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The splitting of cohomology of metacyclic p-groups

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Abstract

Let BP be the p-complete classifying space of a metacyclic p-group P. By using stable homotopy splitting of BP, we study the decomposition of $H^{even}(P;\mathbb{Z})/p$ and $CH^*(BP)/p$.

1 Introduction

Let P be a p-group and BP be its p-completed classifying space of P. We study the stable splitting and splitting of cohomology

$$(*) \quad BP \cong X_1 \vee ... \vee X_i,$$

$$(**) \quad H^*(P) \cong H^*(X_1) \oplus ... \oplus H^*(X_i) \quad (for\ * > 0)$$

where X_i are irreducible spaces in the stable homotopy category. Using the answer of the Segal conjecture by Carlsson, the splitting $(*)$ is given by only using modular representation theory by Nishida [Ni], Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr]. These theorems do not use splittings of cohomology

In particular, Dietz and Dietz-Priddy [Di], [Di-Pr] gave the stable splitting $(*)$ for groups P with $rank_p(P) = 2$ for $p \geq 5$. However it was not used splittings $(**)$ of the cohomology $H^*(P)$, and the cohomologies $H^*(X_i)$ were not given there.

In [Hi-Ya 1,2], we gives the cohomology of $H^*(X_i)$ (and hence $(**)$) for $P = (\mathbb{Z}/p)^2$ and $P = p_+^{1+2}$ the extraspecial p group of order p^3 and exponent p. Their cohomology $H^*(X_i)$ have very complicated but rich structures, in fact p_+^{1+2} is a p-Sylow subgroup of many interesting groups, e.g., $GL_3(\mathbb{F}_p)$ and many simple groups e.g. J_4 for $p = 3$.

In this paper, we give the decomposition of

$$H^*(P) = H^*(P;\mathbb{Z})/(p, \sqrt{0}) \qquad (and\ \ H^{ev}(P) = H^{even}(P;\mathbb{Z})/p)$$

for metacyclic p-groups for odd primes p, while in most cases, $H^*(X_i)$ are easily got and seemed not to have so rich structure as p_+^{1+2}, because they are not p-Sylow subgroups of so interesting groups. Indeed, metacyclic p-groups P are

Swan groups, i.e. for all groups G which have a Sylow p-subgroup isomorphic to P, we have the isomorphism

$$H^*(G) \cong H^*(P)^W \quad for\ some\ W \subset Out(P).$$

However, we believe that it becomes quite clear the relations among splittings of different types of metacyclic p-groups. (We compute the coarse splitting of $H^*(X_i)$ at first, and next more fine splitting $H^*(X'_j)$, in the case $H^*(P) \cong H^*(P')$).

In the last section, we note the relation to the Chow ring $CH^*(BP)/p$ and $H^{even}(P;\mathbb{Z})/p$, and note that the Chow group of the direct summand X_i is represented by some motive.

2 The double Burnside algebra and stable splitting

Let us fix an odd prime p and $k = \mathbb{F}_p$. For finite groups G_1, G_2, let $A_{\mathbb{Z}}(G_1, G_2)$ be the double Burnside group defined by the Grothendieck group generated by (G_1, G_2)-bisets. Each element Φ in $A_{\mathbb{Z}}(G_1, G_2)$ is generated by elements $[Q, \phi] = (G_1 \times_{(Q,\phi)} G_2)$ for some subgroup $Q \le G_1$ and a homomorphism $\phi : Q \to G_2$. In this paper, we use the notation

$$[Q, \phi] = \Phi : G_1 \ge Q \overset{\phi}{\to} G_2.$$

For each element $\Phi = [Q, \phi] \in A_{\mathbb{Z}}(G_1, G_2)$, we can define a map from $H^*(G_2; k)$ to $H^*(G_1; k)$ by

$$x \cdot \Phi = x \cdot [Q, \phi] = Tr_Q^{G_1} \phi^*(x) \quad for\ x \in H^*(G_2; k).$$

When $G_1 = G_2$, the group $A_{\mathbb{Z}}(G_1, G_2)$ has the natural ring structure, and call it the (integral) double Burnside algebra. In particular, for a finite group G, we have an $A_{\mathbb{Z}}(G, G)$-module structure on $H^*(G; k)$ (and $H^*(G; \mathbb{Z})/p$.)

The following lemma is an easy consequence of Quillen's theorem such that the restriction map

$$H^*(G; \mathbb{Z}/p) \to \lim_V H^*(V; \mathbb{Z}/p)$$

is an F-isomorphism (i.e. the kernel and cokernel are nilpotent) where V ranges elementary abelian p-subgroups of G.

Lemma 2.1. *Let $\sqrt{0}$ be the nilpotent ideal in $H^*(G; k)$ (or $H^*(G; \mathbb{Z})/p$). Then $\sqrt{0}$ itself is an $A_{\mathbb{Z}}(G, G)$-module.*

In this paper we consider, at first, the cohomology modulo nilpotents elements, since it is not so complicated from the above Quillen's theorem. Hence we write it simply

$$H^*(G) = H^*(G; \mathbb{Z})/(p, \sqrt{0}).$$

However we also compute $H^{even}(G;\mathbb{Z})/p$ in §4 below.

By the preceding lemma, we see that $H^*(G)$ has the $A_{\mathbb{Z}}(G,G)$-module structure. (Here note that $A_{\mathbb{Z}}(G,G)$ acts on unstable cohomology.) Throughout this paper, we assume that degree $* > 0$ (or we consider $H^*(-)$ as the reduced theory $\tilde{H}^*(-)$). (We consider $H^*(G)$ as an element in $K_0(Mod(A_{\mathbb{Z}}(G,G)))$.)

Let $BG = BG_p$ be the p-completion of the classifying space of G. Recall that $\{BG, BG\}_p$ is the (p-completed) group generated by stable homotopy self maps. It is well known from the Segal conjecture (Carlsson's theorem) that this group is isomorphic to the double Burnside group $A_{\mathbb{Z}}(G_1,G_2)^{\wedge}$ completed by the augmentation ideal.

Since the transfer is represented as a stable homotopy map Tr, an element $\Phi = [Q,\phi] \in A(G_1,G_2)$ is represented as a map $\Phi \in \{BG_1, BG_2\}_p$

$$\Phi : BG_1 \overset{Tr}{\to} BQ \overset{B\phi}{\to} BG_2.$$

(Of course, the action for $x \in H^*(G_2)$ is given by $Tr_Q^{G_1}\phi^*(x)$ as stated.)

Let us write

$$A(G_1,G_2) = A_{\mathbb{Z}}(G_1,G_2) \otimes k \quad (k = \mathbb{Z}/p).$$

Hereafter we consider the cases $G_i = P$; p-groups. Given a primitive idempotents decomposition of the unity of $A(P,P)$

$$1 = e_1 + ... + e_n,$$

we have an indecomposable stable splitting

$$BP \cong X_1 \vee ... \vee X_n \quad with\ e_iBP = X_i.$$

In this paper, an isomorphism $X \cong Y$ for spaces means that it is a stable homotopy equivalence.

Recall that

$$M_i = A(P,P)e_i/(rad(A(P,P)e_i)$$

is a simple $A(P,P)$-module where $rad(-)$ is the Jacobson ideal. By Wedderburn's theorem, the above decomposition is also written as

$$BP \cong \vee_j(\vee_k X_{jk}) = \vee_j m_j X_{j1} \quad where\ m_j = dim(M_j)$$

for $A(P,P)e_{jk}/rad(A(P,P)e_{jk}) \cong M_j$. Therefore the stable splitting of BP is completely determined by the idempotent decomposition of the unity in the double Burnside algebra $A(P,P)$.

For a simple $A(P,P)$-module M, define a stable summand $X(M)$ by

$$e_M = \sum_{M_i \cong M} e_i, \quad X(M) = \vee_{M_{jk}\cong M} X_{jk} = e_M BP.$$

Here $X(M)$ is only defined in the stable homotopy category. (So strictly, the cohomology ring $H^*(X(M))$ is not defined.) However we define $H^*(X(M))$ by

$$H^*(X(M)) = H^*(P) \cdot e_M \quad (= e_M^* H^*(P)\ stabely)$$

where we think $e_M \in A(P,P)$ (rather than $e_M \in \{BP, BP\}$).

Lemma 2.2. *Given a simple $A(P,P)$-module M, there is a filtration of $H^*(X(M))$ such that the associated graded ring $grH^*(X(M))$ is isomorphic to a sum of M, i.e., (for $* > 0$)*

$$grH^*(X(M)) \cong \oplus_{i=1} M[k_i], \qquad 0 \le k_1 \le ... \le k_s \le ...$$

where $[k_s]$ is the operation ascending degree k_s.

From Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr], it is known that each simple $A(P,P)$-module is written as

$$S(P,Q,V) \quad for\ Q \le P,\ and\ V : simple\ k[Out(Q)] - module.$$

(In fact $S(P,Q,V)$ is simple or zero.) Thus we have the main theorem of stable splitting of BP.

Theorem 2.3. *(Benson-Feshbach [Be-Fe], Martino-Priddy [Ma-Pr]) There are indecomposable stable spaces $X_{S(P,Q,V)}$ for $S(P,Q,V) \neq 0$ such that*

$$BP \cong \vee X(S(P,Q,V)) \cong \vee(dimS(P,Q,V))X_{S(P,Q,V)}.$$

3 Metacyclic groups for $p \ge 3$

In this section, we consider metacyclic p groups P for $p \ge 3$

$$0 \to \mathbb{Z}/p^m \to P \to \mathbb{Z}/p^n \to 0.$$

These groups are represented as

$$(*) \quad P = \langle a, b | a^{p^m} = 1, a^{p^{m'}} = b^{p^n}, [a,b] = a^{rp^\ell} \rangle \quad r \neq 0\ mod(p).$$

It is known by Thomas [Th], Huebuschmann [Hu] that $H^{even}(P;\mathbb{Z})$ is generated by Chern classes of complex representations. Let us write

$$\begin{cases} y = c_1(\rho), & \rho : P \to P/\langle a \rangle \to \mathbb{C}^* \\ v = c_{p^{m-\ell}}(\eta), & \eta = Ind_H^P(\xi), \quad \xi : H = \langle a, b^{p^{m-\ell}} \rangle \to \langle a \rangle \to \mathbb{C}^* \end{cases}$$

where ρ, ξ are nonzero linear representations. Then $H^{even}(P;\mathbb{Z})$ is generated by

$$y,\ c_1(\eta), c_2(\xi), ..., c_{p^{m-\ell}}(\eta) = v.$$

(Lemma 3.5 and the explanation just before this lemma in [Ya1].) We can see

$$c_1(\eta) = 0, ..., c_{p^{m-\ell}-1}(\eta) = 0 \quad in\ H^*(P) = H^*(P;\mathbb{Z})/(p, \sqrt{0}).$$

By using Quillen's theorem and the fact that P has just one conjugacy class of maximal abeian p-subgroups, we can prove

Theorem 3.1. *(Theorem 5.45 in [Ya1]) For any metacyclic p-group P in $(*)$ with $p \geq 3$, we have a ring isomorphism*

$$H^*(P) \cong k[y,v], \quad |v| = 2p^{m-\ell}.$$

We now consider the stable splitting.

(I) Non split cases. For a nonsplit metacyclic groups, it is proved that BP itself is irreducible [Di].

(II) Split cases with $(\ell,m,n) \neq (1,2,1)$. We consider a split metacyclic group. it is written as

$$P = M(\ell,m,n) = \langle a,b | a^{p^m} = b^{p^n} = 1, [a,b] = a^{p^\ell} \rangle$$

for $m > \ell \geq max(m-n,1)$.

The outer automorphism is the semidirect product

$$Out(P) \cong (p - group) : \mathbb{Z}/(p-1).$$

The p-group acts trivially on $H^*(P)$, and $j \in \mathbb{Z}/(p-1)$ acts on $a \mapsto a^j$ and so acts on $H^*(P)$ as $j^* : v \mapsto jv$. There are $p-1$ simple $\mathbb{Z}/(p-1)$-modules $S_i \cong k\{v^i\}$. We consider the decomposition by idempotens for $Out(P)$. Let us write $Y_i = e_{S_i}BP$ and

$$H_i^*(P) = H^*(S_i) \cong (dim(S_i))H^*(Y_i) \subset H^*(P).$$

Hence we have the decomposition for $Out(P)$-idempotents

$$H^*(Y_i) = H_i(P) \cong k[y,V]\{v^i\}, \quad V = v^{p-1}.$$

Here we used the notation such that $A\{a,b,...\}$ means the A-free module generated by $a,b,....$

We assume $P \neq M(1,2,1)$. By Dietz, we have splitting

$$(**) \quad BP \cong \vee_{i=0}^{p=2} X_i \vee \vee_{i=0}^{p-2} L(1,i).$$

Here we write $X_i = e_{S(P,P,S_i)}BP$ identifying S_i as the $A(P,P)$ simple module (but not the simple $Out(P)$-module).

The summand $L(1,i)$ is defined as follows. Recall that $H^*(\langle b \rangle) \cong k[y]$. The outer automorphism group is $Out(\langle b \rangle) \cong (\mathbb{Z}/p^n)^*$ and its simple k modules are $S_i' = k\{y^i\}$ for $0 \leq i \leq p-2$. Hence we can decompose

$$B\langle b \rangle \cong \vee_{i=0}^{p-2} L(1,i), \quad H^*(L(1,i) \cong k[Y]\{y^i\} \quad with\ Y = y^{p-1}.$$

Next we consider $L(1.i)$ as a split summand in BP as follows. (Consider the $A(P,P)$-simple module $S(P,\langle b \rangle, S_i')$.) Let $\Phi \in A(P,P)$ be the element defined by the map $\Phi : P \to \langle b \rangle \subset P$ which induced the isomorphism

$$H^*(P)\Phi \cong H^*(\vee_{i=0}^{p-2} L(1,i)) \cong k[y].$$

Thus we can show (since $k[y]$ is invariant under elements in $Out(P)$)

$$(***) \quad Y_i \cong \begin{cases} X_i & i \neq 0 \\ X_0 \vee \vee_{j=0}^{p-2} L(1,j) & i = 0. \end{cases}$$

Theorem 3.2. *Let P be a split metacyclic group with $(\ell, m, n) \neq (1,2,1)$. Then we have*

$$H^*(X_i) \cong \begin{cases} k[y,V]\{v^i\} & i \neq 0 \\ k[y,V]\{V\} & i = 0. \end{cases}$$

Proof. For $i \neq 0$, we have $H_i^*(P) = H^*(Y_i) \cong H^*(X_i)$. Let us use the notation that $A \ominus B = C$ means $A \cong B \oplus C$. Then we see

$$H^*(X_0) \cong H^*(Y_0) \ominus H^*(\vee_{j=0}^{p-2} L(1,j))$$

$$\cong k[y,V] \ominus k[y] \cong k[y,V]\{V\}.$$

□

(III) Split metacycle group with $(\ell, m, n) = (1,2,1)$.

This case $P = p_-^{1+2}$ and its cohomology is the same as (II). But the splitting is given

$$BP \cong \vee_{i=0}^{p=2} X_i \vee \vee_{i=0}^{p-2} L(2,i) \vee \vee_{i=0}^{p-2} L(1,i).$$

Detailed explanation for $L(2,i)$ see [M-P],[Hi-Ya1]. Let $H = \langle b, a^p \rangle$ the maximal elementary abelian subgroup. The space $L(2,i)$ is the transfer ($Tr : BH \to BG$) image of the same named summand of BH. By using the double coset formula

$$Tr_H^P(u^{p-1})|_H = \sum_{i=0}^{p-1} (u+iy)^{p-1} = -y^{p-1}$$

taking the generator u in $H^*(\langle b, a^p \rangle) \cong k[y,u]$.

The group P has just one conjugacy class H of the maximal abelian p-groups. Hence by Quillen's theorem, we have

$$Tr_H^P(u^{p-1}) = -y^{p-1} \quad in\ H^*(P) = H^*(P;\mathbb{Z})/(p, \sqrt{0}).$$

We consider an element $\Phi \in A(P,P)$ defined by $\Phi : P \geq H \subset P$. Then we see

$$Im(Tr_H^P H^*(H)) \supset H^*(P)\Phi = H^*(\vee_{i=0}^{p-2} L(2,i)).$$

Thus we have the isomorphism

$$Y_i \cong \begin{cases} X_i \vee L(2,i) & i \neq 0 \\ X_0 \vee L(2,0) \vee \vee_{j=0}^{p-2} L(1,j) & i = 0. \end{cases}$$

To compute cohomology of irreducible components X_i and $L(2,j)$, we recall the Dickson algebra

$$\mathbb{DA} = k[y,u]^{GL_2(\mathbb{Z}/p)} \cong k[D_1, D_2] \quad with\ D_1 = Y^p + V,\ D_2 = YV.$$

We also write

$$\mathbb{CA} = k[Y,V] \cong \mathbb{DA}\{1, Y, ..., Y^p\},$$

$$\mathbb{CB} = k[Y, D_2] \cong \mathbb{DA}\{1, Y, ..., Y^{p-1}\}.$$

Hence $\mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\}$. Then it is known (see [Hi-Ya1] for details)

$$H^*(L(2,i)) \cong \begin{cases} \mathbb{CB}\{Yd_2^i\} & i \neq 0 \\ \mathbb{CB}\{YD_2\} & i = 0. \end{cases}$$

Theorem 3.3. *Let $P = M(1,2,1) \cong p_-^{1+2}$. Then we have*

$$H^*(X_i) \cong \begin{cases} \mathbb{CA}\{1, ..., \hat{y}^i, ..., y^{p-2}\}\{v^i\} \oplus \mathbb{DA}\{d_2^i\} & i > 0 \\ \mathbb{CA}\{y, , ..., y^{p-2}\}\{V\} \oplus \mathbb{DA} & i = 0. \end{cases}$$

Proof. Let $i \neq 0$. We see

$$H^*(Y_i) \cong k[y, V]\{v^i\} \cong \mathbb{CA}\{1, y, ..., y^{p-2}\}\{v^i\}.$$

The cohomology of the summand X_i is

$$H^*(X_i) \cong H^*(Y_i) \ominus H^*(L(2,i))$$

$$\cong (\mathbb{DA} \oplus \mathbb{CB}\{Y\})\{v^i\}\{1, ..., y^{p-2}\} \ominus \mathbb{CB}\{Yd_2^i\}.$$

Here $v^i y^i = d_2^i$ we have the isomorphism in the theorem for $i \neq 0$.

Next we consider in the case $i = 0$. We have

$$H^*(X_0) \cong H^*(Y_0) \ominus H^*(\vee_j L(1,j)) \ominus H^*(L(2,0))$$

$$\cong \mathbb{CA}\{1, y, ..., y^{p-2}\}\{V\} \ominus \mathbb{CB}\{YD_2\} \cong \mathbb{CA}\{y, ..., y^{p-2}\}\{V\} \oplus B$$

where

$$B = \mathbb{CA}\{V\} \ominus \mathbb{CB}\{YD_2\} \cong \mathbb{CA} \ominus H^*(L(1,0)) \ominus H^*(L(2,0)).$$

We can see $B \cong \mathbb{DA}$ by Lemma 3.4 below. □

Lemma 3.4. *Let $M(2) = L(2,0) \vee L(1,0)$ (as the usual notation of the homotopy theory). Then we have*

$$H^*(M(2)) \cong \mathbb{CB}\{Y\}, \quad \mathbb{CA} \cong \mathbb{DA} \oplus H^*(M(2)).$$

Proof. We can compute

$$H^*(M(2)) \cong k[Y] \oplus \mathbb{CB}\{YD_2\} \cong k[Y] \oplus k[Y, D_2]\{YD_2\}$$

$$\cong (k[Y] \oplus k[Y, D_2]\{D_2\})\{Y\} \cong \mathbb{CB}\{Y\} \quad (\textit{assumed} * > 0).$$

Since $\mathbb{CA} \cong \mathbb{DA} \oplus \mathbb{CB}\{Y\}$, we have the last isomorphism in this lemma. □

4 Nilpotent elements

Let us write $H^{even}(X;\mathbb{Z})/p$ by simply $H^{ev}(X)$ so that

$$H^{ev}(G) = H^*(G) \oplus N(G)$$

where $N(G)$ is the nilpotent ideal in $H^{ev}(G)$.

Since BP is irreducible in nonsplit cases, we only consider in split cases,

$$P = M(\ell, m, n) = \langle a, b | a^{p^m} = b^{p^n} = 1, [a, b] = a^{p^\ell} \rangle$$

for $m > \ell \geq max(m - n, 1)$.

(I) Split metacyclic groups with $\ell > m - n$.

By Diethelm [Di], its mod p-cohomology is

$$H^*(P; \mathbb{Z}/p) \cong k[y, u] \otimes \Lambda(x, z) \quad |y| = |u| = 2, \ |x| = |z| = 1.$$

Of course all elements in $H^*(P;\mathbb{Z})$ are (higher) p-torsion. The additive structure of $H^*(P;\mathbb{Z})/p$ is decided by that of $H^*(P;\mathbb{Z}/p)$ by the universal coefficient theorem. Hence we have additively (but not as rings)

$$H^*(P; \mathbb{Z})/p \cong H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}) \cong k[y, u]\{1, \beta(xz) = yz - ux\}.$$

Since $H^*(P)$ is multiplicatively generated by y and v with $|v| \geq 2p$ from Theorem 4.1, the element u is not integral class (i.e. $u \notin Im(\rho)$ for $\rho : H^*(P;\mathbb{Z}) \to H^*(P;\mathbb{Z}/p)$). Therefore xz is an integral class since

$$H^{even}(P; \mathbb{Z}/p) \cong k[y, u]\{1, xz\}.$$

In $H^4(P;\mathbb{Z}/p)$, the elements y^2, yxz are integral but u^2 is not. Note that $dim(H^4(P;\mathbb{Z})/p) = 3$ and so xzu must be integral. Inductively, we see that

$$x_1 = xz, \ x_2 = xzu, \ ..., \ x_{p^{m-\ell}-1} = xzu^{p^{m-\ell}-2}$$

are integral classes.

The element $u \in H^2(P;\mathbb{Z}/p)$ is defined [Dim] using the spectral sequence

$$E_2^{*,*'} \cong H^*(P/\langle a \rangle; H^*(\langle a \rangle; \mathbb{Z}/p)) \Longrightarrow H^*(P; \mathbb{Z}/p).$$

In fact $u = [u'] \in E_\infty^{0,2}$ identifying $H^2(\langle a \rangle;\mathbb{Z}/2) \cong k\{u'\}$. Hence $u|\langle a \rangle = u'$. On the other hand $v|\langle a \rangle = (u')^{p^{m-\ell}}$ because $v = c_{p^{m-\ell}}(\eta)$ and the total Chern class is

$$\sum c_i(\eta)|\langle a \rangle = (1 + u')^{p^{m-\ell}} = 1 + (u')^{p^{m-\ell}}.$$

Therefore we see $v = u^{p^{m-\ell}} \ mod(y, xz)$ in $H^*(P;\mathbb{Z}/p)$. Thus we get

Theorem 4.1. *Let P be a split metacylic group $M(\ell, m, n)$ with $\ell > m - n$. Then we have*

$$H^{ev}(P) \cong k[y, v]\{1, x_1, ..., x_{p^{m-\ell}-1}\} \quad with \ x_i x_j = 0,$$

that is $N(P) \cong k[y, v]\{x_1, ..., x_{p^{m-\ell}-1}\}$.

These x_i are also defined by Chern classes (from the arguments just before Theorem 4.1), and as $Out(P)$ modules, $x_i \cong S_j$ when $i = j \ mod(p-1)$. Therefore we have

Corollary 4.2. *Let P be a split metacylic group $M(\ell, m, n)$ with $\ell > m - n$. Then*

$$H^{ev}(X_i) \cong H^*(X_i) \oplus k[y, V]\{v^r x_s | r + s = i \ mod(p-1)\}$$

where $1 \leq s \leq p^{m-\ell} - 1$.

(II) Split metacyclic groups $P = M(\ell, m, n)$ with $\ell = m - n$.
By also Diethelm, its mod p-cohomology is

$$H^*(P; \mathbb{Z}/p) \cong k[y, v'] \otimes \Lambda(a_1, ..., a_{p-1}, b, w)/(a_i a_j = a_i y = a_i w = 0)$$

where $|a_i| = 2i - 1, |b| = 1, |y| = 2, |w| = 2p - 1, |v'| = 2p$. So we see

$$H^*(P; \mathbb{Z}/p)/\sqrt{0} \cong k[y, v'].$$

Note that additively $H^*(P; \mathbb{Z})/p \cong H^*(p_-^{1+2}; \mathbb{Z})/p$, which is well known. In particular, we get additively

$$H^{ev}(P) \cong (k[y] \oplus k\{x_1, ..., x_{p-1}\}) \otimes k[v'] \quad (with \ x_i = a_i b)$$

$$\cong (k[y] \oplus k\{x_1, ..., x_{p-1}\}) \otimes k[v]\{1, v', ..., (v')^{p^{m-\ell-1}-1}\}.$$

Therefore $H^{ev}(P)$ is additively isomorphic to

$$H^{ev}(P) \cong \oplus_{i,j} k[v]\{a_i b (v')^j\} \oplus \oplus_j k[v, y]\{(v')^j\}$$

where $1 \leq i \leq p-1$ and $0 \leq j \leq p^{m-\ell-1} - 1$. Here $a_i b(v')^j$ is nilpotent and hence integral class and let $x_{jp+i} = a_i b(v')^j$. The element (v') is not nilpotent and we can take as the integral class wb of dimension $2p$. Let us write $x_{pj} = wb(v')^{j-1}$. Thus we have

Theorem 4.3. *Let P be a split metacylic group $M(\ell, m, n)$ with $\ell = m - n$. Then*

$$H^{ev}(P) \cong k[y, v] \oplus k[y, v]\{x_i | i = 0 \ mod(p)\} \oplus k[v]\{x_i | i \neq 0 \ mod(p)\}$$

where i ranges $1 \leq i \leq p^{m-\ell} - 1$. Here the multiplications are given by $x_i x_j = 0$, $y x_k = 0$ for $k \neq 0 \ mod(p)$.

Hence we have

Corollary 4.4. *Let $P = M(\ell, m, n)$ for $\ell = m - n$. Then*

$$H^{ev}(X_i) = H^*(X_i) \oplus k[y, V]\{v^r x_s | s = 0 \ mod(p), \ r + s = i \ mod(p-1)\}$$

$$\oplus k[V]\{v^r x_s | s \neq 0 \ mod(p), \ r + s = i \ mod(p-1)\}.$$

Let $CH^*(BG)$ be the Chow ring of the classifying space BG (see §5 below for the definition). The following theorem is proved by Totaro, with the assumption $p \geq 5$ but without the assumption of transferred Euler classes (since it holds when $p \geq 5$).

Theorem 4.5. *(Theorem 14.3 in [To2]) Suppose $rank_pP \leq 2$ and P has a faithful complex representation of the form $W \oplus X$ where $dim(W) \leq p$ and X is a sum of 1 dimensional representation. Moreover $H^{ev}(P)$ is genertated by transferred Euler classes. Then we have $CH^*(P)/p \cong H^{ev}(P)$.*

Proof. (See page 179-180 in [To2].) First note the cycle map is surjective, since $H^{ev}(P)$ is generated by transfferd Euler classes. Using the Riemann-Roch theorem without denominators, we can show

$$CH^*(BP)/p \cong H^{2*}(P;\mathbb{Z})/p \quad for * \leq p.$$

By the dimensional conditions of representations $W \oplus X$ and Theorem 12.7 in [To2], we see the following map

$$CH^*(BP)/p \to \prod_V CH^*(BV) \otimes_{\mathbb{Z}/p} CH^{\leq p-1}(BC_P(V))$$

$$\to \prod_V H^*(V;\mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H^{\leq 2(p-1)}(C_G(V);\mathbb{Z}/p)$$

is also injective. Here V ranges elementary abelian p-subgroups of P and $C_P(V)$ is the centralizer group of V in P. So we see that the cycle map is also injective. □

Therefore we have

Corollary 4.6. *Let P be the metacycle group $M(\ell, m, n)$ with $m - \ell = 1$. Then $CH^*(BP)/p \cong H^{ev}(BG)$.*

Totaro computed $CH^*(BP)/p$ for split metacyclic groups with $m - \ell = 1$ in 13.12 in [To]. When P is the extraspecial p-groups of order p^3, the above result is first proved in [Ya2].

For a cohomology theory $h^*(-)$, define the $h^*(-)$-theory toplogical nilpotence degree $d_0(h^*(BG))$ to be the least nonnegative integer d such that the map

$$h^*(BG)/p \to \prod_V h^*(BG) \otimes h^{\leq d}(BC_G(V))/p$$

is injective. Note that $d_0(H^*(BG;\mathbb{Z})) \leq d_0(H^*(BG;\mathbb{Z}/p))$.

Totarto computed in the many cases of groups P with $rank_pP = 2$. In particular, if P is a split metacyclic p-group for $p \geq 3$, then $d_0(H^*(BP;\mathbb{Z}/p)) = 2$ and $d_0(CH^*(BP)) = 1$ when $m - \ell = 1$. Hence $d_0(H^*(P;\mathbb{Z})) = 2$ for these split metacyclic groups P (for $p \geq 3$).

This fact also show easily from Theorem 8.1 and 8.2. Consider the restriction map

$$H^{ev}(P) \to H^{ev}(V) \otimes H^2(P) \quad (where\ V = \langle a^{p^{m-1}} \rangle \subset Z(P) : center)$$

induced the product map $V \times P \to P$. Then the element defined in Theorem 8.1, 8.3

$$c_j = xzu^{j-1} \to \sum_i xzu^i \otimes u^{j-i-1} \equiv u^{j-1} \otimes x_1 \neq 0 \in H^{ev}(V) \otimes H^2(P)$$

for $\ell > m-n$. For $\ell = m-n$ and $n = 1$, we also see that the nilpotent element x_j maps to $ab \otimes u^{j-1}$ (or $wb \otimes u^{j-p-1}$ for $j = 0\ mod(p)$) in $H^{ev}(V) \otimes H^2(P)$. (From the proof of Theorem 2 in [Dim], we see $w|V = zu^{p-1}$.)

5 Motives and stable splitting

For a smooth projective algebraic variety X over $\mathbb{C}$, let $CH^*(X)$ be the Chow ring generated by algebraic cycles of codimension $*$ modulo rational equivalence. There is a natural (cycle) map

$$cl : CH^*(X) \to H^{2*}(X(\mathbb{C}); \mathbb{Z}).$$

where $X(\mathbb{C})$ is the complex manifold of $\mathbb{C}$-rational points of X.

Let V_n be a $G - \mathbb{C}-$vector space such that G acts freely on $V_n - S_n$, with $codim_{V_n} S_n = n$. Then it is known that $(V_n - S_n)/G$ is a smooth quasi-projective algebraic variety. Then Totaro define the Chow ring of BG ([Tol]) by

$$CH^*(BG) = lim_{n \to \infty} CH^*((V_n - S_n)/G).$$

(Note that $H^*(G, \mathbb{Z}) = lim_{n \to \infty} H^*((V_n - S_n)/G)$ also.) Moreover we can approximate $\mathbb{P}^\infty \times BG$ by smooth projective varieties from Godeaux-Serre arguments ([Tol]).

Let P be a $p - group$. By the Segal conjecture, the p-complete automorphism $\{BP, BP\}$ of stable homotopy groups is isomoprphic to $A(P,P)_{\mathbb{Z}_p}$, which is generated by transfers and map induced from homomorphisms. Since $CH^*(BP)$ also has the transfer map, we see $CH^*(BP)$ is an $A(P,P)$-module. For an $A(P,P)$-simple module S, recall e_S is the corresponding idempotent element and $X_S = e_S BP$ the irreducible stable homotopy summand. Let us define

$$CH^*(X_S) = e_S CH^*(BP)$$

so that the following diagram commutes.

$$\begin{array}{ccc} CH^*(BP)_{(p)} & \xrightarrow{cl} & H^{2*}(BP; \mathbb{Z}_{(p)}) \\ \downarrow & & \downarrow \\ CH^*(X_S)_{(p)} & \xrightarrow{cl} & H^{2*}(X_S; \mathbb{Z}_{(p)}). \end{array}$$

For smooth schemes $X.Y$ over a field K, let $Cor(X,Y)$ be the group of finite correspondences from X to Y (which is a $\mathbb{Z}_p$-module on the set of closed subvarieties of $X \times_K Y$ which are finite and surjective over some connected component of X. Let $Cor(K, \mathbb{Z}_p)$ be the category of smooth schemes whose groups of morphisms $Hom(X,Y) = Cor(X,Y)$. Voevodsky constructs the trianगurated categiry $DM = DM(K, \mathbb{Z}_p)$ which contains the category $Cor(K, \mathbb{Z}_p)$ (and *limit* of objects in $Cor(K, \mathbb{Z}_p)$).

Theorem 5.1. *Let S be a simple $A(P,P)$-module. Then there is a motive $M_S \in DM(\mathbb{C}, \mathbb{Z}_p)$ such that*

$$CH^*(M_S) \cong CH^*(X_S) = e_S CH^*(BP).$$

Remark. Of course M_S is (in general) not irreducible, while X_S is irreducible.

The category $Chow^{eff}(K, \mathbb{Z}_p)$ of (effective) pure Chow motives is defined follows. An object is a pair (X,p) where X is a projective smooth variety over K and p is a projector, i.e. $p \in Mor(X,X)$ with $p^2 = p$. Here a morphism $f \in Mor(X,Y)$ is defined as an element $f \in CH^{dim(Y)}(X \times Y)_{\mathbb{Z}_p}$. We say that each $M = (X,p)$ is a (pure) motive and define the Chow ring $CH^*(M) = p^*CH^*(X)$, which is a direct summand of $CH^*(X)$. We we identify that the motive $M(X)$ of X means $(X, id.)$. (The category $DM(K, \mathbb{Z}_p)$ contains the category $Chow^{eff}(K, \mathbb{Z}_p)$.)

It is known that we can approximate $\mathbb{P}^\infty \times BP$ by smooth projective varieties from Godeaux-Serre arguments ([To1]). Hence we can get the following lemma since

$$CH^*(X \times \mathbb{P}^\infty) \cong CH^*(X)[y] \quad |y| = 1.$$

Lemma 5.2. *Let S be a simple $A(P,P)$-module. There are pure motives $M_S(i) \in Chow^{eff}(\mathbb{C}, \mathbb{Z}_p)$ such that*

$$lim_{n\to\infty} CH^*(M_S(i)) \cong CH^*(X_S)[y], \quad deg(y) = 1.$$

Corollary 5.3. *Let P be a split metacycle p-group $M(\ell, m, n)$ with $m - \ell = 1$. Then for each simple $A(P,P)$-module S, there is a motive $M_S \in DM(\mathbb{C}, \mathbb{Z}_p)$ with*

$$CH^*(M_S)/p \cong H^{ev}(X_S) = H^{even}(X_S; \mathbb{Z})/p.$$

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