely:

(1) For every constant function c in $\mathcal{O}_3^{(1)}$, $M(c)$ is a maximal centralizing monoid.

(2) For every minimal majority function m in $\mathcal{O}_3^{(3)}$, $M(m)$ is a maximal centralizing monoid. (The number of such maximal centralizing monoids is 7.)

(3) Conversely, for any maximal centralizing monoid M, there exists a constant function or a minimal majority function f in $\mathcal{O}_3$ which satisfies $M = M(f)$.

A natural question to ask is whether these properties can be generalized from E_3 to E_k ($= \{0, \dots, k-1\}$) for $k > 3$. Item (1) has already been solved affirmatively. For items (2) and (3), however, the questions still remain open.

These facts motivate us to study centralizing monoids with majority functions as their witnesses. In this paper, we shall determine all centralizing monoids on E_3 with a set of ternary majority functions as their witnesses.

3 Majority Functions as Witness

In this section we shall determine all centralizing monoids on E_3 which have witnesses consisting only of ternary majority functions.

3.1 Notation

(1) The set of triples on E_3 consisting of mutually distinct components will be denoted by σ_3, i.e., $\sigma_3 = \{(a,b,c) \in (E_3)^3 \mid \{a,b,c\} = E_3\}$.

(2) A unary function s on E_3 is denoted by s_{abc} if $s(0) = a$, $s(1) = b$, $s(2) = c$ for $a, b, c \in E_3$.

(3) We divide the set of 2-valued unary functions in $\mathcal{O}_3^{(1)}$ into three groups, T_0, T_1 and T_2, as follows:

$$T_0 = \{s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221}\}$$
$$T_1 = \{s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212}\}$$
$$T_2 = \{s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122}\}$$

3.2 Properties for Commutation

3.2.1 Constants

It is evident that every constant function commutes with all majority functions.

3.2.2 2-Valued Unary Function

Lemma 3.1 *Let $m \in \mathcal{O}_3^{(3)}$ be a ternary majority function and $s \in \mathcal{O}_3^{(1)}$ be a 2-valued unary function with range $\{\alpha, \beta\}$ $(\subset E_3)$ satisfying $|s^{-1}(\alpha)| = 2$ (and hence $|s^{-1}(\beta)| = 1$). Then, s and m commute if and only if $m(a,b,c) \in s^{-1}(\alpha)$ for every $(a,b,c) \in \sigma_3$.*

Proof First, assume $s \perp m$ and let (a,b,c) be a triple in σ_3. Commutation of s and m implies $s(m(a,b,c)) = m(s(a), s(b), s(c))$. By assumption on α, two of $s(a), s(b)$ and $s(c)$ are equal to α and, hence, $m(s(a), s(b), s(c)) = \alpha$. Thus, we have $s(m(a,b,c)) = \alpha$ as desired.

Conversely, assume $s(m(a,b,c)) = \alpha$ for any $(a,b,c) \in \sigma_3$. Due to the assumption on α we have $m(s(a), s(b), s(c)) = \alpha$, showing the commutation of s and m on σ_3. On the other hand, $s(m(a,b,c)) = m(s(a), s(b), s(c))$ for any $(a,b,c) \in (E_3)^3 \setminus \sigma_3$. Hence we have $s \perp m$. □

For ternary majority functions $m \in \mathcal{O}_3^{(3)}$, we define three kinds of conditions (A0), (A1), (A2) regarding the values of m on σ_3. Note that the values of m on $(E_3)^3 \setminus \sigma_3$ are determined by definition.

(A0) $m(x,y,z) \in \{1,2\}$ for all $(x,y,z) \in \sigma_3$

(A1) $m(x,y,z) \in \{0,2\}$ for all $(x,y,z) \in \sigma_3$

(A2) $m(x,y,z) \in \{0,1\}$ for all $(x,y,z) \in \sigma_3$

Then, a condition for commutation between ternary majority functions and 2-valued unary functions is obtained as an immediate consequence of Lemma 3.1.

Corollary 3.2 *For a ternary majority function $m \in \mathcal{O}_3^{(3)}$ the following holds.*

(0) *For $s \in T_0$, $s \perp m \iff m$ satisfies (A0)*

(1) *For $s \in T_1$, $s \perp m \iff m$ satisfies (A1)*

(2) *For $s \in T_2$, $s \perp m \iff m$ satisfies (A2)*

3.2.3 Permutation

We consider commutation between permutations and ternary functions of some kind.

Lemma 3.3 *Let $f \in \mathcal{O}_3^{(3)}$ be a ternary function on E_3 satisfying $f(x,y,x) = f(x,x,z) = x$ for all $x,y,z \in E_3$. Then, for each permutation $s \in \mathcal{O}_3^{(1)}$ on E_3 the following holds.*

(1) *Let $\{a,b,c\} = E_3$. For a transposition $s = (a\,b)$, $s \perp f$ if and only if each of the following sets is either $\{a,b\}$ or $\{c\}$:*

$$\{f(a,b,b),\ f(b,a,a)\},\ \{f(a,c,c),\ f(b,c,c)\},\ \{f(c,a,a),\ f(c,b,b)\},$$
$$\{f(a,b,c),\ f(b,a,c)\},\ \{f(a,c,b),\ f(b,c,a)\},\ \{f(c,a,b),\ f(c,b,a)\}$$

(2) *For $s = (0\,1\,2)$ or $(0\,2\,1)$, $s \perp f$ if and only if each of the following triples belongs to $\{(0,1,2),(1,2,0),(2,0,1)\}$:*

$$(\,f(0,1,1),\ f(1,2,2),\ f(2,0,0)\,),$$
$$(\,f(0,1,2),\ f(1,2,0),\ f(2,0,1)\,),$$
$$(\,f(0,2,1),\ f(1,0,2),\ f(2,1,0)\,),$$
$$(\,f(0,2,2),\ f(1,0,0),\ f(2,1,1)\,)$$

The proof is straightforward from the definition of commutation $s \perp f$. Observe that ternary majority functions satisfy the assumption imposed on f in Lemma 3.3.

For ternary majority functions $m \in \mathcal{O}_3^{(3)}$, we define four kinds of conditions, (Ci) ($0 \le i \le 3$), determined by the values of m on σ_3. First, for every $i \in \{0,1,2\}$, (Ci) is given as follows.

(Ci) Each of the following sets is $\{i\}$ or $\{a,b\}$ where $\{i,a,b\} = E_3$:

$$\{m(i,a,b),\ m(i,b,a)\},\ \{m(a,i,b),\ m(b,i,a)\},\ \{m(a,b,i),\ m(b,a,i)\}.$$

Then, (C3) is the following.

(C3) Each of the following triples belongs to $\{(0,1,2),(1,2,0),(2,0,1)\}$.

$$(m(0,1,2),\ m(1,2,0),\ m(2,0,1)),\ \ (m(0,2,1),\ m(1,0,2),\ m(2,1,0))$$

We have the conditions connecting permutations and properties (Ci)'s.

Corollary 3.4 *For a ternary majority function $m \in \mathcal{O}_3^{(3)}$ the following holds.*

(0) *For $s=(1\,2)$, $s \perp m \iff m$ satisfies (C0)*

(1) *For $s=(0\,2)$, $s \perp m \iff m$ satisfies (C1)*

(2) *For $s=(0\,1)$, $s \perp m \iff m$ satisfies (C2)*

(3) *For $s \in \{(0\,1\,2),\ (0\,2\,1)\}$, $s \perp m \iff m$ satisfies (C3)*

3.2.4 Combined Conditions

We shall investigate combinations of conditions from (Ai)'s ($0 \le i \le 2$) and (Cj)'s ($0 \le j \le 3$).

(i) **A-group**
Combined conditions of (Ai)'s are the following. For $i, j \in E_3$ such that $0 \le i < j \le 2$,

$$(\mathrm{A}i)\wedge(\mathrm{A}j) \qquad m(\sigma_3) \;=\; E_3 \setminus \{i,\, j\}.$$

Obviously, no majority functions satisfy all three conditions (A0)∧(A1)∧(A2).

The collection of conditions (Ai), $0 \le i \le 2$, and (Ai)∧(Aj), $0 \le i < j \le 2$, will be called A-group.

(ii) **C-group**
Concerning combined conditions of (Ci) ($0 \le i \le 3$) for majority functions, any combination of two, three or four conditions of (Ci) yields the same condition, which we denote by (C*).

$$(\mathrm{C}^*) \qquad m|_{\sigma_3} = e^3_\ell|_{\sigma_3} \ \text{ for some } \ \ell \in \{1,2,3\}$$

(Note that, in general, $f|_S$ denotes the restriction of f to a subset S of the domain of f.)

The collection of conditions (Ci), $1 \le i \le 4$, and (C*) will be called C-group.

(iii) **Compounds of A-group and C-group**
What remains to be considered is the collection of compound cases: one from A-group and one from C-group.

For mutually distinct $i, j, \ell \in E_3$ (that is, $\{i, j, \ell\} = E_3$), assuming $i < j$, the compound condition of (Ai)∧(Aj) and (Cℓ) is as follows.

$$((\mathrm{A}i)\wedge(\mathrm{A}j))\wedge(\mathrm{C}\ell) \qquad m(\sigma_3) \;=\; \{\ell\}$$

This is the same as $m(\sigma_3) \;=\; E_3 \setminus \{i, j\}$, which appeared above as (Ai)∧(Aj). Moreover, combined conditions (Ai)∧(Cℓ) for $i, \ell \in E_3$ with $i \ne \ell$ also result in the same condition as above.

Next, the combined condition (Ai)∧(Ci) for $i \in E_3$ is as follows.

(Ai)∧(Ci) Each of the following sets is $\{a, b\}$:
$\{m(i,a,b),\ m(i,b,a)\}$, $\{m(a,i,b),\ m(b,i,a)\}$,
$\{m(a,b,i),\ m(b,a,i)\}$

For all other combined conditions such as (A0)∧(C3), (A1)∧(C3), etc., no majority function satisfies such sets of conditions.

3.2.5 Results

We present all centralizing monoids which have witnesses of ternary majority functions. It is readily, and interestingly, verified that every centralizing monoid which has a non-empty set of majority functions as its witness has a singleton set (of a majority function) as its witness.

Centralizing monoids on E_3 having majority functions as their witnesses

(1) $L_0 = \{\, c_0,\, c_1,\, c_2 \,\} \cup \{\, \mathrm{id},\, s_{021} \,\}$
Number of elements: 5
Property of witness: (C0)
Example of witness: $m(0,1,2) = m(0,2,1) = m(1,0,2) = m(2,0,1) = 0$,
$m(1,2,0) = 1,\ m(2,1,0) = 2$

(2) $L_1 = \{ c_0, c_1, c_2 \} \cup \{ \mathrm{id}, \mathrm{s}_{210} \}$
Number of elements: 5
Property of witness: (C1)
Example of witness: $m(1,0,2) = m(1,2,0) = m(0,1,2) = m(2,1,0) = 1$,
$m(0,2,1) = 0$, $m(2,0,1) = 2$

(3) $L_2 = \{ c_0, c_1, c_2 \} \cup \{ \mathrm{id}, \mathrm{s}_{102} \}$
Number of elements: 5
Property of witness: (C2)
Example of witness: $m(2,0,1) = m(2,1,0) = m(0,2,1) = m(1,2,0) = 2$,
$m(0,1,2) = 0$, $m(1,0,2) = 1$

(4) $L_3 = \{ c_0, c_1, c_2 \} \cup \{ \mathrm{id}, \mathrm{s}_{120}, \mathrm{s}_{201} \}$
Number of elements: 6
Property of witness: (C3)
Example of witness: $m(0,1,2) = 0$, $m(1,2,0) = 1$, $m(2,0,1) = 2$,
$m(0,2,1) = 1$, $m(1,0,2) = 2$, $m(2,1,0) = 0$

(5) $L_4 = \{ c_0, c_1, c_2 \} \cup \{ \mathrm{id}, \mathrm{s}_{021}, \mathrm{s}_{102}, \mathrm{s}_{120}, \mathrm{s}_{201}, \mathrm{s}_{210} \}$
Number of elements: 9
Property of witness: (C*)
Example of witness: $m(0,1,2) = 0$, $m(1,2,0) = 1$, $m(2,0,1) = 2$,
$m(0,2,1) = 0$, $m(1,0,2) = 1$, $m(2,1,0) = 2$

(6) $M_0 = \{ c_0, c_1, c_2 \} \cup \{ s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221} \} \cup \{ \mathrm{id} \}$
Number of elements: 10
Property of witness: (A0)
Example of witness: $m(0,1,2) = 1$, $m(1,2,0) = 1$, $m(2,0,1) = 2$,
$m(0,2,1) = 1$, $m(1,0,2) = 1$, $m(2,1,0) = 2$

(7) $M_1 = \{ c_0, c_1, c_2 \} \cup \{ s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212} \} \cup \{ \mathrm{id} \}$
Number of elements: 10
Property of witness: (A1)
Example of witness: $m(0,1,2) = 0$, $m(1,2,0) = 0$, $m(2,0,1) = 2$,
$m(0,2,1) = 0$, $m(1,0,2) = 0$, $m(2,1,0) = 2$

(8) $M_2 = \{ c_0, c_1, c_2 \} \cup \{ s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122} \} \cup \{ \mathrm{id} \}$
Number of elements: 10
Property of witness: (A2)
Example of witness: $m(0,1,2) = 0$, $m(1,2,0) = 0$, $m(2,0,1) = 1$,
$m(0,2,1) = 0$, $m(1,0,2) = 0$, $m(2,1,0) = 1$

(9) $LM_0 = \{ c_0, c_1, c_2 \} \cup \{ s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221} \} \cup \{ \mathrm{id}, \mathrm{s}_{021} \}$
Number of elements: 11
Property of witness: (A0) + (C0)
Example of witness: $m(0,1,2) = 1$, $m(0,2,1) = 2$, $m(1,0,2) = 2$, $m(2,0,1) = 1$,
$m(1,2,0) = 1$, $m(2,1,0) = 2$

(10) $LM_1 = \{ c_0, c_1, c_2 \} \cup \{ s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212} \} \cup \{ \mathrm{id}, \mathrm{s}_{210} \}$
Number of elements: 11
Property of witness: (A1) + (C1)
Example of witness: $m(1,0,2) = 0$, $m(1,2,0) = 2$, $m(0,1,2) = 2$, $m(2,1,0) = 0$
$m(0,2,1) = 0$, $m(2,0,1) = 2$

(11) $LM_2 = \{ c_0, c_1, c_2 \} \cup \{ s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122} \} \cup \{ \mathrm{id}, \mathrm{s}_{102} \}$
Number of elements: 11
Property of witness: (A2) + (C2)
Example of witness: $m(2,0,1) = 0,\ m(2,1,0) = 1,\ m(0,2,1) = 1,\ m(1,2,0) = 0$
$m(0,1,2) = 0,\ m(1,0,2) = 1$

(12) $LM_3 = \{ c_0, c_1, c_2 \} \cup \{ s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212}, s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122} \}$
$\cup \{ \mathrm{id}, \mathrm{s}_{021} \}$
Number of elements: 17
Property of witness: (A1) + (A2) + (C0)
Example of witness: $m(0,1,2) = 0,\ m(0,2,1) = 0,\ m(1,0,2) = 0,\ m(1,2,0) = 0,$
$m(2,0,1) = 0,\ m(2,1,0) = 0$

(13) $LM_4 = \{ c_0, c_1, c_2 \} \cup \{ s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221}, s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122} \}$
$\cup \{ \mathrm{id}, \mathrm{s}_{210} \}$
Number of elements: 17
Property of witness: (A0) + (A2) + (C1)
Example of witness: $m(0,1,2) = 1,\ m(0,2,1) = 1,\ m(1,0,2) = 1,\ m(1,2,0) = 1,$
$m(2,0,1) = 1,\ m(2,1,0) = 1$

(14) $LM_5 = \{ c_0, c_1, c_2 \} \cup \{ s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221}, s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212} \}$
$\cup \{ \mathrm{id}, \mathrm{s}_{102} \}$
Number of elements: 17
Property of witness: (A0) + (A1) + (C2)
Example of witness: $m(0,1,2) = 2,\ m(0,2,1) = 2,\ m(1,0,2) = 2,\ m(1,2,0) = 2,$
$m(2,0,1) = 2,\ m(2,1,0) = 2$

(15) $K = \{ c_0, c_1, c_2 \} \cup \{ \mathrm{id} \}$
Number of elements: 4
Property of witness: None in A-group and C-group
Example of witness: $m(0,1,2) = 0,\ m(0,2,1) = 1,\ m(1,0,2) = 1,\ m(1,2,0) = 2,$
$m(2,0,1) = 2,\ m(2,1,0) = 2$

To summarize briefly:

Proposition 3.5 *The number of centralizing monoids on E_3 which have sets of majority functions as their witnesses is* 15.

We observe that the number 15 is remarkably small as compared with the number of ternary majority functions, that is, $3^6 = 729$.

Remark Similar results were obtained for semiprojections in place of majority functions (See [GMR15]).

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