数理解析研究所講究録
第 1967 巻 2015 年 75-81

ENDO-TRIVIAL MODULES AND WEAK HOMOMORPHISMS

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ABSTRACT. This is a report on some recent joint work with Jon F. Carlson, which appears in [CT4]. It is an expanded version of a talk given at the RIMS workshop *Cohomology of finite groups and related topics*, February 18–20, 2015.

1. INTRODUCTION

Endo-trivial modules for a finite group G over a field k of prime characteristic p play a significant role in modular representation theory. They have been classified in case G is a p-group [CT2, CT3] and various results have appeared since for some specific families of groups [CHM, CMN1, CMN2, CMN3, CMT1, CMT2, CMT3, LM2, Ma, MT, NR]. Another line of research is concerned with the classification of all endo-trivial modules which are simple [Ro, LMS, LM1].

The abelian group $T(G)$ is finitely generated and its torsion-free part is essentially known (see [CMT3] for details). The question remains of describing its torsion subgroup, written $TT(G)$. Let S be a Sylow p-subgroup of G, which we suppose nontrivial. The subgroup

$$K(G) = \mathrm{Ker}\,\{\,\mathrm{Res}^G_S : T(G) \longrightarrow T(S)\,\}$$

is easily seen to be finite and it is known to be equal to the whole torsion subgroup $TT(G)$ in most cases. Specifically, this happens whenever S is not cyclic, generalized quaternion, or semi-dihedral (because, if we exclude these three cases, then $T(S)$ is torsion-free by [CT3]). The excluded cases are treated in [CMT2], [Ka] and [MT].

The problem is now to describe $K(G)$ and this uses a new approach, introduced by Balmer in [Ba]. He shows that $K(G)$ is isomorphic to the group $A(G)$ of all weak homomorphisms $G \to k^\times$ (defined below in Section 3). One can then use $A(G)$ instead of $K(G)$. Let $N := N_G(S)$ be the normalizer of S. It is not hard to show that $A(G)$ embeds via restriction into $A(N)$ and the problem is to describe its image. Moreover, because N has a nontrivial normal p-subgroup, it turns out that every weak homomorphisms $N \to k^\times$ is actually an ordinary group homomorphism, so that $A(N) = \mathrm{Hom}(N, k^*)$, the abelian group of all one-dimensional representations of N.

The main difficulty is to describe the image of the injective map

$$\mathrm{Res}^G_N : A(G) \longrightarrow A(N) = \mathrm{Hom}(N, k^*)\,.$$

By duality of abelian groups, this image is of the form $\mathrm{Hom}(N/J, k^*)$ for some normal subgroup J such that N/J is abelian and of order prime to p. In other words, the image is the dual group $(N/J)^*$ of the abelian group N/J. So the problem of describing $K(G)$ comes down to the following specific question : What is J ?

A system of subgroups $\{\rho^i(S)\}$ was introduced in [CMN3] (for solving the problem when G is a general linear group) and was further considered in [CT4] for an arbitrary finite group G (see Section 4 for the definition). We have a nested sequence of subgroups

$$S \subseteq \rho^1(S) \subseteq \rho^2(S) \subseteq \ldots \subseteq N = N_G(S),$$

and we let $\rho^\infty(S)$ be the limit of the system, namely the union of all $\rho^i(S)$.

The following theorem was proved in [CT4] :

Theorem 1.1. *With the notation above,* $\rho^\infty(S) \subseteq J$.

The question of equality is open and is stated as a conjecture in [CT4] :

Conjecture 1.2. $J = \rho^\infty(S)$.

Finally, the main theorem of [CT4] gives a positive answer in a special case :

Theorem 1.3. *If a Sylow p-subgroup S of G is abelian, then $J = \rho^2(S) = \rho^\infty(S)$. In other words, $K(G)$ is isomorphic to the dual group $\big(N_G(S)/\rho^2(S)\big)^*$ of the abelian group $N_G(S)/\rho^2(S)$.*

In view of the direct description of the subgroup $\rho^2(S)$, this theorem completes the classification of all torsion endo-trivial modules when a Sylow p-subgroup is abelian.

2. Restriction of endo-trivial modules

Let k denote an algebraically closed field of prime characteristic p and let G be a finite group. We assume that G has order divisible by p and we let S be a Sylow p-subgroup of G. Recall that a kG-module M is *endo-trivial* if its endomorphism algebra $\mathrm{End}_k(M)$ is isomorphic (as a kG-module) to the direct sum of the trivial module k and a projective kG-module. In other words, M is endo-trivial if and only if $\mathrm{Hom}_k(M,M) \cong M^\vee \otimes M \cong k \oplus (\mathrm{proj})$, where $M^\vee$ denotes the k-dual of M. Any endo-trivial module M splits as the direct sum $M = M_0 \oplus (\mathrm{proj})$ for an indecomposable endo-trivial kG-module M_0, which is unique up to isomorphism. We let $T(G)$ be the set of equivalence classes of endo-trivial kG-modules for the equivalence relation

$$M \sim L \iff M_0 \cong L_0 .$$

The tensor product induces an abelian group structure on the set $T(G)$. The identity element is the class of the trivial module, while the inverse of the class of a module M is the class of the dual module $M^\vee$. By a theorem of Puig, the group $T(G)$ is known to be a finitely generated abelian group.

A useful fact is the following.

Lemma 2.1. *Let M be a kG-module.*

(a) *M is endo-trivial if and only if $M{\downarrow}^G_E$ is endo-trivial for every elementary abelian p-subgroup E of G.*
(b) *If M satisfies the condition $M{\downarrow}^G_S \cong k\oplus(\mathrm{proj})$, where S is a Sylow p-subgroup of G, then M is endo-trivial.*

The proof of (a) uses Chouinard's theorem. It appears in Lemma 2.9 of [CT1] for p-groups, but the proof is the same for any finite group. Statement (b) follows immediately from (a).

We now let $K(G)$ be the kernel of the restriction map $\mathrm{Res}^G_S : T(G) \longrightarrow T(S)$, where S is a Sylow p-subgroup of G. In other words, the class of an endo-trivial kG-module M belongs to $K(G)$ if and only if M has *trivial Sylow restriction*, that is, $M{\downarrow}^G_S \cong k \oplus (\mathrm{proj})$. This implies in particular that, if M is indecomposable, then M has vertex S and trivial source. Note also that if a kG-module M satisfies the condition $M{\downarrow}^G_S \cong k \oplus (\mathrm{proj})$, then M is necessarily endo-trivial by Lemma 2.1, and its class lies in $K(G)$.

Lemma 2.2. *Let $K(G)$ be the kernel of the restriction map $\mathrm{Res}^G_S : T(G) \longrightarrow T(S)$.*

(a) *$K(G)$ is a finite subgroup of $T(G)$.*
(b) *$K(G)$ is the entire torsion subgroup $TT(G)$ of $T(G)$, provided S is not cyclic, generalized quaternion, or semi-dihedral.*

This is proved in Lemma 2.3 of [CMT1]. The first part is easy and is due to the fact that there are finitely many indecomposable kG-modules with trivial source. The second statement is much deeper and depends on the fact that, by the main result of [CT3], $T(S)$ is torsion-free if S is not cyclic, generalized quaternion, or semi-dihedral.

We now introduce some notation. For any finite group H, we let H' be the smallest normal subgroup of H such that H/H' is an abelian p'-group. In other words $H' = [H, H]S$ is the subgroup of H generated by the commutator subgroup $[H, H]$ and by a Sylow p-subgroup S of H. Clearly, H' is in the kernel of any group homomorphism $H \to k^\times$, where $k^\times$ denotes the group of nonzero elements of k. Since k contains all p'-roots of unity (because k is algebraically closed by assumption), H' is actually the intersection of the kernels of all group homomorphisms $H \to k^\times$. In other words, $\mathrm{Hom}(H, k^\times) \cong (H/H')^*$, the dual group of the abelian group H/H'.

The next result is a straightforward application of the Mackey formula. The details appear in Lemma 2.6 of [MT].

Lemma 2.3. *Suppose that a finite group H has a nontrivial normal p-subgroup. If M is an indecomposable kH-module with trivial Sylow intersection (so that M is endo-trivial and its class is in $K(H)$), then M has dimension one. In other words $K(H) \cong \mathrm{Hom}(H, k^\times) \cong (H/H')^*$.*

Our next result is an easy application of the Green correspondence. For details, see Proposition 2.6 in [CMN1].

Lemma 2.4. *Let S be a Sylow p-subgroup of G and let $N = N_G(S)$.*

(a) *The restriction map $\mathrm{Res}^G_N : T(G) \to T(N)$ is injective, induced by the Green correspondence.*
(b) *In particular, the restriction map $\mathrm{Res}^G_N : K(G) \to K(N)$ is injective.*

By Lemma 2.3, we know that $K(N)$ consists of the classes of all one-dimensional representations of N. The main problem is to know which of them are in the image of the restriction map from $K(G)$. In other words, given a one-dimensional kN-module U, we need to know when its Green correspondent M is endo-trivial.

Definition 2.5. *We define J to be the intersection of the kernels of all the one-dimensional kN-modules U such that the Green correspondent of U is an endo-trivial kG-module. Thus J is a subgroup of N, which can also be characterized as the intersection of the kernels of all the one-dimensional kN-modules U whose class lies in the image of the restriction $\mathrm{Res}^G_N : T(G) \to T(N)$.*

In other words, we obtain the following.

Lemma 2.6. *The image of the restriction map $\mathrm{Res}^G_N : K(G) \to K(N)$ is equal to $(N/J)^* \cong \mathrm{Hom}(N/J, k^\times)$, as a subgroup of $\mathrm{Hom}(N, k^\times) \cong K(N)$. In other words, $K(G) \cong (N/J)^* \cong \mathrm{Hom}(N/J, k^\times)$.*

We see that the problem of characterizing the group $K(G)$ is equivalent to the question of finding the subgroup J.

3. Weak homomorphisms

In [Ba], Balmer provided a new characterization of the group $K(G)$ in terms of the group of weak homomorphisms. As above, S denotes a Sylow p-subgroup of G.

Definition 3.1. *A map $\chi : G \to k^\times$ is called a* weak homomorphism *if it satisfies the following three conditions:*

(a) *If $s \in S$, then $\chi(s) = 1$.*
(b) *If $g \in G$ and $S \cap {}^gS = \{1\}$, then $\chi(g) = 1$.*
(c) *If $a, b \in G$ and if $S \cap {}^aS \cap {}^{ab}S \neq \{1\}$, then $\chi(ab) = \chi(a)\chi(b)$.*

The set $A(G)$ of all weak homomorphisms is an abelian group under the usual product of maps.

Theorem 3.2. *(Balmer [Ba]) The groups $K(G)$ and $A(G)$ are isomorphic.*

Balmer's isomorphism is explicit and is described in [Ba]. Things become easy for a group H having a nontrivial normal p-subgroup, thanks to the third condition in Definition 3.1.

Lemma 3.3. *Suppose that a finite group H has a nontrivial normal p-subgroup. Then every weak homomorphism $\chi : H \to k^\times$ is a group homomorphism.*

Consequently, we obtain Balmer's isomorphism in this special case :

$$A(H) \cong \mathrm{Hom}(H, k^\times) \cong (H/H')^* \cong K(H).$$

In view of Balmer's isomorphism, Lemma 2.4 and Lemma 2.6 can now be restated.

Lemma 3.4. *Let S be a Sylow p-subgroup of G and let $N = N_G(S)$.*

(a) *The restriction map $\mathrm{Res}^G_N : A(G) \to A(N)$ is injective.*
(b) *The image of the restriction map $\mathrm{Res}^G_N : A(G) \to A(N)$ is equal to $(N/J)^* = \mathrm{Hom}(N/J, k^\times)$, as a subgroup of $\mathrm{Hom}(N, k^\times) = A(N)$.*

It would be interesting to have a direct proof of (a), using only the definition of weak homomorphisms. Note that we still face the problem of finding what is the subgroup J.

4. A system of local subgroups

For any nontrivial subgroup Q of a Sylow p-subgroup S, we define a sequence of subgroups $\{\rho^i(Q) \mid i \geq 1\}$ inductively as follows :

$$\rho^1(Q) := N_G(Q)'.$$

As before, $N_G(Q)'$ is the product of the commutator subgroup of $N_G(Q)$ and a Sylow p-subgroup of $N_G(Q)$. Note that $Q \subseteq \rho^1(Q) \subseteq N_G(Q)$. For $i \geq 2$, we let

$$\rho^i(Q) := < N_G(Q) \cap \rho^{i-1}(R) \mid \{1\} \neq R \subseteq S >,$$

the subgroup generated by all the subgroups $N_G(Q) \cap \rho^{i-1}(R)$, for all nontrivial subgroups R of S. This contains $\rho^{i-1}(Q)$, so we have a nested sequence of subgroups

$$Q \subseteq \rho^1(Q) \subseteq \rho^2(Q) \subseteq \rho^3(Q) \subseteq \ldots \subseteq N_G(Q).$$

Since G is finite, the sequence eventually stabilizes and we let $\rho^\infty(Q)$ be the limit subgroup of the sequence $\{\rho^i(Q) \mid i \geq 1\}$, namely their union.

The following observation was made in [CMN3].

Proposition 4.1. *Let $\chi : G \to k^\times$ be a weak homomorphism. If $x \in \rho^i(Q)$ for some $i \geq 1$ and for some nontrivial subgroup $Q \subseteq S$, then $\chi(x) = 1$.*

In the case that $i = 1$, the statement is a trivial consequence of Lemma 3.3 applied to $H = N_G(Q)$. Then the proposition is proved by induction (see Proposition 4.1 in [CT4] for details).

Applying this to the case of the Sylow p-subgroup S, we now obtain the following theorem about the subgroup J defined in 2.5.

Theorem 4.2. $\rho^\infty(S) \subseteq J$.

Proof. Recall that J is the intersection of the kernels of all the one-dimensional kN-modules U such that the Green correspondent of U is an endo-trivial kG-module. Let U be one of them and let M be its Green correspondent. Then M is endo-trivial and $M{\downarrow}^G_N \cong U \oplus (\text{proj})$. In order to translate this information in terms of weak homomorphisms, we let $\chi : G \to k^\times$ be the weak homomorphism corresponding to the class of M under Balmer's isomorphism $K(G) \cong A(G)$. Then $\chi|_N : N \to k^\times$ is a homomorphism (by Lemma 3.3) and this corresponds to the one-dimensional kN-module U under Balmer's isomorphism $K(N) \cong \mathrm{Hom}(N, k^\times) = A(N)$. By Proposition 4.1 above, χ vanishes on $\rho^\infty(S)$ and therefore

$$\rho^\infty(S) \subseteq \mathrm{Ker}(\chi|_N) = \mathrm{Ker}(U).$$

This holds for every U as above, so $\rho^\infty(S)$ is contained in the intersection of the corresponding kernels, that is, $\rho^\infty(S) \subseteq J$. □

Theorem 4.2 is essentially proved in [CT4], except that the result is stated in the following equivalent form (see Theorem 4.3 in [CT4]).

Theorem 4.3. *Suppose that M is a kG-module with trivial Sylow restriction, i.e.* $M{\downarrow}^G_S \cong k \oplus (\text{proj})$. *Then* $M{\downarrow}^G_{\rho^\infty(S)} \cong k \oplus (\text{proj})$.

Proof. Since $M{\downarrow}^G_S \cong k \oplus (\text{proj})$, the module M must be endo-trivial, by Lemma 2.1. Then we have $M{\downarrow}^G_N \cong U \oplus (\text{proj})$ for some indecomposable endo-trivial kN-module U. By Lemma 2.3, U has dimension 1. By Theorem 4.2, $\rho^\infty(S) \subseteq \text{Ker}(U)$ and therefore $U{\downarrow}^N_{\rho^\infty(S)} \cong k$. It follows that

$$M{\downarrow}^G_{\rho^\infty(S)} = M{\downarrow}^G_N{\downarrow}^N_{\rho^\infty(S)} \cong \big(U \oplus (\text{proj})\big){\downarrow}^N_{\rho^\infty(S)} \cong k \oplus (\text{proj})\,,$$

as required. □

5. A conjecture

We conjecture that the inclusion in Theorem 4.2 is an equality. This is Conjecture 5.5 in [CT4].

Conjecture 5.1. $J = \rho^\infty(S)$.

Evidence for this conjecture is based on numerous examples (see Section 8 in [CT4]), as well as on the following positive answer in the special case when G has an abelian Sylow p-subgroup.

Theorem 5.2. *Suppose that a Sylow p-subgroup S of G is abelian. Let $N = N_G(S)$.*

(a) *The image of the restriction map* $\text{Res}^G_N : A(G) \to A(N)$ *consists exactly of all group homomorphisms* $N_G(S) \to k^\times$ *having* $\rho^2(S)$ *in their kernel.*

(b) $K(G) \cong A(G) \cong \big(N_G(S)/\rho^2(S)\big)^*$.

(c) $J = \rho^2(S) = \rho^\infty(S)$.

This theorem is proved in [CT4]. Note that (b) and (c) follow immediately from (a), using Lemma 2.6 and Lemma 3.4. Thus the important part is the proof of (a). Starting from a group homomorphism $N_G(S) \to k^\times$ having $\rho^2(S)$ in its kernel, one has to extend it to a weak homomorphism $G \to k^\times$. Thanks to the assumption that S is abelian, this is made possible by using an explicit form of the fact that $N_G(S)$ controls fusion (Burnside's theorem). The property that S is abelian is also used in the fact that, for any subgroup Q of S, the group S is contained in $N_G(Q)$, allowing for a Frattini argument. We refer to [CT4] for more details.

Acknowledgements

The author is very grateful to Prof. Fumihito Oda for his invitation, supported by Kinki University, Osaka.

The author would like also to thank Prof. Akihiko Hida for giving the opportunity to speak at the RIMS workshop *Cohomology of finite groups and related topics*, February 18–20, 2015.

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