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EQUIVARIANT INTERTWINING OPERATORS FOR TWISTED MODULES

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ABSTRACT. We outline how work of Nagatomo-Tsuchiya and Frenkel-Szczesny on conformal blocks can be used to describe intertwining operators for twisted modules of regular vertex operator algebras. As an application in moonshine, the simple current property in the holomorphic case allows for the construction of abelian intertwining algebras and Lie algebras with actions of large finite groups.

CONTENTS

INTRODUCTION

One of the key goals in the Borcherds-Höhn program for generalized moonshine is a way to construct, for each element g in the monster simple group, an infinite dimensional Lie algebra equipped with a certain well-behaved projective action of the centralizer of g [1]. To date, the only known way to construct such Lie algebras with exceptional diagram symmetry is by using a string quantization functor, so we wish to assemble a suitable input object for the functor. This paper is concerned with the last substantial remaining piece of the assembly process. The proofs and fine details, assuming I finish writing them, should appear as a series of papers starting with [Carnahan$\geq$2015a] and culminating in [Carnahan$\geq$2015z].

The main result to be outlined is the following:

Theorem 1. *Let V be a holomorphic C_2-cofinite vertex operator algebra, and let G be a finite group of automorphisms of V. If $g_1, g_2, g_3 \in G$, such that $g_3 = g_1 g_2 = g_2 g_1$, and M_i are irreducible g_i-twisted V-modules for $i = 1, 2, 3$, then the space of intertwining operators of type $\binom{M_3}{M_1 M_2}$ is one dimensional.*

The proof follows from a common generalization of the following work:

(1) In [Zhu-1994], we find that for modules of a C_2-cofinite vertex operator algebra, the space of intertwining operators is isomorphic to a space of genus zero 3-point conformal blocks.
(2) In [Frenkel-Szczesny-2004], the formalism of conformal blocks is generalized to allow insertions of (admissible) twisted modules.
(3) In [Nagatomo-Tsuchiya-2005], we find that genus zero conformal blocks for modules of a regular vertex operator algebra satisfy several convenient permanence properties.

[1]The program is implicit in [Hoehn-2003], where Höhn generalized Borcherds's proof [Borcherds-1992] of the Conway-Norton conjecture [Conway-Norton-1979] for $V^\natural$ to the 2A case of Norton's generalized moonshine conjecture [Norton-1987]. The precise conditions on the action are given at the ends of [Carnahan-2010] and [Carnahan-2012].

In my dissertation [Carnahan-2007], I sketched such a generalization by following the pattern behind these papers. However, a substantial problem was that for any step whose proof was skipped in the source material, I assumed it would work without change in the more general setting. I have been thoroughly filling in these details, and this process has expanded the length of the treatment by a surprisingly large factor.

As it happens, for our applications in generalized moonshine, we need a stronger statement, involving intertwining operators between more than 3 twisted modules and some equivariance. However, the methods of proof we have chosen apply to both the theorem stated here and the stronger form.

1. Technical input

I'll describe the primary mathematical ingredients in the proof.

1.1. **Vertex operator algebras and twisted modules.** A vertex operator algebra (over $\mathbb{C}$) of central charge $c \in \mathbb{C}$ is a complex vector space V equipped with distinguished vectors 1 and ω, together with a multiplication map $m_z : V \otimes V \to V((z))$, satisfying the following conditions:

1. Identity: For all $v \in V$, $m_z(1 \otimes v) = vz^0$, and $m_z(v \otimes 1) \in v + zV[[z]]$.
2. Conformal element: Left-multiplication by ω yields a power series $\sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ of operators $L(n) \in \operatorname{End} V$ satisfying the Virasoro relations: $[L(m)L(n)] = (m-n)L(m+n) + \frac{n^3-n}{12}\delta_{m+n,0}c \cdot Id_V$, for some fixed $c \in \mathbb{C}$.
3. Translation: The operator $L(-1)$ satisfies the condition $L(-1)m_z(u \otimes v) - m_z(u \otimes L(-1)v) = \frac{d}{dz}m_z(u \otimes v)$ for all $u, v \in V$.
4. Grading: The operator $L(0)$ acts semisimply on V, and the eigenvalues form a set of integers that are bounded below, each with finite multiplicity.
5. Locality: The following diagram commutes:

$$\begin{tikzcd}
V\otimes V\otimes V \arrow[rr, "\tau_{(12)}"] \arrow[d, "1\otimes m_w"'] & & V\otimes V\otimes V \arrow[d, "1\otimes m_z"] \\
V\otimes V((w)) \arrow[dr] \arrow[dd, "m_z((w))"'] & & V\otimes V((z)) \arrow[dl] \arrow[dd, "m_w((z))"] \\
& V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}] \arrow[dl, "\iota_{z,w}"'] \arrow[dr, "\iota_{w,z}"] & \\
V((z))((w)) & & V((w))((z))
\end{tikzcd}$$

Vertex operator algebras were introduced in [Frenkel-Lepowsky-Meurman-1988] as an enhancement of vertex algebras, which were introduced in [Borcherds-1986]. Instead of a Virasoro action, a vertex algebra just has $L(-1)$, and does not necessarily satisfy any finiteness conditions.

A vertex operator algebra is C_2-cofinite if the subspace of V spanned by the z^1-coefficients of all pairwise multiplications has finite codimension in V. As it happens, this condition is equivalent to many natural finiteness conditions on the representation theory of V - see e.g., the beginning of [Miyamoto-2004] for a list.

If g is a finite-order automorphism of a vertex operator algebra V of central charge c, then a g-twisted module is a vector space M, equipped with an action map $act_z : V \otimes M \to M((z^{1/|g|}))$, satisfying the following conditions:

1. Identity: $act_z(1 \otimes x) = xz^0$ for all $x \in M$.
2. Conformal element: We define the operators $L(n)^M$ by $act_z(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)^M xz^{-n-2}$, and demand that they describe an action of the Virasoro algebra on M, of central charge c.

(3) Translation: $L(-1)^M act_z(u \otimes x) - act_z(u \otimes L(-1)^M x) = act_z(L(-1)u \otimes x)$.
(4) Grading: $L(0)^M$ acts semisimply on M, so that $M = \bigoplus_{n \in \mathbb{C}} M_n$ is an eigenvalue decomposition. We demand that the eigenvalues have finite multiplicity, and are bounded below in each coset of $\mathbb{Z}$.
(5) Compatibility with multiplication - The following diagram commutes:

$$\begin{tikzcd}
 & & V \otimes V \otimes M \arrow[dll, "m_{z-w}\otimes 1"'] \arrow[drr, "1\otimes act_w"] & & \\
V((z-w)) \otimes M \arrow[drr] \arrow[dd, "act_w"'] & & & & V \otimes M((w^{1/N})) \arrow[dll] \arrow[dd, "act_z"] \\
 & & M[[z^{1/N}, w^{1/N}]][z^{-1}, w^{-1}, (z-w)^{-1}] \arrow[dll, "\iota_{w,z-w}"'] \arrow[drr, "\iota_{z,w}"] & & \\
M((w^{1/N}))((z-w)) & & & & M((z^{1/N}))((w^{1/N}))
\end{tikzcd}$$

(6) Monodromy: For all $u \in V$ and $v \in M$, $act_z(gu \otimes v) = act_{e^{2\pi i}z}(u \otimes v)$. In other words, if $gu = e^{2\pi i k/N}u$, then $act_z(u \otimes v) \in z^{k/N}M((z))$.

Twisted modules were essentially introduced in [Frenkel-Lepowsky-Meurman-1988] as part of the construction of the monster vertex algebra.

A vertex operator algebra V is called rational if the category of 1-twisted V-modules satisfies complete reducibility, and we say that V is holomorphic if it is rational and any irreducible object is isomorphic to V. Examples of holomorphic C_2-cofinite vertex operator algebras include the monster vertex algebra [Dong-1994, Dong-Li-Mason-2000], and the vertex algebra attached to any positive definite even unimodular lattice [Dong-1993].

Let V be a vertex algebra with an action of a group G, and let g_1 and g_2 be commuting finite order elements of G. Let M_1 be a g_1-twisted V-module, let M_2 be a g_2-twisted V-module, and let M_3 be a g_1g_2-twisted V-module. An intertwining operator of type $\binom{M_3}{M_1 M_2}$ is a map $I_z : M_1 \otimes M_2 \to M_3\{z\}$ satisfying the following conditions:

(1) Translation covariance: $L(-1)^{M_3} I_z(u \otimes v) - I_z(u \otimes L(-1)^{M_2} v) = \frac{d}{dz} I_z(u \otimes v)$.
(2) V-compatibility: For elements $r_1, r_2 \in \mathbb{Q}/\mathbb{Z}$, let V^{r_1,r_2} denote the subspace of V on which g_1 acts by $e^{2\pi i r_1}$ and g_2 acts by $e^{2\pi i r_2}$. Then the map $act_z \circ (1 \otimes I_w) : V^{r_1,r_2} \otimes M_1 \otimes M_2 \to z^{r_1+r_2} M_3((z))\{w\}$ and the map $I_w \circ (act_{z-w} \otimes 1) : V^{r_1,r_2} \otimes M_1 \otimes M_2 \to (z-w)^{r_1} M_3\{w\}((z-w))$ factor through the space $z^{r_2}(z-w)^{r_1} M_3[[z]]\{w\}[z^{-1}, (w-z)^{-1}]$ where they coincide.
(3) Truncation: For any $u \in M_1$, $v \in M_2$, and any $c \in \mathbb{C}$, there exists $n_c \in \mathbb{Z}$ such that for all $n < n_c$, the z^{c+n} term in $I_z(u \otimes v)$ vanishes.

As I understand, these were introduced informally in [Feingold-Frenkel-Ries-1991] and formally in [Xu-1995]. One may define a notion of intertwining operator between twisted modules whose twisting elements don't commute, but the V-compatibility axiom becomes more complicated due to braiding.

1.2. **Logarithmic geometry.** The theory of logarithmic structures introduced by Fontaine, Illusie, and K. Kato (see [Kato-1989] or [Ogus-20??]) is especially well-suited to the study of degeneration of geometry. In particular, we employ their theory to generalize functorial constructions on smooth curves to the regime of mild singularities, and to canonically extend sheaves on moduli spaces of smooth curves to partial compactifications.

A logarithmic structure on a scheme $(X, \mathcal{O}_X)$ is a sheaf of commutative monoids M_X equipped with a monoid sheaf map $\alpha_X : M_X \to \mathcal{O}_X$ satisfying the condition that the restriction of α_X to $\alpha_X^{-1}\mathcal{O}_X^\times$ is an isomorphism. When α_X is injective, sections of the sheaf M_X may be viewed as functions f that admit a reasonable notion of logarithmic differential $d\log f = \frac{df}{f}$. That is, even if f itself is not invertible in a region, we add formal elements of the form df/f to the sheaf of differentials. The standard example of a logarithmic structure is given by letting X be a smooth variety, choosing a divisor with normal crossings $D \subset X$, and for each open U, setting $M_X(U)$ to be the set of functions $f \in \mathcal{O}_X(U)$ that are invertible away from $U \cap D$. This example shows up for us in three ways:

(1) X is a moduli space (e.g., of stable curves), and D is the boundary divisor, parametrizing the singular objects (e.g., curves with nodal singularities).
(2) X is a curve, and D is a marked point in the schematically smooth locus. With the corresponding logarithmic structure, the marked point is called a log point.
(3) X is the universal curve over a moduli space of stable curves, and D is the union of nodal fibers and some marked points.

One of the distinguishing advantages of logarithmic structures in geometry is that a suitable logarithmic structure can turn a mildly singular space into a smooth space, in the sense of infinitesimal lifting criteria. In particular, a generically smooth family of curves that has a nodal degeneration admits a canonical logarithmic structure, such that the corresponding morphism of log schemes is "log-smooth". Conversely, F. Kato introduced a notion of "log curve" in [Kato-2000] as a proper morphism of log schemes that is log-smooth of relative dimension 1 and satisfies a monoid-theoretic integrality condition, and showed that any log curve is uniquely described by a semistable curve equipped with an unordered set of marked points. To be precise, for any (g,n) in the stable range $(2g-3+n \geq 0)$, if we denote by u the functor from log schemes to schemes that forgets log structures, the functor $(X \to S) \mapsto (u(X) \to u(S))$ taking a stable log curve of type (g,n) to the underlying genus g curve but retaining the n unordered markings induces an equivalence of stacks $u(\mathcal{M}_{g,n}^{\log}) \cong [\mathcal{M}_{g,n}/S_n]$.

We will use an enhanced version of Kato's log curves, inspired by a combination of ideas from [Abramovich-Vistoli-2002], [Jarvis-Kaufmann-Kimura-2005], and [Olsson-2007]. Our curves have the following features:

(1) The curves $X \to S$ may be stacky along markings, i.e., forgetting the log structure yields a balanced twisted curve in the sense of [Abramovich-Vistoli-2002].
(2) The twisted curves are equipped with a representable G-torsor $C \to X$, for G a fixed finite group.
(3) Marked points may have trivial relative log structure.
(4) Marked points with trivial log structure are endowed with a lift to the log scheme $Frame^{\infty}_{C/S}$ of infinite-order frames.

Here, for $C \to S$ an integral log-smooth morphism of fs log schemes of relative dimension one, $Frame^{\infty}_{C/S}$ represents the moduli problem $\underline{\mathrm{Hom}}_S^{fnr}(D_\infty \times S, C)$ of formally unramified maps from the formal disc $D_\infty = \mathrm{Spf}\,\mathbb{C}[[z]]$ to C. Because of the log-smoothness, $Frame^{\infty}_{C/S}$ naturally forms a torsor over C under the group $\mathrm{Aut}^0_S(D_\infty \times S)$ of pointed automorphisms of the disc. Aside from the description in terms of moduli of formally unramified maps, one may combine the results in [Vojta-2007] and [Dutter-2010] to form the infinite jet log scheme $J^{\infty}_{C/S} = \underline{\mathrm{Hom}}_S(D_\infty \times S, C)$ as the relative spectrum of a sheaf of algebras on C. This is a fiber bundle associated to the equivariant open embedding $\mathrm{Aut}^0_S(D_\infty \times S) \subset \mathrm{End}^0_S(D_\infty \times S)$, so the frame torsor is an open dense log subscheme of the jet log scheme that is strict affine over C.

1.3. **The moduli stack of twisted log curves with G-covers with framed points.**

Let G be a finite group, and let g, n, k be non-negative integers such that if $n = 0$, then $2g-3+k \geq 0$. A complex twisted log curve of type (g,n,k) with log-pointed G-torsor and n marked points with infinite order frames is a diagram

$$\begin{tikzcd}
Frame^{\infty}(C/S) \arrow[drr, "a"] & & \\
 & & C \arrow[d, "b"] \\
S_{\mathbb{N}} \arrow[urr, "\{y_j\}_{j=1}^{k}"'] \arrow[d] & & X \arrow[dll, "c"] \\
S \arrow[uuu, bend left, "\{p_i\}_{i=1}^{n}"] & &
\end{tikzcd}$$

where:

(1) C and S are fs log schemes over $\operatorname{Spec}\mathbb{C}$.
(2) $c : X \to S$ is a log-smooth integral Deligne-Mumford morphism of fs log stacks, such that the canonical map $\bar{X} \to S$ from the coarse moduli space is a log-smooth proper morphism of log schemes whose geometric fibers are connected curves.
(3) $C \to S$ is an integral log-smooth proper morphism of relative dimension 1.
(4) $b : C \to X$ is a G-torsor (in particular, étale), and the stabilizer of any node in C has a balanced action on the tangent space.
(5) $S_{\mathbb{N}} \to S$ is the base change of the log point.
(6) $y_j : S_{\mathbb{N}} \to C$ are strict closed embeddings into the schematically smooth locus with disjoint G-orbits, such that the union of G-orbits is the union of log points on C, and $c \circ b \circ y_j$ is equal to the previous map $S_{\mathbb{N}} \to S$.
(7) $p_i : S \to Frame^{\infty}(C/S)$ are sections of the composite $c \circ b \circ a : Frame^{\infty}(C/S) \to C \to S$ whose images in X are pairwise disjoint. Furthermore, the coarse moduli space $X \to \bar{X}$ is an isomorphism away from the images of p_i.

A 1-morphism of complex twisted log curves of type (g, n, k) with log-pointed G-torsors and finfinite order frames is a pullback diagram of log-stacks. A 2-morphism of 1-morphisms is the identity on log-schemes, and a natural isomorphism on X. We write $\mathcal{M}_{g,n,k}^{G,\log,\infty-fr}$ to denote the category whose objects are complex twisted log curves of type (g, n, k) with log-pointed G-torsors and infinite order frames, and whose morphisms are 2-isomorphism classes of 1-morphisms.

The fibered category $\mathcal{M}_{g,n,k}^{G,\log,\infty-fr}$ over fs log schemes is a logarithmic Deligne-Mumford stack of infinite type, and is a torsor under $(\operatorname{Aut}^0 D_\infty)^n$ over $\mathcal{M}_{g,n,k}^{G,\log}$, whose underlying stack is equivalent to the smooth Deligne-Mumford stack of $(g, n+k)$-curves with pointed admissible G-covers, defined in [Jarvis-Kaufmann-Kimura-2005].

1.4. **Logarithmic localization and canonical Lie algebras.** Beilinson-Bernstein localization is an enhancement of the usual gluing of sheaves by descent to the situation where one has infinitesimal gluing data [Beilinson-Bernstein-1993]. Recall from [SGA1] Exp. 8 that if we are given an affine group scheme K, a K-torsor $Z \to X$, and a K-equivariant quasicoherent sheaf $\mathcal{F}$ on Z, then fpqc descent implies $\mathcal{F}$ is the pullback of a unique quasicoherent sheaf $\bar{\mathcal{F}}$ on X (to be precise, $\bar{\mathcal{F}}$ is equipped with an isomorphism between its pullback to Z and $\mathcal{F}$, and these data taken together are unique up to unique isomorphism). If one also has a Lie algebra $\mathfrak{g}$, and a Harish-Chandra structure on the pair $(\mathfrak{g}, K)$, one may consider situations where Z is also endowed with a compatible infinitesimal action of $\mathfrak{g}$, and $\mathcal{F}$ is equivariant under the $(\mathfrak{g}, K)$-action. If the action of $\mathfrak{g}$ on Z is transitive, i.e., one has an algebroid surjection to the tangent sheaf, then the sheaf $\bar{\mathcal{F}}$ is endowed with a (projectively) flat connection. We extend this work to consider infinitesimal descent on log schemes, and we obtain relative log-connections.

Recall that a vertex operator algebra V is equipped with an action of the Virasoro algebra, spanned by operators $\{L_n\}_{n\in\mathbb{Z}}$ and a central element. Because of the constraints on the L_0-spectrum, the operators $\{L_n\}_{n\geq 0}$ are integrable, so V is a representation of the group scheme $\mathrm{Aut}^0 D_\infty$ of pointed automorphisms of the disc. Given a twisted log curve with G-torsor $(C \to X \xrightarrow{c} S)$, we then have an action of the Harish-Chandra pair $(\mathrm{Der}\, D_\infty, G\times \mathrm{Aut}^0 D_\infty)$ on V, and on the $G\times\mathrm{Aut}^0 D_\infty$-torsor $Frame^\infty_{C/S} \to X$, so localization yields a quasicoherent sheaf $\mathcal{V}$ with a canonical relative log-connection ∇ on X/S. The idea behind the construction of $\mathcal{V}$ by the torsor of infinite order frames dates back to work by Gelfand and Kazhdan around 1970 - see [Gelfand-Fuks-Kazhdan-1972]. However, I have yet to see a mathematically precise treatment in the literature for varieties in families, even in the non-logarithmic setting.

1.5. **What is a conformal block?** In our axiomatization, following [Frenkel-Ben-Zvi-2004], a conformal block on a curve with framed points is a map that outputs a number given input vectors from the inserted modules. The map from modules to the complex numbers is not only multilinear over the complex numbers, but is a map of modules of a certain Lie algebra formed from the curve. The extra compatibilities enforced by the Lie algebra structure encode the Ward identities of the chiral conformal field theory. In their formalism, we start with a complex algebraic curve X equipped with framed points, and remove the locus of special points to get $\overset{\circ}{X}$. Using the quasi-conformal structure of the vertex algebra V, one forms a Lie algebra $\mathrm{Lie}_X(V)$, together with $\mathrm{Lie}_X(V)$-modules $\bigotimes \mathcal{M}_i$ from the V-modules assigned to the framed points in X. The space of conformal blocks is $\mathrm{Hom}_{\mathrm{Lie}_X(V)}(\bigotimes \mathcal{M}_i, \mathbb{C})$.

We now describe our version of the Frenkel-Szczesny method. Let $\overset{\circ}{X}\subset X$ denote the substack given by removing the images of the framed points from the twisted curve. The quotient $\mathcal{O}_S$-module $\mathrm{Lie}_{C\to X\to S}(V) = c_*((\mathcal{V}\otimes\Omega_{\overset{\circ}{X}/S})/(\nabla\mathcal{V}))$ has a natural quasicoherent $\mathcal{O}_S$-Lie algebra structure. The formation of this Lie algebra is étale local on $C\to X$, so it essentially amounts to the canonical functor from vertex algebras to Lie algebras given in [Borcherds-1986], together with tensor products with structure sheaves.

Given a framed point $p : S \to Frame^\infty_{C/S}$, we may attach a canonical locally constant section $g_p \in G_S(S)$ (which is just an element of G when S is connected), namely the positively oriented generator of the cyclic stabilizer of the the marked point in C. For any g_p-twisted $V\otimes\mathcal{O}_S$-module M, we define a $\mathrm{Lie}_{C\to X\to S}(V)$-module $\mathcal{M}_p$ as the $\mathcal{O}_S$-module $p^*((c\circ b\circ a)^*M)\cong M$, and defining the action by using the frame to embed $\mathrm{Lie}_{C\to X\to S}(V)$ in a suitably sheafified and completed form of the Lie algebra $V[g]$ defined by [Dong-Li-Mason-1998].

If we attach such modules to every framed point, we have a $\mathrm{Lie}_{C\to X\to S}(V)$-module structure on the tensor product $\mathcal{O}_S$-module $\bigoplus_p \mathcal{M}_p$. The main players in our story are the quasicoherent $\mathcal{O}_S$-module $(\bigoplus_p \mathcal{M}_p)/(\mathrm{Lie}_{C\to X\to S}(V)\cdot\bigoplus_p\mathcal{M}_p)$ of coinvariants, and the $\mathcal{O}_S$-module $\mathrm{Hom}_{\mathrm{Lie}_{C\to X\to S}(V)}(\bigotimes_p\mathcal{M}_p,\mathcal{O}_S)$ of conformal blocks.

In the universal case, we have a quasi-coherent sheaf of Lie algebras on the moduli stack $\mathcal{M}^{G,\log,\infty-fr}_{g,n,k}$, but it is invariant under change of frames at points. Similarly, the structures of modules attached to framed points are equivariant with respect to changing higher-order frame data (indeed, if we don't impose the admissibility assumption of [Frenkel-Szczesny-2004], even logarithmic modules work well here). Furthermore, (following [Frenkel-Ben-Zvi-2004] Chapter 17) the infinitesimal dependence of the Lie algebra on the moduli of $(C\to X\to S)$ gives a weakly equivariant connection on the lift of this sheaf to the determinant $\mathbb{G}_m$-torsor. Localization then yields a quasicoherent sheaf with connection that has logarithmic singularities on the boundary of the moduli stack $\widetilde{\mathcal{M}^{G,\log,1-fr}_{g,n,k}}$ whose objects are twisted log curves with G-torsors and first order frames at points together with a trivialization of the determinant bundle. In the case that V is a regular vertex operator algebra, some analysis of nodal degeneration following [Nagatomo-Tsuchiya-2005] yields the result that the sheaf is finite rank locally free, and therefore so is the dual sheaf of conformal blocks.

Recently, I have been made aware of an alternative method for constructing $\mathcal{V}$, used in upcoming work by Hashimoto and Tsuchiya. They apply a method of stabilization in families to obtain the relevant Lie algebras. Since their locally free sheaves agree with mine away from a codimension 2 subspace of the universal curve, both constructions yield isomorphic Lie algebras and sheaves of conformal blocks.

2. The proof in brief

Now that we have defined conformal blocks, we can make the following claim: Given a C_2-cofinite vertex operator algebra and g_i-twisted modules M_i (for $i = 1, 2, 3$ and $g_3 = g_1g_2 = g_2g_1$) there is an isomorphism between the vector space of intertwining operators of type $\binom{M_3}{M_1 M_2}$ and the vector space of conformal blocks on the projective line $\mathbb{P}^1$ with insertions of M_1 over zero, M_2 over 1, and the contragradient $M_3^\vee$ over infinity. The proof is quite similar to that in [Zhu-1994].

Genus zero conformal blocks satisfy the following key properties in the case of a rational vertex operator algebra V satisfying the C_2-cofiniteness condition. This list is a generalization of the main results of [Nagatomo-Tsuchiya-2005] to the setting of twisted modules and logarithmic schemes.

(1) Vector bundle with log connection: Given a fixed list of irreducible twisted modules to be inserted, the spaces of conformal blocks form a finite rank locally free sheaf with log connection on the (log) moduli stack of twisted genus zero curves with Galois covers and framed insertion points. The sheaf descends to a connected moduli space, so one obtains non-canonical isomorphisms of conformal blocks spaces for any pair of insertion configurations.
(2) Propagation of vacua: Inserting the vacuum module at a non-special point yields a canonical isomorphism of spaces of conformal blocks.
(3) Duality: Given the insertion of a g-twisted module M_0 over zero and a g^{-1}-twisted module M_∞ over infinity, the space of two-point genus zero conformal blocks is isomorphic to $\mathrm{Hom}_V(M_0^\vee, M_\infty)$, where $M_0^\vee$ denotes the g^{-1}-twisted contragradient module.
(4) Logarithmic points: Adding a log point to the non-special locus has the same effect on conformal blocks as inserting the distinguished $V - A_0(V)$-bimodule $M(0)$, where $A_0(V)$ is Zhu's zero-mode algebra (which is finite dimensional and semisimple under our hypotheses).
(5) Nodal factorization: If A and B are smooth curves with log points, and $A + B$ is a curve obtained by gluing A and B transversally at a log point, then $Conf_{A+B} \cong Conf_A \otimes_{A_0(V)} Conf_B$.

When V is holomorphic and C_2-cofinite, then $A_0(V)$ is isomorphic to a matrix ring, and $M(0)$ is isomorphic to the tensor product of V with the unique simple $A_0(V)$-module W. This fact makes it easy to understand what happens to conformal blocks spaces when we glue curves together. Let A and B be twisted log curves with torsors and framed points, equipped with twisted module insertions and conformal blocks spaces $Conf_A$ and $Conf_B$, respectively. Puncturing each curve at a smooth point, we get new curves A', B', and by inserting $M(0)$ at the smooth points, we get $A_0(V)$-modules $Conf_{A'} \cong Conf_A \otimes_{\mathbb{C}} W$ and $Conf_{B'} \cong Conf_B \otimes_{\mathbb{C}} W$. These are isomorphic to the $A_0(V)$-modules $Conf_A^{log}$ and $Conf_B^{log}$, where we replace the punctures with log points. Gluing the two curves together at their new log points yields a nodal log curve $A + B$, and $Conf_{A+B} \cong Conf_A^{log} \otimes_{A_0(V)} Conf_B^{log} \cong Conf_A \otimes_{\mathbb{C}} Conf_B$.

We may now use this property to describe the intertwining operators of type $\binom{M_3}{M_1 M_2}$. We translate the problem to the question of describing the space of conformal blocks on the curve A with insertions of the three irreducible twisted modules $M_1, M_2, M_3^\vee$. We build a curve B with insertions $M_1^\vee, M_2^\vee, M_3$. From the discussion in the previous paragraph, $Conf_{A+B} \cong Conf_A \otimes_{\mathbb{C}} Conf_B$. By deformation and degeneration (using the fact that conformal blocks form a locally free sheaf), we may split $A + B$ with its three pairs of dual modules into separate genus zero curves, each carrying one dual pair. We find that $Conf_{A+B} \cong \bigotimes_{i=1}^{3} Conf_{\mathbb{P}^1}(M_i, M_i^\vee)$. By the duality property, the right side is a tensor product of one-dimensional vector spaces, so both $Conf_A$ and $Conf_B$ are one dimensional.

These ideas can be generalized in a few directions:

(1) We may consider intertwining operators with more than 2 inputs. Like the case of 3-point conformal blocks, when the twisting of the insertions is appropriate, we end up with one dimensional spaces of intertwining operators.
(2) If our twisting is given by elements of an abelian group, we want to understand how various eigenspaces in the twisted modules fuse. This amounts to working out how the action of the group interacts with the twisting, and we find that eigenspaces end up with fusion order that depends on the $L(0)$-eigenvalue modulo $\mathbb{Z}$.
(3) Some directions that we plan to consider in future work are: expanding to non-rational vertex algebras, and conformal blocks on higher genus curves with possibly non-nodal singularities. There are already results in this direction in unpublished work of Hashimoto and Tsuchiya - they replace the $V - A_0(V)$-bimodule here with a "regular bimodule" of Zhu's enveloping algebra $U(V)$, described in [Matsuo-Nagatomo-Tsuchiya-2005].

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