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Nonsymmetric Askey-Wilson polynomials and Q-polynomial distance-regular graphs

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1 Nonsymmetric Askey-Wilson polynomials

Throughout this paper we assume q is not a root of unity. For $a \in \mathbb{C}$,

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

where $n = 0, 1, 2, \ldots$. For $a_1, a_2, \ldots, a_r \in \mathbb{C}$,

$$(a_1, a_2, \ldots, a_r; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_r;q)_n.$$

Throughout this section, let $a, b, c, d \in \mathbb{C}^*$ be such that

$$ab, ac, ad, bc, bd, cd, abcd \notin \{q^{-m} \mid m = 0, 1, 2, \ldots\}. \tag{1}$$

We now recall the Askey-Wilson polynomials [1]. For $n = 0, 1, 2, \ldots$ define a polynomial

$$\begin{aligned} p_n(z+z^{-1}; a, b, c, d \mid q) &:= \sum_{i=0}^{\infty} \frac{(q^{-n}, abcdq^{n-1}, az, az^{-1}; q)_i}{(ab, ac, ad, q; q)_i} q^i \\ &= {}_4\phi_3\left(\begin{matrix} q^{-n},\ abcdq^{n-1},\ az,\ az^{-1} \\ ab,\ ac,\ ad \end{matrix} \middle|\, q,\ q \right). \end{aligned} \tag{2}$$

The last equality follows from the definition of basic hypergeometric series [3, p. 4]. Observe that $(q^{-n};q)_i = 0$ if $i > n$. We call p_n the n-th *Askey-Wilson polynomials*. Consider the *monic* Askey-Wilson polynomials

$$P_n = P_n[z; a, b, c, d \mid q] := \frac{(ab, ac, ad; q)_n}{a^n(abcdq^{n-1}; q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n},\ abcdq^{n-1},\ az,\ az^{-1} \\ ab,\ ac,\ ad \end{matrix} \middle|\, q,\ q \right).$$

Let $\mathcal{L}$ denote the space of the Laurent polynomials with a variable z. By a *symmetric polynomial* f *in* $\mathcal{L}$ we mean $f[z] = f[z^{-1}]$. Note that P_n is symmetric. The *nonsymmetric Askey-Wilson polynomials* [4] are defined by

$$E_{-n} = P_n - Q_n \qquad (n = 1, 2, \ldots), \tag{3}$$

$$E_n = P_n - \frac{ab(1-q^n)(1-cdq^{n-1})}{(1-abq^n)(1-abcdq^{n-1})} Q_n \qquad (n = 0, 1, 2, \ldots), \tag{4}$$

where $Q_n = a^{-1}b^{-1}z^{-1}(1-az)(1-bz)P_{n-1}[z; qa, qb, c, d \mid q]$.

The *double affine Hecke algebra* (DAHA) of type $(C_1^\vee, C_1)$, denoted by $\tilde{\mathfrak{H}}$ [4,6], is defined by the generators Z, Z^{-1}, T_0, T_1 and relations

$$(T_1 + ab)(T_1 + 1) = 0, \qquad (T_0 + q^{-1}cd)(T_0 + 1) = 0,$$
$$(T_1 Z + a)(T_1 Z + b) = 0, \qquad (qT_0 Z^{-1} + c)(qT_0 Z^{-1} + d) = 0.$$

The algebra $\tilde{\mathfrak{H}}$ has a faithful representation on $\mathcal{L}$, which is called the *basic representation* [4, §3]:

$$(Zf)[z] := zf[z],$$
$$(T_1f)[z] := \frac{(a+b)z-(1+ab)}{1-z^2}f[z] + \frac{(1-az)(1-bz)}{1-z^2}f[z^{-1}],$$
$$(T_0f)[z] := \frac{q^{-1}z((cd+q)z-(c+d)q)}{q-z^2}f[z] - \frac{(c-z)(d-z)}{q-z^2}f[qz^{-1}].$$

Let $Y = T_1T_0$. By [4, Theorem 4.1], each of $E_{\pm n}$ is the eigenfunction for Y;

$$YE_{-n} = q^{-n}E_{-n} \qquad (n = 1, 2, \ldots) \tag{5}$$
$$YE_n = q^{n-1}abcdE_n \qquad (n = 0, 1, 2, \ldots). \tag{6}$$

2 Q-polynomial distance-regular graphs

In this section we review some preliminaries regarding Q-polynomial distance-regular graphs. Let X denote a nonempty finite set. Let Γ denote a simple connected graph with vertex X. For $x \in X$ define $\Gamma_i(x) := \{y \in X \mid \partial(x,y) = i\}$, where ∂ is the shortest path-length distance function. Let $D := \max\{\partial(x,y) \mid x, y \in X\}$, called *diameter*. Assume that Γ has $D \geq 3$. We say that Γ is *distance-regular* whenever for $0 \leq i \leq D$ and vertices $x, y \in X$ with $\partial(x,y) = i$, the numbers $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|, b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|, c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ are independent of x and y. The constants a_i, b_i, c_i are called the *intersection numbers* of Γ. Let $Mat_X(\mathbb{C})$ be the $\mathbb{C}$-algebra consisting of square matrices indexed by X. Define the matrix $A_i \in Mat_X(\mathbb{C})$ by $(A_i)_{xy} = 1$ if $\partial(x,y) = i$ and 0 otherwise. It is called the *i-th distance matrix* of Γ. In particular, $A = A_1$ is called the *adjacency matrix*. Let M be the subalgebra of $Mat_X(\mathbb{C})$ generated by A, called the *adjacency algebra*, so every element in M forms a polynomial in A. For $0 \leq i \leq D$ there is a polynomial $f_i \in \mathbb{C}[x]$ such that $\deg(f_i) = i$ and $f_i(A) = A_i$ (P-polynomial property).

We recall the notion of Q-polynomial property. By [2, p. 127], the $\{A_i\}_{i=0}^D$ forms a basis for M. Since A generates M, A has $D+1$ mutually distinct (real) eigenvalues, denoted by $\theta_0, \theta_1, \ldots, \theta_D$. Let $E_i \in Mat_X(\mathbb{C})$ denote the orthogonal projection onto the eigenspace of θ_i $(0 \leq i \leq D)$. Remark that $E_0, E_1, \ldots, E_D$ are the primitive idempotents of M. Γ is said to be *Q-polynomial* with respect to $E_0, E_1, \ldots, E_D$ if there exists $f_i^* \in \mathbb{C}[x]$ such that $\deg(f_i^*) = i$ and $f_i^*(E_1) = E_i$, where the multiplication of M is under the entrywise product. For the rest of this paper, we assume that Γ is a Q-polynomial distance-regular graph.

By a *clique* of Γ we mean a nonempty subset $C \subset X$ such that any two distinct vertices in C are adjacent each other. We say that C is *Delsarte* whenever $|C| = 1 - k/\theta_{min}$, where k is a valency of Γ and θ_{min} is the minimum eigenvalue of A. We assume that Γ contains a Delsarte clique C. Fix a vertex $x \in C$. Consider $\Gamma_i = \Gamma_i(x)$ $(0 \leq i \leq D)$ and $C_i := \{v \in X \mid \partial(v, C) = i\}$ $(0 \leq i \leq D-1)$. For $0 \leq i \leq D-1$, define the subset $C_i^{\pm} \subset X$ to be $C_i^- := C_i \cap \Gamma_i$ and $C_i^+ := C_i \cap \Gamma_{i+1}$. Note that $\{C_i^{\pm}\}_{i=0}^{D-1}$ is a partition of X. Define $\mathbf{W}$ to be the subspace of $\mathbb{C}^X$ spanned by the characteristic vectors $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$. It turns out that the $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ forms a basis for $\mathbf{W}$. Observe that $\hat{x}, \hat{C} \in \mathbf{W}$.

Lemma 2.1. [5, Lemma 5.23] *For $0 \leq i \leq D-1$,*

$$\hat{C}_i^- = \sum_{j=0}^{i} A_j\hat{x} - \sum_{j=0}^{i-1} \hat{C}_j, \qquad \hat{C}_i^+ = \sum_{j=0}^{i} \hat{C}_j - \sum_{j=0}^{i} A_j\hat{x}.$$

We recall the *Terwilliger algebra* (or the subconstituent algebra) of Γ (see [8]). Define $A^* = A^*(x) := |X|\mathrm{diag}(E_1\hat{x}) \in Mat_X(\mathbb{C})$, called the *dual adjacency matrix* of Γ with respect to x. The Terwilliger algebra $T = T(x)$ with respect to x is the subalgebra of $Mat_X(\mathbb{C})$ generated by A, A^*. We define $\widetilde{A}^* = \widetilde{A}^*(C) = \frac{|X|}{|C|}\mathrm{diag}(E_1\hat{C}) \in Mat_X(\mathbb{C})$, called the *dual adjacency matrix* of Γ with respect to C. The Terwilliger algebra $\widetilde{T} = \widetilde{T}(C)$ with respect to C is the subalgebra of $Mat_X(\mathbb{C})$ generated by $A, \widetilde{A}^*$ [7]. Using this two algebras, we define the generalized Terwilliger algebra $\mathbf{T} = \mathbf{T}(x, C)$ that is generated by $T, \widetilde{T}$ [5]. Note that $\mathbf{W}$ has a module structure for both T and $\widetilde{T}$, and so it is a $\mathbf{T}$-module [5, Proposition 5.25]. The T-submodule (resp. $\widetilde{T}$-submodule) of $\mathbf{W}$ generated by $\hat{x}$ (resp. $\hat{C}$) will be called the *primary T-module* (resp. primary $\widetilde{T}$-module), denoted by $M\hat{x}$ (resp. $M\hat{C}$). The $\{A_i\hat{x}\}_{i=0}^{D}$ (resp. $\{\hat{C}_i\}_{i=0}^{D-1}$) is a basis for $M\hat{x}$ (resp. $M\hat{C}$).

Let $\{\theta_i\}_{i=0}^{D}$ (resp. $\{\theta_i^*\}_{i=0}^{D}$) denote the eigenvalue sequence of A (resp. A^*). Γ is said to have *q-Racah type* whenever for $0 \leq i \leq D$

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i}, \tag{7}$$

$$\theta_i^* = \theta_0^* + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}. \tag{8}$$

Then there are the corresponding scalars s, s^*, r_1, r_2 with $r_1r_2 = ss^*q^{D+1}$ and some constraints; see [9]. For the rest of this paper we assume that Γ has q-Racah type. In what follows, whenever we encounter square roots, these are interpreted as follows. We fix square roots $s^{1/2}, s^{*1/2}, r_1^{1/2}, r_2^{1/2}$ such that $r_1^{1/2}r_2^{1/2} = s^{1/2}s^{*1/2}q^{(D+1)/2}$.

3 Polynomials F_i and $\widetilde{F}_i$

3.1

Recall the polynomials $\{f_i\}_{i=0}^{D}$ from the first paragraph in §2. This polynomial sequence satisfies the following 3-term recursion:

$$xf_i = b_{i-1}f_{i-1} + a_if_i + c_{i+1}f_{i+1} \qquad (0 \leq i \leq D), \tag{9}$$

where $f_{-1} = 0$ and $f_{D+1} = 0$. It is readily to see that $f_i(A)\hat{x} = A_i\hat{x}$. We normalize the polynomials $f_i (0 \leq i \leq D)$ as follows.

$$F_i := f_i/k_i, \tag{10}$$

where $k_i = b_0b_1\cdots b_{i-1}/c_1c_2\cdots c_i$. Then (9) becomes

$$xF_i = c_iF_{i-1} + a_iF_i + b_iF_{i+1} \qquad (0 \leq i \leq D).$$

By [10, Theorem 23.2], it follows that for $0 \leq i \leq D$

$$F_i(x) = \sum_{j=0}^{i} \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{j-1}^*)}{\varphi_1\varphi_2\cdots\varphi_j}(x-\theta_0)(x-\theta_1)\cdots(x-\theta_{j-1}), \tag{11}$$

where $\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-D-1})(1-r_1q^i)(1-r_2q^i)$.

Until further notice, we put the scalars $a, b, c, d \in \mathbb{C}^*$ such that

$$a = \left(\frac{r_1r_2}{s^*q^D}\right)^{1/2}, \qquad b = \left(\frac{s^*}{r_1r_2q^D}\right)^{1/2}, \qquad c = \left(\frac{s^*r_2q^{D+2}}{r_1}\right)^{1/2}, \qquad d = \left(\frac{s^*r_1q^{D+2}}{r_2}\right)^{1/2}. \tag{12}$$

For $0 \leq i \leq D$, consider the Askey-Wilson polynomial $p_i(y+y^{-1}) = p_i(y+y^{-1}; a, b, c, d \mid q)$. The following lemma explains how the polynomial F_i is related to the Askey-Wilson polynomial p_i.

Lemma 3.1. *Let x be of the form*

$$h(sq)^{1/2}(y+y^{-1}) + (\theta_0 - h - hsq), \tag{13}$$

where y is indeterminate. Then

$$F_i(x) = p_i(y+y^{-1}), \qquad i = 0, 1, 2, \ldots D. \tag{14}$$

Proof. We compute both sides of the equation (14). First we compute the right-hand side in (14). Apply (12) to (2) and use the equation $r_1 r_2 = ss^* q^{D+1}$ to get

$$\sum_{j=0}^{i} \frac{(q^{-i};q)_j (s^* q^{i+1};q)_j (s^{1/2}q^{1/2}y;q)_j (s^{1/2}q^{1/2}y^{-1};q)_j}{(r_1 q;q)_j (r_2 q;q)_j (q^{-D};q)_j (q;q)_n} q^j.$$

We now compute the left-hand side in (14). Put (13) for x in (11) and simplify it. Then the result follows. ■

3.2

Recall the partition $\{C_i\}_{i=0}^{D-1}$ of X from above Lemma 2.1. For $0 \le i \le D-1$ and $z \in C_i$, define $\widetilde{c}_i := |\Gamma_1(z) \cap C_{i-1}|$, $\widetilde{a}_i := |\Gamma_1(z) \cap C_i|$, $\widetilde{b}_i := |\Gamma_1(z) \cap C_{i+1}|$; see [5, §4]. With these parameters, define $\widetilde{f}_i \in \mathbb{C}[x] (0 \le i \le D-1)$ by $\widetilde{f}_0 = 1$ and

$$x\widetilde{f}_i = \widetilde{b}_{i-1}\widetilde{f}_{i-1} + \widetilde{a}_i \widetilde{f}_i + \widetilde{c}_{i+1}\widetilde{f}_{i+1} \qquad (0 \le i \le D-1), \tag{15}$$

where $\widetilde{f}_{-1} = 0$ and $\widetilde{f}_D = 0$. By construction, we have

$$\widetilde{f}_i(A)\hat{C} = \hat{C}_i. \tag{16}$$

In a similar manner to (10), we define the sequence of polynomials $\widetilde{F}_0, \widetilde{F}_1, \ldots \widetilde{F}_{D-1}$ by

$$\widetilde{F}_i := \widetilde{f}_i / \widetilde{k}_i, \tag{17}$$

where $\widetilde{k}_i = \widetilde{b}_0 \widetilde{b}_1 \cdots \widetilde{b}_{i-1} / \widetilde{c}_1 \widetilde{c}_2 \cdots \widetilde{c}_i$. Then (15) becomes

$$x\widetilde{F}_i = \widetilde{c}_i \widetilde{F}_{i-1} + \widetilde{a}_i \widetilde{F}_i + \widetilde{b}_i \widetilde{F}_{i+1} \qquad (0 \le i \le D-1).$$

By [10, Theorem 23.2] and using [5, Theorem 4.21], it follows that for $0 \le i \le D-1$

$$\widetilde{F}_i(x) = \sum_{j=0}^{i} \frac{(\widetilde{\theta}_i^* - \widetilde{\theta}_0^*)(\widetilde{\theta}_i^* - \widetilde{\theta}_1^*) \cdots (\widetilde{\theta}_i^* - \widetilde{\theta}_{j-1}^*)}{\widetilde{\varphi}_1 \widetilde{\varphi}_2 \cdots \widetilde{\varphi}_j} (x - \widetilde{\theta}_0)(x - \widetilde{\theta}_1) \cdots (x - \widetilde{\theta}_{j-1}),$$

where $\widetilde{\varphi}_i = \widetilde{h}\widetilde{h}^* q^{1-2i}(1-q^i)(1-q^{i-D})(1-\widetilde{r}_1 q^i)(1-\widetilde{r}_2 q^i)$.

Define the scalars $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in \mathbb{C}^*$ by

$$\widetilde{a} = a, \qquad \widetilde{b} = bq, \qquad \widetilde{c} = c, \qquad \widetilde{d} = d. \tag{18}$$

With these parameters, for $n = 0, 1, 2, \ldots, D-1$ define a polynomial $\widetilde{p}_n = \widetilde{p}_n[y; \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \mid q]$ by

$$\begin{aligned} \widetilde{p}_n &:= {}_4\phi_3 \left(\begin{matrix} q^{-n},\ \widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}q^{n-1},\ \widetilde{a}y,\ \widetilde{a}y^{-1} \\ \widetilde{a}\widetilde{b},\ \widetilde{a}\widetilde{c},\ \widetilde{a}\widetilde{d} \end{matrix} \,\middle|\, q,\ q \right) \\ &= {}_4\phi_3 \left(\begin{matrix} q^{-n},\ abcdq^{n},\ ay,\ ay^{-1} \\ abq,\ ac,\ ad \end{matrix} \,\middle|\, q,\ q \right) = p_n[y; a, bq, c, d \mid q]. \end{aligned}$$

Note that the monic of $\widetilde{p}$ is

$$\widetilde{P}_n = \widetilde{P}_n[y; \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \mid q] = \frac{(\widetilde{a}\widetilde{b}, \widetilde{a}\widetilde{c}, \widetilde{a}\widetilde{d}; q)_n}{\widetilde{a}^n (\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}q^{n-1}; q)_n} \widetilde{p}_n = \frac{(abq, ac, ad; q)_n}{a^n (abcdq^n; q)_n} p_n[y; a, bq, c, d \mid q]$$

The following lemma explains how the $\widetilde{F}_i$ is related to the $\widetilde{p}_i$, that is the analogue of Lemma 3.1.

Lemma 3.2. *Let x be of the form*

$$h(sq)^{1/2}(y + y^{-1}) + (\theta_0 - h - hsq),$$

where y is indeterminate. Then

$$\widetilde{F}_i(x) = \widetilde{p}_i(y + y^{-1}), \qquad i = 0, 1, 2, \ldots D-1.$$

Proof. Similar to Lemma 3.1. ■

4 The universal DAHA of type $(C_1^\vee, C_1)$

For notational convenience, define $\mathbb{I} := \{0,1,2,3\}$. The *universal DAHA* of type $(C_1^\vee, C_1)$ [11, Definition 3.1] is the $\mathbb{C}$-algebra $\hat{H}_q$ defined by generators $\{t_n^{\pm 1}\}_{n\in\mathbb{I}}$ and relations

$$\text{(i) } t_n t_n^{-1} = t_n^{-1} t_n = 1 \quad (n \in \mathbb{I}); \qquad \text{(ii) } t_n + t_n^{-1} \text{ is central} \quad (n \in \mathbb{I}); \qquad \text{(iii) } t_0 t_1 t_2 t_3 = q^{-1/2}.$$

In [5, §11] we discussed that $\mathbf{W}$ has an $\hat{H}_q$-module structure in detail. In this paper, for our purpose we will twist $\mathbf{W}$ via a certain $\mathbb{C}$-algebra automorphism of $\hat{H}_q$. Recall the $\hat{H}_q$-module $\mathbf{W}$ from [5, §11]. Consider a $\mathbb{C}$-algebra automorphism $\sigma : \hat{H}_q \to \hat{H}_q$ that sends

$$t_0 \mapsto t_1, \qquad t_1 \mapsto t_0, \qquad t_2 \mapsto t_0^{-1} t_3 t_0, \qquad t_3 \mapsto t_1 t_2 t_1^{-1}.$$

Observe that $\sigma^2 = id$. There exists an $\hat{H}_q$-module structure on $\mathbf{W}$, called $\mathbf{W}$ *twisted via* σ, that behaves as follows: for all $h \in \hat{H}_q, w \in \mathbf{W}$, the vector $h.w$ computed in $\mathbf{W}$ twisted via σ coincides with the vector $h^\sigma.w$ computed in the original $\hat{H}_q$-module $\mathbf{W}$. For the rest of this paper, we regard an $\hat{H}_q$-module $\mathbf{W}$ as the $\hat{H}_q$-module $\mathbf{W}$ twisted via σ. Define the following elements in $\hat{H}_q$:

$$\mathbf{Y} = t_0 t_1, \qquad \mathbf{X} = t_3 t_0, \qquad \widetilde{\mathbf{X}} = t_1 t_2 = q^{-1/2} t_0^{-1} t_3^{-1},$$
$$\mathbf{A} = \mathbf{Y} + \mathbf{Y}^{-1}, \qquad \mathbf{B} = \mathbf{X} + \mathbf{X}^{-1}, \qquad \widetilde{\mathbf{B}} = \widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}^{-1}.$$

We describe the $\hat{H}_q$-module $\mathbf{W}$ in detail. Recall the parameters r_1, r_2, s, s^*, D from the last paragraph in §2.

Definition 4.1. [5, Definition 11.1]

(a) For $1 \le i \le D-1$, the (2×2)-matrix $t_0(i)$ is

$$\begin{bmatrix} \frac{q^{D/2}(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} + \frac{1}{q^{D/2}} & \frac{q^{D/2}(q^{i-D}-1)(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \\ \frac{(1-q^i)(1-s^*q^{D+i+1})}{q^{D/2}(1-s^*q^{2i+1})} & \frac{(q^i-1)(1-s^*q^{D+i+1})}{q^{D/2}(1-s^*q^{2i+1})} + \frac{1}{q^{D/2}} \end{bmatrix}$$

and

$$t_0(0) = \left[\frac{1}{q^{D/2}} \right], \qquad t_0(D) = \left[\frac{1}{q^{D/2}} \right].$$

(b) For $0 \le i \le D-1$, the (2×2)-matrix $t_1(i)$ is

$$\begin{bmatrix} \frac{1}{(s^*r_1r_2)^{1/2}} \left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^* \right) & -\left(\frac{s^*}{r_1r_2}\right)^{1/2} \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \\ \frac{1}{(s^*r_1r_2)^{1/2}} \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} & \left(\frac{s^*}{r_1r_2}\right)^{1/2} \left(1 - \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \right) \end{bmatrix}.$$

(c) $0 \le i \le D-1$, the (2×2)-matrix $t_2(i)$ is

$$\begin{bmatrix} \frac{1}{q^{i+1}(r_1r_2)^{1/2}} \left(1 - \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \right) & \frac{s^*q^{i+1}}{(r_1r_2)^{1/2}} \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \\ -\frac{1}{s^*q^{i+1}(r_1r_2)^{1/2}} \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} & \frac{q^{i+1}}{(r_1r_2)^{1/2}} \left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^* \right) \end{bmatrix}.$$

(d) For $1 \le i \le D-1$, the (2×2)-matrix $t_3(i)$ is

$$\begin{bmatrix} \frac{1}{q^i(s^*q)^{1/2}} \left(\frac{(q^i-1)(1-s^*q^{D+i+1})}{q^{D/2}(1-s^*q^{2i+1})} + \frac{1}{q^{D/2}} \right) & \frac{1}{q^i(s^*q)^{1/2}} \left(\frac{q^{D/2}(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \right) \\ q^i(s^*q)^{1/2} \left(\frac{(q^i-1)(1-s^*q^{D+i+1})}{q^{D/2}(1-s^*q^{2i+1})} \right) & q^i(s^*q)^{1/2} \left(\frac{q^{D/2}(1-q^{i-D})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} + \frac{1}{q^{D/2}} \right) \end{bmatrix}$$

and

$$t_3(0) = \left[(s^*q^{D+1})^{1/2} \right], \qquad t_3(D) = \left[\frac{1}{(s^*q^{D+1})^{1/2}} \right].$$

Define the block diagonal matrices $\mathcal{T}_n (n \in \mathbb{I})$:

$$\mathcal{T}_0 = \text{blockdiag}\Big[t_0(0), t_0(1), \dots, t_0(D-1), t_0(D)\Big];$$
$$\mathcal{T}_1 = \text{blockdiag}\Big[t_1(0), t_1(1), \dots, t_1(D-1)\Big];$$
$$\mathcal{T}_2 = \text{blockdiag}\Big[t_2(0), t_2(1), \dots, t_2(D-1)\Big];$$
$$\mathcal{T}_3 = \text{blockdiag}\Big[t_3(0), t_3(1), \dots, t_3(D-1), t_3(D)\Big].$$

Then $\mathbf{W}$ has a module structure for $\hat{H}_q$ such that for $n \in \mathbb{I}$ the matrix $\mathcal{T}_n$ represents the generator t_n with respect to the ordered basis $\{\hat{C}_i^-, \hat{C}_i^+\}_{i=0}^{D-1}$ [5, §11].

Remark 4.2. In [5, Definition 11.2] we defined the scalars $\{k_n\}_{n\in\mathbb{I}}$. On $\mathbf{W}$, the scalars $\{k_n\}_{n\in\mathbb{I}}$ are defined by

$$k_0 = \left(\frac{1}{q^D}\right)^{1/2}, \qquad k_1 = \left(\frac{r_1 r_2}{s^*}\right)^{1/2}, \qquad k_2 = \left(\frac{r_2}{r_1}\right)^{1/2}, \qquad k_3 = \left(s^* q^{D+1}\right)^{1/2}$$

Remark 4.3. The above module structure for $\hat{H}_q$ on $\mathbf{W}$ was determined by the parameters q, s, s^*, r_1, r_2, D. Denote $\mathbf{W} = \mathbf{W}_{q,s,s^*,r_1,r_2,D}$. Using the relation (12) we can replace the parameters s, s^*, r_1, r_2, D by a, b, c, d. Then the module structure for $\hat{H}_q$ on $\mathbf{W}$ is described with the parameters q, a, b, c, d. We denote by $\mathbf{W}_{q,a,b,c,d}$. Since the diameter D disappears in $\mathbf{W}_{q,a,b,c,d}$, we can extend this finite dimensional module to an infinite dimensional module in an algebraic aspect; see Appendix.

The following theorem shows how $\hat{H}_q$ is related to Γ. Recall the elements $A, A^*, \widetilde{A}^*$ in $\mathbf{T}$ from §2 and the elements $\mathbf{A}, \mathbf{B}, \widetilde{\mathbf{B}}$ in $\hat{H}_q$. Recall that $\mathbf{W}$ is a $\mathbf{T}$-module as well as an $\hat{H}_q$-module twisted via σ.

Theorem 4.4. [5, Theorem 12.1] *On* $\mathbf{W}$,

(i) *A acts as* $h(sq)^{1/2}\mathbf{A} + (\theta_0 - h - hsq)$;

(ii) *A^* acts as* $h^*(s^*q)^{1/2}\mathbf{B} + (\theta_0^* - h^* - h^*s^*q)$;

(iii) *$\widetilde{A}^*$ acts as* $\widetilde{h}^*(\widetilde{s}^*q)^{1/2}\widetilde{\mathbf{B}} + (\widetilde{\theta}_0^* - \widetilde{h}^* - \widetilde{h}^*\widetilde{s}^*q)$.

5 Nonsymmetric Laurent polynomials $\varepsilon_i^{\pm}$

In this section we construct the nonsymmetric Laurent polynomials $\varepsilon_i^{\pm}$ using the $\hat{H}_q$-module $\mathbf{W}$. We begin with the following lemma.

Lemma 5.1. *Let*

$$g[y] = m(1 - by^{-1}),$$

where

$$b = \left(\frac{s^*}{r_1 r_2 q^D}\right)^{1/2}, \qquad m = \frac{1 - s^*q^2}{(1 - s^*q/r_1)(1 - s^*q/r_2)}.$$

Then on $\mathbf{W}$, *we have*

$$g[\mathbf{Y}].\hat{x} = \hat{C}.$$

Lemma 5.1 tells that the element $g[\mathbf{Y}]$ maps $\hat{x}$ to $\hat{C}$ on $\mathbf{W}$. Our next goal is to find the element in $\hat{H}_q$ that maps $\hat{x}$ to $\hat{C}_i^-$ for $0 \le i \le D-1$. Recall Lemma 2.1 that

$$\hat{C}_i^- = \sum_{j=0}^{i} A_j \hat{x} - \sum_{j=0}^{i-1} \hat{C}_j = \sum_{j=0}^{i} f_j(A)\hat{x} - \sum_{j=0}^{i-1} \widetilde{f}_j(A)\hat{C}, \tag{19}$$

where the last equality is obtained by the comment below (9) and by (16). By (10) and (17), the right-hand side in (19) becomes

$$\sum_{j=0}^{i} k_j F_j(A)\hat{x} - \sum_{j=0}^{i-1} \widetilde{k}_j \widetilde{F}_j(A)\hat{C}. \tag{20}$$

By applying Theorem 4.4 (i) to (20), we find

$$\begin{aligned}
\hat{C}_i^- &= \sum_{j=0}^{i} k_j F_j\Big(h(sq)^{1/2}\mathbf{A} + (\theta_0 - h - hsq)\Big)\hat{x} \\
&\quad - \sum_{j=0}^{i-1} \widetilde{k}_j \widetilde{F}_j\Big(h(sq)^{1/2}\mathbf{A} + (\theta_0 - h - hsq)\Big)\hat{C} \\
\text{(by Lemma 3.1, Lemma 3.2)} &= \sum_{j=0}^{i} k_j p_j(\mathbf{A})\hat{x} - \sum_{j=0}^{i-1} \widetilde{k}_j \widetilde{p}_j(\mathbf{A})\hat{C} \\
\text{(by Lemma 5.1)} &= \left(\sum_{j=0}^{i} k_j p_j(\mathbf{A}) - g[\mathbf{Y}]\sum_{j=0}^{i-1} \widetilde{k}_j \widetilde{p}_j(\mathbf{A})\right)\hat{x}.
\end{aligned} \tag{21}$$

Note that $\mathbf{A} = \mathbf{Y} + \mathbf{Y}^{-1}$. Similarly we find

$$\hat{C}_i^+ = \left(g[\mathbf{Y}]\sum_{j=0}^{i} \widetilde{k}_j \widetilde{p}_j(\mathbf{A}) - \sum_{j=0}^{i} k_j p_j(\mathbf{A})\right)\hat{x}. \tag{22}$$

Motivated by (21), (22) we make a following definition.

Definition 5.2. For $i = 0, 1, 2, \ldots, D-1$ define the polynomials $\varepsilon_i^{\pm}$ in $\mathbb{C}[y, y^{-1}]$ by

$$\varepsilon_i^- := \sum_{j=0}^{i} k_j p_j(y + y^{-1}) - (m - bmy^{-1})\sum_{j=0}^{n-1} \widetilde{k}_j \widetilde{p}_j(y + y^{-1}),$$

$$\varepsilon_i^+ := (m - bmy^{-1})\sum_{j=0}^{i} \widetilde{k}_j \widetilde{p}_j(y + y^{-1}) - \sum_{j=0}^{n} k_j p_j(y + y^{-1}),$$

where

$$k_j = b_0 b_1 \cdots b_{j-1}/c_1 c_2 \cdots c_j, \qquad \widetilde{k}_j = \widetilde{b}_0 \widetilde{b}_1 \cdots \widetilde{b}_{j-1}/\widetilde{c}_1 \widetilde{c}_2 \cdots \widetilde{c}_j,$$

$$b = \left(\frac{s^*}{r_1 r_2 q^D}\right)^{1/2}, \qquad m = \frac{(1 - s^* q^2)}{(1 - s^* q/r_1)(1 - s^* q/r_2)}.$$

On $\mathbf{W}$ we observe that $\varepsilon_i^-[\mathbf{Y}].\hat{x} = \hat{C}_i^-$ and $\varepsilon_i^+[\mathbf{Y}].\hat{x} = \hat{C}_i^+$. Using the relations (12) and (18) we can replace the parameters r_1, r_2, s, s^*, D by the parameters a, b, c, d.

Lemma 5.3. *Referring to Definition 5.2, for* $0 \le i \le D-1$

$$\varepsilon_i^- = \sum_{j=0}^{i} \frac{(abcd;q)_{2j}}{a^j(q, bc, bd, cd; q)_j} P_j(y + y^{-1}) - (m - bmy^{-1})\sum_{j=0}^{i-1} \frac{(abcdq;q)_{2j}}{a^j(q, bcq, bdq, cd; q)_j} \widetilde{P}_j(y + y^{-1}), \tag{23}$$

$$\varepsilon_i^+ = (m - bmy^{-1})\sum_{j=0}^{i} \frac{(abcdq;q)_{2j}}{a^j(q, bcq, bdq, cd; q)_j} \widetilde{P}_j(y + y^{-1}) - \sum_{j=0}^{i} \frac{(abcd;q)_{2j}}{a^j(q, bc, bd, cd; q)_j} P_j(y + y^{-1}), \tag{24}$$

where

$$m = \frac{1 - abcd}{(1 - bc)(1 - bd)}.$$

We give some comments on $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$. The ε_i^- has the highest degree i and the lowest degree $-i$. By Lemma 5.3 the ε_i^- has of the form

$$\frac{(abcd;q)_{2i}}{a^i(q,bc,bd,cd;q)_i}y^i+\cdots+\frac{(abcd;q)_{2i}}{a^i(q,bc,bd,cd;q)_i}\left(1+\frac{ab(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}}\right)y^{-i}.$$

The ε_i^+ has the highest degree i and the lowest degree $-i-1$. By Lemma 5.3 the ε_i^+ has of the form

$$\frac{(abcd;q)_{2i}}{a^i(q,bc,bd,cd;q)_i}\left(\frac{(1-abcdq^{2i})}{(1-bcq^i)(1-bdq^i)}-1\right)y^i+\cdots+(-1)\frac{(abcd;q)_{2i+2}}{a^{i+1}(q,bc,bd,cd;q)_{i+1}}\frac{ab(1-q^{i+1})(1-cdq^i)}{1-abcdq^{2i+1}}y^{-i-1}.$$

Therefore the set $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$ is linearly independent in $\mathbb{C}[y,y^{-1}]$.

Remark 5.4. Let V denote a subspace of $\mathbb{C}[y,y^{-1}]$ spanned by $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$. Note that $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$ is a basis for V. Observe that the space V is isomorphic to the space $\mathbf{W}$ via an isomorphism that sends $\varepsilon_i^{\pm}$ to $\hat{C}_i^{\pm}$, respectively. View an $\hat{H}_q$-module $\mathbf{W}$ as $\mathbf{W}_{q,a,b,c,d}$ from Remark 4.3. By these comments we can endow a module structure for $\hat{H}_q$ to V, that is, the matrix representing t_n with respect to $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$ coincides with the matrix representing t_n with respect to $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$ for $n\in\mathbb{I}$.

6 How $\varepsilon_i^{\pm}$ are related to $E_{\pm i}$

For the rest of this paper, we set the parameters $a,b,c,d\in\mathbb{C}^*$ that satisfy (1), not involved to the parameters r_1,r_2,s,s^*,D any longer. Referring to Lemma 5.3, for $i=0,1,2,\dots$ define the (infinite) sequence of Laurent polynomials $\mathcal{E}_i^{\pm}$ in $\mathbb{C}[y,y^{-1}]$ by

$$\mathcal{E}_i^- := \sum_{j=0}^{i}\frac{(abcd;q)_{2j}}{a^j(q,bc,bd,cd;q)_j}P_j(y+y^{-1})-(m-bmy^{-1})\sum_{j=0}^{i-1}\frac{(abcdq;q)_{2j}}{a^j(q,bcq,bdq,cd;q)_j}\widetilde{P}_j(y+y^{-1}), \tag{25}$$

$$\mathcal{E}_i^+ := (m-bmy^{-1})\sum_{j=0}^{i}\frac{(abcdq;q)_{2j}}{a^j(q,bcq,bdq,cd;q)_j}\widetilde{P}_j(y+y^{-1})-\sum_{j=0}^{i}\frac{(abcd;q)_{2j}}{a^j(q,bc,bd,cd;q)_j}P_j(y+y^{-1}). \tag{26}$$

Observe that $\mathcal{E}_i^-=\varepsilon_i^-$ and $\mathcal{E}_i^+=\varepsilon_i^+$ for $0\le i\le D-1$. By the comment below Lemma 5.3, we find that the set $\{\mathcal{E}_i^{\pm}\}_{i\ge 0}$ is a basis for $\mathbb{C}[y,y^{-1}]$. Moreover, by Remark 4.3 and 5.4, we can find a module structure for $\hat{H}_q$ on $\mathbb{C}[y,y^{-1}]$; see the Appendix. Identify $\mathcal{L}$ with $\mathbb{C}[y,y^{-1}]$ via a map $z\mapsto y, z^{-1}\mapsto y^{-1}$. On $\mathcal{L}$, for $i\ge 1$ the action of t_0 on the set $\{\mathcal{E}_{i-1}^+,\mathcal{E}_i^-\}$ is

$$\begin{aligned}
t_0.\mathcal{E}_{i-1}^+ &= \left(\frac{(1-abq^i)(1-abcdq^{i-1})}{(ab)^{1/2}(1-abcdq^{2i-1})}+(ab)^{1/2}\right)\mathcal{E}_{i-1}^+ + (ab)^{1/2}\frac{(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}}\mathcal{E}_i^-,\\
t_0.\mathcal{E}_i^- &= -\frac{(1-abq^i)(1-abcdq^{i-1})}{(ab)^{1/2}(1-abcdq^{2i-1})}\mathcal{E}_{i-1}^+ + \left(-\frac{(ab)^{1/2}(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}}+(ab)^{1/2}\right)\mathcal{E}_i^-,
\end{aligned}$$

and $t_0.\mathcal{E}_0^-=(ab)^{1/2}\mathcal{E}_0^-$. We compare this action with the action of T_1 on $\{E_{-i},E_i\}$ in the basic representation of $\tilde{\mathfrak{H}}$.

Theorem 6.1. *On $\mathcal{L}$, for $i\ge 1$ the matrix representing T_1 with respect to*

$$\{E_{-i},E_i\}$$

coincides with the matrix representing $-(ab)^{-1/2}t_0$ with respect to

$$\left\{\mathcal{E}_{i-1}^+,\frac{(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}}\mathcal{E}_i^-\right\}.$$

The matrix is

$$\begin{bmatrix} -\frac{1+ab-abcdq^{i-1}-abq^i}{1-abcq^{2i-1}} & \frac{(1-q^i)(1-abq^i)(1-cdq^{i-1})(1-abcdq^{i-1})}{(1-abcdq^{2i-1})^2} \\ -ab & -\frac{abq^{i-1}(cd+q-cdq^i-abcdq^i)}{1-abcdq^{2i-1}} \end{bmatrix}.$$

On the $\hat{H}_q$-module $\mathcal{L}$, the action of $\mathbf{X}$ on the set $\{\mathcal{E}^+_{i-1}, \mathcal{E}^-_i\}_{i\geq 1}$ is

$$\mathbf{X}.\mathcal{E}^+_{i-1} = q^{-i+\frac{1}{2}}(abcd)^{-1/2}\mathcal{E}^+_{i-1},$$
$$\mathbf{X}.\mathcal{E}^-_i = q^{i-\frac{1}{2}}(abcd)^{1/2}\mathcal{E}^-_i.$$

We compare these actions with (5), (6).

Theorem 6.2. *On $\mathcal{L}$, for $i \geq 1$ the matrix representing Y with respect to*

$$\{E_{-i}, E_i\}$$

coincides with the matrix representing $q^{-1/2}(abcd)^{1/2}\mathbf{X}$ with respect to

$$\left\{\mathcal{E}^+_{i-1}, \frac{(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}}\mathcal{E}^-_i\right\}.$$

The matrix is

$$diag\Big(q^{-i}, q^{i-1}abcd\Big).$$

7 Appendix

Recall the $\hat{H}_q$-module $\mathbf{W}_{q,a,b,c,d}$ from Remark 4.3. In this Appendix we display this module structure explicitly, and extend this finite dimensional module to an infinite dimensional module, which was discussed below the line (26). First, consider the free parameters $a, b, c, d \in \mathbb{C}^*$ that satisfy the condition (1).

Definition 7.1. (a) For $1 \leq i \leq D-1$, the (2×2)-matrix $\tau_0(i)$ is

$$(ab)^{-1/2}\begin{bmatrix} \frac{(1-abq^i)(1-abcdq^{i-1})}{1-abcdq^{2i-1}} + ab & -\frac{(1-abq^i)(1-abcdq^{i-1})}{1-abcdq^{2i-1}} \\ \frac{ab(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}} & -\frac{ab(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}} + ab \end{bmatrix}$$

and

$$\tau_0(0) = \begin{bmatrix} (ab)^{1/2} \end{bmatrix}.$$

(b) For $0 \leq i \leq D-1$, the (2×2)-matrix $\tau_1(i)$ is

$$(ab^{-1})^{1/2}\begin{bmatrix} \frac{(1-bcq^i)(1-bdq^i)}{1-abcdq^{2i-1}} + \frac{b}{a} & -\frac{b}{a}\frac{(1-adq^i)(1-acq^i)}{1-abcdq^{2i}} \\ \frac{(1-bcq^i)(1-bdq^i)}{1-abcdq^{2i}} & \frac{b}{a}\left(1 - \frac{(1-adq^i)(1-acq^i)}{1-abcdq^{2i}}\right) \end{bmatrix}.$$

(c) $0 \leq i \leq D-1$, the (2×2)-matrix $\tau_2(i)$ is

$$(cd)^{-1/2}\begin{bmatrix} \frac{1}{aq^i}\left(1 - \frac{(1-adq^i)(1-acq^i)}{1-abcdq^{2i}}\right) & \frac{bcdq^i(1-adq^i)(1-acq^i)}{1-abcdq^{2i}} \\ -\frac{1}{bq^i}\frac{(1-bcq^i)(1-bdq^i)}{1-abcdq^{2i}} & acdq^i\left(\frac{(1-bcq^i)(1-bdq^i)}{1-abcdq^{2i}} + \frac{b}{a}\right) \end{bmatrix}.$$

(d) For $1 \le i \le D-1$, the (2×2)-matrix $\tau_3(i)$ is

$$(cdq^{-1})^{-1/2}\begin{bmatrix} \frac{1}{q^i}\left(1-\frac{(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}}\right) & \frac{1}{abq^i}\frac{(1-abq^i)(1-abcdq^{i-1})}{1-abcdq^{2i-1}} \\ -\frac{abcdq^{i-1}(1-q^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}} & cdq^{i-1}\left(\frac{(1-abq^i)(1-cdq^{i-1})}{1-abcdq^{2i-1}}+ab\right) \end{bmatrix}$$

and

$$\tau_3(0) = \left[\ (cdq^{-1})^{1/2}\ \right].$$

We consider the block diagonal matrices $\mathbf{T}_n (n \in \mathbb{I})$:

$$\mathbf{T}_0 = \mathrm{blockdiag}\Big[\tau_0(0), \tau_0(1), \dots, \tau_0(D-1), [\ (ab)^{1/2}\ ]\Big], \qquad \mathbf{T}_1 = \mathrm{blockdiag}\Big[\tau_1(0), \tau_1(1), \dots, \tau_1(D-1)\Big],$$

$$\mathbf{T}_3 = \mathrm{blockdiag}\Big[\tau_3(0), \tau_3(1), \dots, \tau_3(D-1), [\ (cdq^{-1})^{-1/2}\ ]\Big], \quad \mathbf{T}_2 = \mathrm{blockdiag}\Big[\tau_2(0), \tau_2(1), \dots, \tau_2(D-1)\Big].$$

Then the $\hat{H}_q$-module $\mathbf{W}_{q,a,b,c,d}$ from Remark 4.3 is described as follows; the matrix $\mathbf{T}_n$ represents the generator t_n with respect to $\{\hat{C}_i^{\pm}\}_{i=0}^{D-1}$.

In remark 5.4 we saw that the V has a module structure for $\hat{H}_q$ with respect to a basis $\{\varepsilon_i^{\pm}\}_{i=0}^{D-1}$. We extend an $\hat{H}_q$-module V to $\mathcal{L}$ as follows. Consider the infinite matrix

$$\mathrm{blockdiag}\Big[\tau_n(0), \tau_n(1), \tau_n(2), \cdots\Big] \qquad (n \in \mathbb{I}). \tag{27}$$

Then $\mathcal{L}$ has a module structure for $\hat{H}_q$ such that (27) represents the generator t_n with respect to $\{\mathcal{E}_i^-, \mathcal{E}_i^+\}_{i\ge 0}$.

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