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A direct sum decomposition of the $kG(p^r)$-submodule generated by the highest weight vector of a certain Weyl G-module

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1 Introduction

In modular representation theory of finite or algebraic groups, the representation theory of algebraic groups plays an important role to study main representations of a finite Chevalley group in the defining characteristic. For example, any simple module for a finite Chevalley group comes from a simple module for the corresponding algebraic group.

A finite Chevalley group has certain induced modules called principal series modules in the defining characteristic, which are as important as simple or projective modules. A principal series module is defined as an induced module from a one-dimensional module for a (finite) Borel subgroup. On the other hand, the corresponding algebraic group also has important modules which are called Weyl modules.

Actually, it is known that the principal series modules are closely related to the Weyl modules 'above' the Steinberg module. This fact was first observed by Pillen (§3 Theorem 2), and after that, the author generalized this result with a weaker assumption on the characteristic of the field (§3 Theorem 3). In this article, we report that this result holds without the assumption, and is best possible (§3 Theorem 4).

2 Preliminaries

Let G be a simply connected and simple algebraic group over an algebraically closed field k of characteristic $p > 0$, which is defined and split over the finite field $\mathbb{F}_p$, and set $q = p^r$. We fix a maximal split torus T and a Borel subgroup B containing T. We shall use the following standard notation:

(1) $X := \mathrm{Hom}(T, k^\times)$: the character group.
(2) $\Phi(\subset X)$: the root system relative to the pair (G, T).
(3) Φ^+ : the set of positive roots where B corresponds to $-\Phi^+$.
(4) $\Delta := \{\alpha_1, \cdots, \alpha_l\} \subseteq \Phi^+$: the set of simple roots.
(5) s_α : the reflection for $\alpha \in \Phi^+$ in the euclidean space $\mathbb{E} := X \otimes_\mathbb{Z} \mathbb{R}$.
(6) $W := N_G(T)/T = \langle s_\alpha | \alpha \in \Delta \rangle$: the Weyl group.
(7) $l(w)$: the length of a reduced expression of $w \in W$
(i.e. $w = s_{\beta_1} \cdots s_{\beta_t}$ with $\beta_i \in \Delta$ $(1 \leq i \leq t)$ and t minimal $\Rightarrow t = l(w)$).
(8) $\dot{w}$: a fixed representative of $w \in W$.

(9) w_0 : the longest element of W which satisfies $l(w_0) = |\Phi^+|$ and $w_0^2 = 1$.
(10) $B^+ := \dot{w}_0 B \dot{w}_0{}^{-1}$: the Borel subgroup opposite to B.
(11) U, U^+ : the unipotent radicals of B and B^+.
(12) $U_\alpha := U^+ \cap \dot{s_\alpha}^{-1} U \dot{s_\alpha}$, $U_{-\alpha} := \dot{s_\alpha}^{-1} U_\alpha \dot{s_\alpha}$ for $\alpha \in \Phi^+$.
(13) $T_\alpha := T \cap \langle U_\alpha, U_{-\alpha}\rangle$ for $\alpha \in \Delta$.
(14) $W_J := \langle s_\alpha | \alpha \in J\rangle$ for $J \subseteq \Delta$.
(15) $w_{0,J}$: the longest element of W_J.
(16) $\langle\cdot,\cdot\rangle$: a W-invariant inner product on $\mathbb{E} = X \otimes_{\mathbb{Z}} \mathbb{R}$.
(17) $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$: the coroot of $\alpha \in \Phi$.
(18) $\omega_i := \omega_{\alpha_i}$: the fundamental weight for $\alpha_i \in \Delta$ (i.e. $\langle \omega_i, \alpha_j^\vee\rangle = \delta_{i,j}$ for any j) (then $X = \sum_{i=1}^l \mathbb{Z}\omega_i$, and a weight $\sum_{i=1}^l c_i\omega_i$ is often written as $(c_1, \cdots, c_l)$).
(19) $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \sum_{i=1}^l \omega_i$ and $\rho_J := \sum_{\alpha\in J}\omega_\alpha$ for $J \subseteq \Delta$.
(20) $X^+ := \sum_{i=1}^l \mathbb{Z}_{\geq 0}\omega_i$: the set of dominant weights.
(21) $V_\lambda := \{v \in V | tv = \lambda(t)v, \forall t \in T\}$: the weight space of weight $\lambda \in X$ in a T-module V.
(22) k_λ : the one-dimensional T-module of weight $\lambda \in X$.
(23) $H^0(\lambda) := \mathrm{Ind}_B^G k_\lambda$: the induced G-module with highest weight $\lambda \in X^+$.
(24) $V(\lambda) := H^0(-w_0\lambda)^*$: the Weyl G-module with highest weight $\lambda \in X^+$ ($*$ denotes the k-dual).
(25) $L(\lambda) := \mathrm{soc}_G H^0(\lambda)$: the simple G-module with highest weight $\lambda \in X^+$.

The set $\{L(\lambda) | \lambda \in X^+\}$ forms the non-isomorphic simple G-modules, where $L(0) \cong k$ (the one-dimensional trivial module) and $L((p^n-1)\rho) \cong \mathrm{St}_n$ (the n-th Steinberg module).

Example 1. Consider the case $G = \mathrm{SL}_2(k)$. Let $E = \left\{\begin{pmatrix} a \\ b\end{pmatrix} \middle| a, b \in k\right\}$ be the natural G-module. Then $H^0(\lambda) \cong \mathrm{Sym}^\lambda(E)$ (the λ-th symmetric power) for $\lambda \in X^+ = \mathbb{Z}_{\geq 0}$. In particular, $E \cong H^0(1)$. Moreover, if $0 \leq \lambda \leq p-1$, then $H^0(\lambda) \cong L(\lambda)$, and if $p \leq \lambda \leq 2p-2$, then $H^0(\lambda)$ has just two composition factors with

$$H^0(\lambda)/\mathrm{rad}_G(H^0(\lambda)) \cong L(2p-2-\lambda)$$

and

$$\mathrm{rad}_G(H^0(\lambda)) = \mathrm{soc}_G(H^0(\lambda)) \cong L(\lambda).$$

Let us explain an alternative definition of Weyl modules. We use the following notation:

(26) $\mathfrak{g}_{\mathbb{C}}$: the simple complex Lie algebra with the same root system as G.
(27) $\{e_\alpha, h_\beta | \alpha \in \Phi, \ \beta \in \Delta\}$: a Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$.
(28) $\mathcal{U}$: the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.
(29) $\mathcal{U}_{\mathbb{Z}}$: the subring of $\mathcal{U}$ generated by all $e_\alpha^{(m)} = e_\alpha^m/m!$ with $\alpha \in \Phi$ and $m \in \mathbb{Z}_{\geq 0}$.

Then the (associative) k-algebra $\mathcal{U}_k = k \otimes_{\mathbb{Z}} \mathcal{U}_{\mathbb{Z}}$ is called the hyperalgebra of G, and the following hold:

· The category of finite dimensional $\mathcal{U}_k$-modules can be identified with that of finite dimensional (rational) G-modules.

· Let $V_{\mathbb{C}}(\lambda)$ be the simple $\mathfrak{g}_{\mathbb{C}}$-module with highest weight λ and v a highest weight vector of $V_{\mathbb{C}}(\lambda)$. Then $V(\lambda) \cong k \otimes_{\mathbb{Z}} (\mathcal{U}_{\mathbb{Z}} v)$ as $\mathcal{U}_k$-modules (i.e. as G-modules).

3 Simple modules and principal series modules for the finite Chevalley group $G(p^r)$

We introduce the following notation:

(30) $X_r := \{\sum_{i=1}^{l} c_i\omega_i \in X^+ | c_i < q, \forall i\}$: the set of q-restricted weights.
(31) $F : G \to G$: the standard Frobenius map relative to $\mathbb{F}_p$.
(32) $G(q) := G^{F^r} = \{g \in G | F^r(g) = g\}$: the corresponding finite Chevalley group.
(33) $H(q) := H \cap G(q)$ for a subgroup H of G (for example, $T(q)$, $B(q)$, $U(q)$, $\cdots$).

The set $\{L(\lambda) | \lambda \in X_r\}$ forms the non-isomorphic simple $kG(q)$-modules, where the r-th Steinberg module $L((q-1)\rho) = \mathrm{St}_r$ is the unique simple projective $kG(q)$-module.

For a finite group H, set $\overline{H} := \sum_{h\in H} h \in kH$. For $\lambda \in X$, the principal series module $M_r(\lambda)$ is defined as

$$M_r(\lambda) := kG(q)\varepsilon_\lambda \quad (\cong kG(q) \otimes_{kB^+(q)} k_\lambda = \mathrm{Ind}_{B^+(q)}^{G(q)} k_\lambda),$$

where $\varepsilon_\lambda := \sum_{t\in T(q)} \lambda(t^{-1})\ t\ \overline{U^+(q)} \in kG(q)$. Note that

$$M_r(\lambda) \overset{G(q)}{\cong} M_r(\mu) \Leftrightarrow k_\lambda \overset{T(q)}{\cong} k_\mu \Leftrightarrow \lambda \equiv \mu \ (\mathrm{mod}\ (q-1)X).$$

Therefore, there are $(q-1)^l$ non-isomorphic principal series modules, and λ in the symbol $M_r(\lambda)$ can be regarded as an element of the quotient group $\Lambda = X/(q-1)X$.

4 Main result

We introduce two sets of simple roots for each $\lambda = \sum_{i=1}^{l} c_i\omega_i \in X_r$:

$$I_0(\lambda) := \{\alpha_i \in \Delta | c_i = 0\},$$

$$I_{q-1}(\lambda) := \{\alpha_i \in \Delta | c_i = q-1\}.$$

A direct sum decomposition of each principal series module was characterized by Sawada.

Theorem 1 ([4, (3.10) Theorem]). For $\lambda \in X_r$, the $kG(q)$-module $M_r(\lambda)$ is decomposed as

$$M_r(\lambda) \cong \bigoplus_{J' \subseteq I_0(\lambda)} \bigoplus_{J \subseteq I_{q-1}(\lambda)} Y(\lambda + (q-1)\rho_{J'} - (q-1)\rho_J),$$

where $Y(\mu)$ is the indecomposable direct summand of $M_r(\lambda)$ with $Y(\mu)/\mathrm{rad}_{kG(q)}Y(\mu) \cong L(\mu)$.

A relation between principal series $kG(q)$-modules and Weyl G-modules was first observed by Pillen in 1997.

Theorem 2 ([3, Theorem 1.2]). Suppose that $q > 2h-1$ (h : Coxeter number) and $\lambda = \sum_{i=1}^{l} c_i\omega_i \in X_r$, and let v be a highest weight vector of $V(\lambda + (q-1)\rho)$. Then

$$kG(q)v \cong M_r(\lambda) \iff c_i > 0, \forall i.$$

In 2012, the author succeeded in generalizing Pillen's result.

Theorem 3 ([5, Theorem 2.1]). Suppose that $q > h+1$ (h : Coxeter number) and $\lambda \in X_r$, and let v be a highest weight vector of $V(\lambda + (q-1)\rho)$. Then

$$kG(q)v \cong \bigoplus_{J \subseteq I_{q-1}(\lambda)} Y(\lambda + (q-1)\rho_{I_0(\lambda)} - (q-1)\rho_J),$$

where each $Y(\mu)$ is the indecomposable summand of $M_r(\lambda)$ as in Theorem 1.

Recently, it turned out that this result did not require the assumption on q. Now we have the best possible result.

Theorem 4 ([7, Theorem 5.1]). Theorem 3 holds for any $q (= p^r)$.

Example. Consider the case $G = \mathrm{SL}_5(k)$. In this case we have $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (the standard numbering of type A_4). Actually, the example in [6, §2] contains an error. Here we give a corrected and somewhat generalized example. We shall take $\lambda = 2\omega_3 + (q-1)\omega_4 = (0, 0, 2, q-1)$, but suppose $q \neq 2$ and $q \neq 3$ so that $\lambda \in X_r$ and $2 \neq q-1$. Then $I_0(\lambda) = \{\alpha_1, \alpha_2\}$ and $I_{q-1}(\lambda) = \{\alpha_4\}$, and $M_1(0, 0, 2, q-1)$ is decomposed as

$$Y(0,0,2,0) \oplus Y(0,0,2,q-1) \oplus Y(0,q-1,2,0) \oplus Y(0,q-1,2,q-1)$$
$$\oplus Y(q-1,0,2,0) \oplus Y(q-1,0,2,q-1) \oplus Y(q-1,q-1,2,0) \oplus Y(q-1,q-1,2,q-1).$$

In the formula the entries of λ being 0 or $q-1$ 'split' into 0 and $q-1$. Moreover, a highest weight vector v of $V(\lambda+(q-1)\rho) = V(q-1,q-1,2+(q-1),2(q-1))$ generates

$$kG(q)v \cong Y(q-1,q-1,2,0) \oplus Y(q-1,q-1,2,q-1).$$

Note that the entries of λ being $q-1$ 'split' into 0 and $q-1$ in this formula, and those being 0 'change' into $q-1$ (the decomposition of $kG(n)v$ in [6, §2 Example] is not correct).

5 A sketch of the proof of Theorem 4

Here we shall give a sketch of the proof of Theorem 4. See [7, §5] for details.

In this section, we always choose a representative $\dot{w}$ of $w \in W$ as in [7, §2]. For $w \in W$ and $\chi \in \Lambda = X/(q-1)X$, the $kG(q)$-homomorphism

$$\mathcal{T}_w : M_r(\chi) \to kG(q), \quad \varepsilon_\chi \mapsto \varepsilon_\chi \dot{w}^{-1}\overline{U^+_{w^{-1},-}(q)},$$

where $U^+_{\sigma,-} = U^+ \cap \dot{\sigma}^{-1}U\dot{\sigma}$ for $\sigma \in W$, is a well-defined map into $M_r(w\chi)$ (i.e. $\mathcal{T}_w : M_r(\chi) \to M_r(w\chi)$). This map $\mathcal{T}_w$ amounts to $T_{\dot{w}}$ in [1], and to $A_{\dot{w}^{-1}}$ in [4].

Set $I_0^\Lambda(\chi) := \{\alpha \in \Delta | \chi \equiv 1 \text{ on } T_\alpha(q)\}$ for $\chi \in \Lambda$.

Property of $\mathcal{T}_w$. Let $\chi \in \Lambda$. Then the following hold:
(1) The set $\{\mathcal{T}_{s_\alpha} | \alpha \in I_0^\Lambda(\chi)\}$ generates the endomorphism algebra $\mathbb{E}_\chi = \mathrm{End}_{kG(q)}(M_r(\chi))$, which is a 0-Hecke algebra of type $(W_{I_0^\Lambda(\chi)}, I_0^\Lambda(\chi))$.
(2) $\mathcal{T}_w = \mathcal{T}_{s_{i_1}} \cdots \mathcal{T}_{s_{i_j}}$ for a reduced expression $w = s_{i_1} \cdots s_{i_j}$.
(3) $\mathcal{T}_w(\varepsilon_\chi) = \overline{U^+_{w,-}(q)}\dot{w}^{-1}\varepsilon_{w\chi}$.

If $\lambda \in X_r$, then let $\bar{\lambda}$ be the image of λ in $\Lambda = X/(q-1)X$. Note that $I_0^\Lambda(\bar{\lambda}) = I_0(\lambda) \cup I_{q-1}(\lambda)$.

For $J \subseteq I_0^\Lambda(\bar{\lambda})$, let π_J be the projection

$$M_r(\lambda) \to Y(\lambda + (q-1)\rho_{I_0(\lambda)-I_0(\lambda)\cap J} - (q-1)\rho_{I_{q-1}(\lambda)\cap J}).$$

Then $\sum_{J \subseteq I_0^\Lambda(\bar{\lambda})} \pi_J$ is a decomposition of the identity map $1 \in \mathbb{E}_\lambda = \mathrm{End}_{kG(q)}(M_r(\lambda))$ into the orthogonal primitive idempotents.

Define a $kG(q)$-homomorphism

$$f : M_r(\lambda) \to V(\lambda) \otimes_k \mathrm{St}_r, \ \varepsilon_\lambda \mapsto v = v_\lambda \otimes v_{(q-1)\rho},$$

where $v_\lambda \in V(\lambda)$ and $v_{(q-1)\rho} \in \mathrm{St}_r$ are highest weight vectors. The image $\mathrm{Im}f$ can be regarded as a $kG(q)$-submodule of $V(\lambda+(q-1)\rho)$ (i.e. $f: M_r(\lambda) \to V(\lambda+(q-1)\rho)$).

To prove Theorem 4, we only have to show the following:

(a) If $J \cap I_0(\lambda) \neq \emptyset$, then f is zero on $\mathrm{Im}\pi_J$.
(b) If $J \cap I_0(\lambda) = \emptyset$ (i.e. $J \subseteq I_{q-1}(\lambda)$), then f is injective on $\mathrm{Im}\pi_J$.

Here we shall prove only (a). It is more complicated to prove (b).

Proof of (a). Set $s_i := s_{\alpha_i}$ and $\mathcal{T}_i := \mathcal{T}_{s_i}$ for simplicity.

For $J \subseteq I_0^\Lambda(\bar{\lambda})$, set $e_J := \sum_{w \in W_J} \mathcal{T}_w$ and $o_J := (-1)^{l(w_{0,J})}\mathcal{T}_{w_{0,J}}$. By Norton's result [2, 4.21 Theorem], we have $e_J o_{\hat{J}} \mathbb{E}_\lambda = \pi_J \mathbb{E}_\lambda$ ($\hat{J} = I_0^\Lambda(\bar{\lambda}) - J$), and so there exists an element $a \in \mathbb{E}_\lambda$ such that $\pi_J = e_J o_{\hat{J}} a$.

Suppose that $J \cap I_0(\lambda) \neq \emptyset$. To prove (a) we need to show that $f(\pi_J(\varepsilon_\lambda)) = 0$. Choose a reduced expression $w_{0,J} = s_{i_1} \cdots s_{i_s}$ with $\alpha_{i_1} \in J \cap I_0(\lambda)$. Then we can write

$$e_J = (1+\mathcal{T}_{i_1}) \cdots (1+\mathcal{T}_{i_s}),$$

and there exists $b \in \mathbb{E}_\lambda$ such that $\pi_J = (1+\mathcal{T}_{i_1})b$. Since $\mathbb{E}_\lambda$ is generated by the $\mathcal{T}_j$'s with $\alpha_j \in I_0^\Lambda(\bar{\lambda})$, Property (3) of $\mathcal{T}_w$ implies that there exists $x_b \in kG(q)$ such that $b(\varepsilon_\lambda) = x_b \varepsilon_\lambda$.

Now we use the following two formulas:

(c) $\mathcal{T}_\alpha(\varepsilon_\lambda) = \overline{U_\alpha(q)} \dot{s_\alpha}^{-1} \varepsilon_\lambda \qquad (\forall \alpha \in I_0^\Lambda(\bar{\lambda}))$.
(d) $\overline{U_\alpha(q)} \dot{s_\alpha}^{-1} v = -v \qquad (\forall \alpha \in \Delta)$.

The formula (c) is a special case of $w = s_\alpha$ ($\alpha \in I_0^\Lambda(\bar{\lambda})$) in Property (3) of $\mathcal{T}_w$. By (c), we have

$$\pi_J(\varepsilon_\lambda) = (1+\mathcal{T}_{i_1})b(\varepsilon_\lambda) = x_b \cdot (1+\mathcal{T}_{i_1})(\varepsilon_\lambda) = x_b(1+\overline{U_{\alpha_{i_1}}(q)}\dot{s_{i_1}}^{-1})\varepsilon_\lambda.$$

So it is enough to show that $(1+\overline{U_{\alpha_{i_1}}(q)}\dot{s_{i_1}}^{-1})v = 0$. But this comes from (d), and (a) is proved. □

References

[1] R. W. Carter and G. Lusztig, *Modular representations of finite groups of Lie type*, Proc. London Math. Soc. **32** (1976), 347-384.

[2] P. N. Norton, *0-Hecke algebras*, J. Austral. Math. Soc. Ser. A **27** (1979), 337-357.

[3] C. Pillen, *Loewy series for principal series representations of finite Chevalley groups*, J. Algebra **189** (1997), 125-149.

[4] H. Sawada, *A characterization of the modular representations of finite groups with split (B, N) pairs*, Math. Z. **155** (1977), 29-41.

[5] Y. Yoshii, *A generalization of Pillen's theorem for principal series modules*, Proc. Amer. Math. Soc. **140** (2012), 3761-3768.

[6] Y. Yoshii, *Weyl modules and Principal series modules*, in "Topics in Combinatorial Representation Theory", Surikaisekikenkyusho Kokyuroku **1795** (2012), pp. 83-87.

[7] Y. Yoshii, *A generalization of Pillen's theorem for principal series modules II*, J. Algebra **429** (2015), 177-191.