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Continuous bands on a real interval *

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1 Semigroups on a real interval

A *semigroup* S *on* a real interval I is a semigroup $S=(I, *)$ such that its operation $*: I\times I\rightarrow I$ is continuous with respect to the ordinary topology and compatible with the ordinary order. For $a\in \mathbb{R}\cup\{-\infty\}$ and $b\in \mathbb{R}\cup\{+\infty\}$ with $a\leq b$, let

$$\mathbb{R}(a, b)=\{x\in \mathbb{R}\,|\,a<x<b\}$$

and

$$\mathbb{R}[a, b]=\{x\in \mathbb{R}\,|\,a\leq x\leq b\}.$$

The following results are classical.

Theorem 1 (Abel 1826, Aczél 1949). *Any group on* $\mathbb{R}$ *is isomorphic to* $(\mathbb{R}, +)$.

Theorem 2 (Craigen & Pales 1989). *There are exactly three cancellative semigroups on* $\mathbb{R}_{+}=\mathbb{R}(0, +\infty)$ *up to isomorphism. They are* $(\mathbb{R}_{+}, \times)$, $(\mathbb{R}_{+}, +)$ *and* $(\mathbb{R}_{+}, \star)$, *where* $\star$ *is given by* $x\star y=x+y+1$ *for* $x, y\in \mathbb{R}_{+}$.

On the other hand there are many non-cancellative semigroups on a real interval I.

Example 1. *Let* $x, y\in I$.

(a) *The null semigroup: For a fixed* $a\in I$ *define the operation* $*$ *on* I *by*

$$x*y=a.$$

(b) *The left (resp. right) zero semigroup: Define* $*$ *by*

$$x*y=x \quad (\text{resp. } y).$$

(c) *The max (resp. min) semigroup: Define* $*$ *by*

$$x*y=\max\{x, y\} \quad (\text{resp. } \min\{x, y\}).$$

In the above example, the semigroups $(I, *)$ given in **(b)** and **(c)** are bands, that is, all the elements of them are idempotents.

*This is a final version and will not appear elsewhere.

2 Bands

Let I be a real interval, and $B = (I, *)$ be a band on I. B is a semigroup on I in which all the elements are idempotents, that is, $* : I \times I \to I$ is continuous and order compatible and $x * x = x$ for all $x \in I$.

Two bands are *isomorphic* if there is a continuous isomorphism between them. They are *order-isomorphic* if there is an order-preserving continuous isomorphism.

For $e \in S$, the left (resp. right) transformation L_e (resp. R_e), defined by

$$L_e(x) = e * x, \quad (\text{resp. } R_e(x) = x * e)$$

for $x \in I$, is a continuous increasing function. Since e is an idempotent, we see

$$L_e(L_e(x)) = L_{e^2}(x) = L_e(x).$$

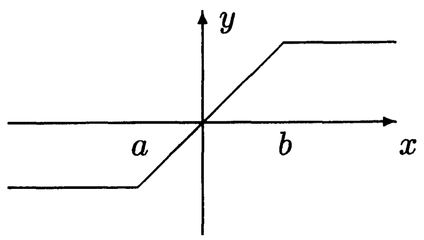
Hence, we have

$$\mathrm{Im}(L_e) = \{x \in I \mid L_e(x) = x\}.$$

Because L_e is continuous and increasing, there are $a \in I \cup \{-\infty\}$ and $b \in I \cup \{+\infty\}$ such that $a \leq b$, $\mathrm{Im}(L_e) = \mathbb{R}[a, b]$, and

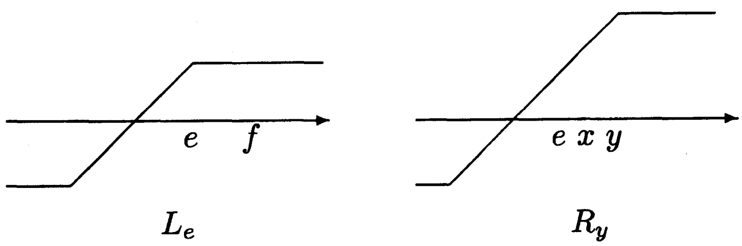
$$L_e(x) = \begin{cases} a & \text{if } x < a \\ x & \text{if } a \leq x \leq b \\ b & \text{if } x > b \end{cases}$$

for $x \in I$. So, the graph of L_e is of the following shape:



Similar results also holds for R_e.

Now, suppose that $e * f = e < f$ for $e, f \in B$. Then, $\mathrm{Im}(L_e) = \mathbb{R}[a, e]$ for $a \in \mathbb{R} \cup \{-\infty\}$. Thus, for any $y \geq e$ we see $e * y = L_e(y) = e$. Hence, $e \in \mathrm{Im}(R_y)$, and therefore, $\mathbb{R}[e, y] \subset \mathrm{Im}(R_y)$. This implies that $x * y = x$ for all x with $e \leq x \leq y$ (see the graphs of L_e and R_y below).

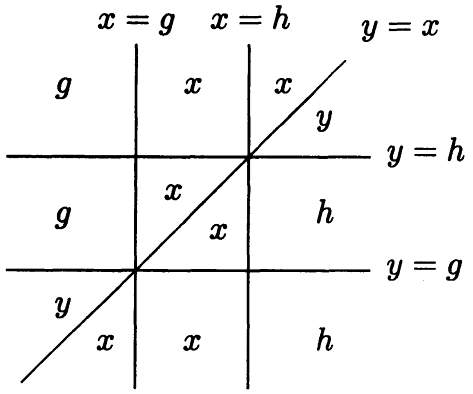


Repeating these arguments we can determine the structure of B.

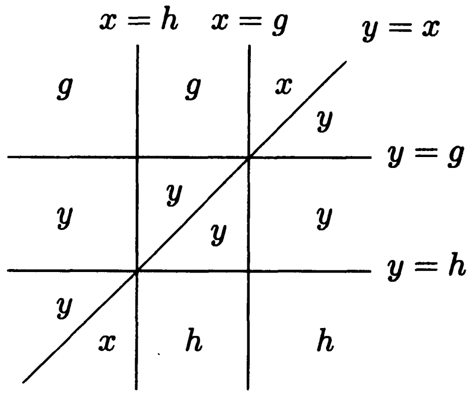
Let $g, h \in \mathbb{R} \cup \{-\infty, +\infty\}$, and let $B(I, g, h)$ be the semigorup on I with the operation $*$ given by

$$x * y = \begin{cases} y & \text{if } x \leq y \leq g, \\ x & \text{if } g \leq x \leq y, \\ g & \text{if } x \leq g \leq y, \\ x & \text{if } y \leq x \leq h, \\ y & \text{if } h \leq y \leq x, \\ h & \text{if } y \leq h \leq x, \end{cases} \tag{1}$$

for $x, y \in I$. We describe the operation table of $*$ on xy-plane by entering $x * y$ at the position (x, y). The operation given in (1) is exhibited as below (in the cases $g < h$ and $h < g$, the frames are the same but the contents are different).



$B(g, h)$, $g < h$



$B(g, h)$, $g > h$

3 Main results

Let $\alpha \in \mathbb{R} \cup \{-\infty\}$ and $\beta \in \mathbb{R} \cup \{+\infty\}$ such that $\alpha < \beta$, and let I be an (open, closed or half open) inteval between α and β. The results ([4]) on bands on $\mathbb{R}$ are valid on an arbitrary interval I.

Theorem 3. *For any $g, h \in \mathbb{R}[\alpha, \beta]$, $B(I, g, h)$ is a band on I, and any band on I is equal to $B(I, g, h)$ for some g, h.*

Suppose that $\alpha < 0$ and $1 < \beta$. We abbreviate $B(I, g, h)$ as $B(g, h)$.

Theorem 4. **(a)** *There are exactly 11 distinct bands on I up to order-isomorphism. They are*

$$B(0,1),\ B(1,0),\ B(0,0),\ B(\alpha,0),\ B(0,\beta),$$
$$B(\beta,0),\ B(0,\alpha),\ B(\alpha,\beta),\ B(\beta,\alpha),\ B(\beta,\beta),\ B(\alpha,\alpha).$$

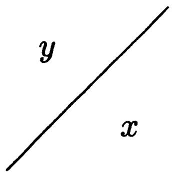
(b) *There are exactly 8 distinct bands on I up to isomorphism. They are*

$$B(0,1),\ B(1,0),\ B(0,0),\ B(\alpha,0),\ B(\beta,0),\ B(\alpha,\beta),\ B(\beta,\alpha),\ B(\beta,\beta).$$

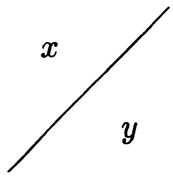
(c) *There are exactly 5 distinct bands on I up to isomorphism and anti-isomorphism. They are*

$$B(0,1),\ B(0,0),\ B(\alpha,0),\ B(\alpha,\beta),\ B(\beta,\beta).$$

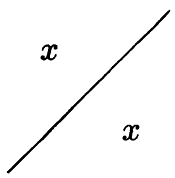
The operation tables of the special cases in our theorems are shown below:



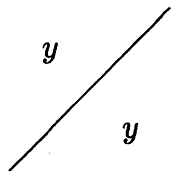
$B(\beta,\beta)$: max



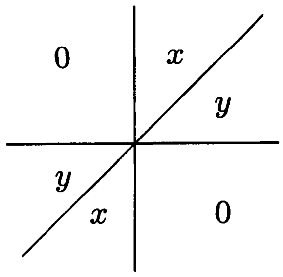
$B(\alpha,\alpha)$: min



$B(\alpha,\beta)$: left zero



$B(\beta,\alpha)$: right zero



B(0, 0): max-min

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