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Remarks on the cuspidal simple module of $GL(p,q)$ over a field of characteristic p with $p \mid q-1$

北海道教育大学・旭川校 奥山 哲郎
Tetsuro Okuyama
Hokkaido University of Education, Asahikawa Campus

1 Introduction and Results

Let p be an odd prime and k be an algebraically closed field of characteritic p. Let q be a prime power with $p \mid q-1$ and set $\widetilde{G}=GL(p,q)$, $G=\widetilde{G}/Z(\widetilde{G})=PGL(p,q)$. The principal block algebra $B_0(k\widetilde{G})$ has a unique cuspidal simple module (of dimension $\prod_{i=1}^{p-1}(q^i-1)$) and it is known to be periodic as a kG-module. Our purpose in this talk is to describe the support variety of the cuspidal simple module. For this, we apply the so called a cohomological pushout method of constructing endotrivial modules developed by Carlson and Thévenaz who classified such modules for p-groups. For the method refer the articles by Carlson [8], by Carlson, Mazza and Thévenaz [10] and see the refferences in the article on classification theorem.

1.1 p-Local subgroups of $GL(p,q)$

Let p^n be the exact power of p dividing $q-1$. Then a Sylow p-subgroup $\widetilde{P}$ of $\widetilde{G}$ is isomorphic to $\mathbb{Z}_{p^n} \wr \mathbb{Z}_p$. Write $\widetilde{P}=\widetilde{Q} \rtimes \langle \tilde{a} \rangle$ where $\widetilde{Q}$ is a Sylow p-subgroup of the group of diagonal matrices in $\widetilde{G}$ and $\tilde{a}$ is a permutation matrix corresponding to a suitable cyclic permutation of length p. We know that $Z_2(\widetilde{P}) \cong \langle \tilde{b} \rangle \times Z(\widetilde{P})$ for some element $\tilde{b} \in \widetilde{Q}$ of order p and $\widetilde{E} ;= \langle \tilde{a},\ \tilde{b} \rangle \cong p_+^{1+2}$. We have

$$N_{\widetilde{G}}(\widetilde{Q})=\widetilde{D} \rtimes \widetilde{W} \tag{1.1}$$

where $\widetilde{D}$ is the group of diagonal matrices in $\widetilde{G}$ and $\widetilde{W} \cong \Sigma_p$ is the group of permutation matrices of degree p, the Weyl group of $\widetilde{G}$. Set $\widetilde{A}=\langle \tilde{a} \rangle$. Then $\widetilde{A} \subset \widetilde{W}$ and $N_{\widetilde{W}}(\widetilde{A})=\widetilde{H} \ltimes \widetilde{A}$ for some subgroup $\widetilde{H} \subset \Sigma_{p-1}$ such that $\widetilde{H} \cong \mathbb{Z}_{p-1}$. And we have

$$N_{\widetilde{G}}(\widetilde{P})=Z(\widetilde{G})(\widetilde{Q} \rtimes (\widetilde{A} \rtimes \widetilde{H})) \subset N_{\widetilde{G}}(\widetilde{Q}) \tag{1.2}$$

$N_{\widetilde{G}}(\widetilde{E})$ has a subgroup $\widetilde{L}$ such that $\widetilde{L} \cong SL(2,p)$ and

$$N_{\widetilde{G}}(\widetilde{E})=Z(\widetilde{G}) * (\widetilde{E} \rtimes \widetilde{L}) \tag{1.3}$$

Let P, Q and E be the images in the factor group $G=\widetilde{G}/Z(\widetilde{G})=PGL(p,q)$ of $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{E}$, respectively. P is a Sylow p-subgroup of G.

$E=\langle a,\ b \rangle$ is an (maximal) elementary abelian p-subgroup of G of rank 2 where a and b are images in G of $\tilde{a}$ and $\tilde{b}$, respectively. Note that Q is of index p in P. $N_G(Q)$ and $N_G(E)$ (and $N_G(P)$) controll the p-fusion in G. For these facts, see the paper by Alperin and Fong [2]. We shall give the elements a and b concretely, below.

As we denote by P the image of $\widetilde{P}$ in G, in the following we shall denote the images of various subgroups $\widetilde{K}$ of $\widetilde{G}$ in G by K.

1.2 The cuspidal simple module

The Weyl group W is isomorphic to Σ_p which is a Coxeter group of type A_{p-1}. For a subset J of the generating set for W, let $\widetilde{G}_J$ be the corresponding standard parabolic subgroup of $\widetilde{G}$. And the set of parbolic subgroups defines so called the Tits Building for $\widetilde{G} = GL(p,q)$. Associated with the building, we have a complex of $k\widetilde{G}$-modules (actually, of kG-modules) of the following form ;

$$\cdots \to 0 \to X^{p-2} \xrightarrow{f} X^{p-3} \to \cdots \to X^0 \to k_G \to 0 \to \cdots \tag{1.4}$$

where $X^i = \oplus \sum_{|J|=p-2-i} k_{\widetilde{G}_J} \uparrow^{\widetilde{G}}$. In particular, $X^{p-2} = k_{\widetilde{B}_0} \uparrow^{\widetilde{G}}$ where $\widetilde{B}_0$ is a Borel subgroup of $\widetilde{G}$.

In our setting that $q-1 \equiv 0 \pmod p$, we have the following situation.

The sequence is exact except at X^{p-2}. Each term X^i is a Q-projective permutation kG-module and the sequence is Q-split. In particular, the sequence is the first $p-1$ terms of a Q-projective resolution of k_G (although, terms are not minimal). This fact is shown by , for example, by an investigation by Cabanes and Rickard [7]. The Brauer character of $\operatorname{Ker} f$ is a modular reduction of the Steinberg character. By results of James (Lemma 3.4 [13]) and Geck, Hiss and Malle (Theorem 4.2 [12]), $\operatorname{Top}\operatorname{Ker} f$ is a cuspidal simple kG-module whose Brauer character is the modular reduction of an ordinary cuspidal irreducible character (of degree $\prod_{i=1}^{p-1}(q^i-1)$.) and projective when restricted to Q. For these facts, see also a study by Geck [11].

Set $S = \operatorname{Top}\ker f$, the cuspidal simple kG-module.

1.3 A Q-projective resolution of k_G

The principal block $B_0(kG)$ has a close relation with the group algeba $k\Sigma_p$ (and with the Hecke algebra $\operatorname{End}_{kG}(k_{B_0}\uparrow^G)$). A minimal Q-projective resolution of k_G is described as follws.

By studies of Dipper, James and others (see [11]), we can label the set of the projective indecomposable module of the Hecke algebra $\operatorname{End}_{kG}(k_{B_0}\uparrow^G)$ by the set of p-regular partitions of the number p. The set bijectively corresponds to the set of indecomposable direct summand of $k_{B_0}\uparrow^G$. And in our setting of the present note, each such summand has a simple top (and simple socle).

For each integer $0 \leqq i \leqq p-2$, let $P_Q(i)$ be an indecomposable direct summand of $k_{B_0}\uparrow^G$ corresponding to the partition $(1^{p-i}, 1, \cdots, 1)$. Then $P_Q(i)$ has the following Lewy (and socle) structure;

$$\begin{array}{l} P_Q(0) = \begin{matrix} S_0 \\ S_1 \\ S_0 \end{matrix}, \quad P_Q(1) = \begin{matrix} S_1 \\ S_0 \oplus S_2 \\ S_1 \end{matrix}, \quad \cdots, \\ P_Q(i) = \begin{matrix} S_i \\ S_{i-1} \oplus S_{i+1} \\ S_i \end{matrix}, \quad \cdots, \quad P_Q(p-2) = \begin{matrix} S_{p-2} \\ S_{p-3} \oplus S \\ S_{p-2} \end{matrix} \end{array} \tag{1.5}$$

where $S_0 = k_G, S_1, \cdots, S_{p-2}$ are simple kG-modules (which have P as vertex and belong to the principal block algebra $B_0(kG)$). The cuspidal simple module S appears in the composition factors of $P_Q(p-2)$.

The complex (1.4) is isomorphic to the following complex of kG-modules;

$$\cdots \to 0 \to P_Q(p-2) \xrightarrow{\pi_{p-2}} P_Q(p-3) \xrightarrow{\pi_{p-3}} \cdots \xrightarrow{\pi_2} P_Q(1) \xrightarrow{\pi_1} P_Q(0) \xrightarrow{\pi_0} S_0 \to 0 \to \cdots \tag{1.6}$$

where the maps π_i's are uniquely determined map (up to scalar) by the shapes of modules $P_Q(i)$'s. Thus the complex (1.6) (and (1.4)) is the first $p-1$ terms of a Q-projective resolution of $k_G = S_0$ and $\Omega_Q^{p-1}k_G = \Omega_Q^{p-1}S_0 = \operatorname{Ker}\pi_{p-2}$. Thus

$$\Omega_Q^{p-1}k_G = \begin{matrix} S \\ S_{p-2} \end{matrix} \tag{1.7}$$

Here we denote a Q-projective sygyzy of a kG-module V by $\Omega_Q V$. I learned from Kunugi and Miyachi that the situation above mentioned occurs.

One of our main results is the following theorem. Set $N = N_G(P)$. Then $N = P \rtimes H$ and $H \cong \mathbb{Z}_{p-1} \cong GF(p)^{\times}$. Remember that we set $A = \langle\, a\, \rangle$.

Theorem 1.1 *$\Omega_Q^{2(p-1)}k_G$ is an endotrivial kG-module and its Green correspondent is $\Omega^{-2(p-1)}\Omega_A^{2(p-1)}k_N$. Furthermore, there exists a kG-module M such that we have exact sequences of kG-modules of the following forms ;*

$$0 \to k_G \to \Omega_Q^{2(p-1)}k_G \to M \to 0, \qquad 0 \to \Omega^{p-2}S \to M \oplus \ proj \ \to \Omega^{p-1}S \to 0 \tag{1.8}$$

Remark 1.2 *The Green correspondent of $\Omega_Q^{p-1}k_G$ is $\Omega^{-(p-1)}\Omega_A^{p-1}\varepsilon_N$ and is endotriial, where ε_N is a one dimensional kN-module with $\varepsilon_N^2 = k_N$. However, $\Omega_Q^{p-1}k_G$ is not endotrivial.*

1.4 A vertex of S

S is periodic because $S\downarrow_Q$ is projective and Q is of index p in P. If we set $G_0 ;= PSL(p,q) \subset G$, then G/G_0 is of order p and we see that $S\downarrow_{G_0}$ is a direct sum of p nonisomorphic simple kG_0-modules, that is, S is induced from a kG_0-module. And then we can see that E is a minimal elementary abelian p-subgroup of G such that $S\downarrow_E$ is not projective. By a result of Benson [5] and the fact that $C_G(E) = E$, we have the following proposition. Let $K \subset L \cong SL(2,p)$ be a cyclic subgroup of L of order $p+1$ (which is uniquely determined up to L-conjugate) and set $N_0 = E \rtimes K \subset N_G(E)$.

Proposition 1.3 *E is a vertex of S. And there exists an indecomposable kN_0-module T such that $T\uparrow^G = S \oplus proj$.*

1.5 The support variety $V_G(S)$ of S

We have $\Omega k_E = \langle\ (a-1),\ (b-1)\ \rangle_{kE} = J(kE)$. Let $\lambda_0,\ \mu_0 \in H^2(E,k)$ be the Bocksteins of the following elements in $H^1(E,k) \cong \mathrm{Hom}_{kE}(\Omega k_E, k)$,

$$\begin{cases}(a-1) & \mapsto 1\\ (b-1) & \mapsto 0\end{cases}, \qquad \begin{cases}(a-1) & \mapsto 0\\ (b-1) & \mapsto 1\end{cases}, \quad \text{respectively}$$

so that $H^*(E,k) = k[\lambda_0,\ \mu_0] + \sqrt{0}$. And set

$$\rho_0 = \prod\nolimits_{x\in GF(p^2)-GF(p)}(\mu_0 - x\lambda_0) \in H^{2p(p-1)}(E,k)$$

Theorem 1.4 *The support variety $V_G(S)$ of S is given by*

$$V_G(S) = \mathrm{res}_E^*(V_E(\rho_0))$$

, where res_E^* is the map from $V_E(k) \to V_G(k)$ induced by the restriction map $\mathrm{res}_E : H^*(G,k) \to H^*(E,k)$.

The first half of Section 2 is devoted to describe p-local structures of G. In the latter half of the section, we construct some cohomology elements in $H^*(G,k)$ and some endotrivial kG-module making use of the cohomological pushout method. Proofs of Theorem 1.1 and 1.4 will be given in Section 3.

2 Subgroups of G and $H^*(G,k)$

We shall define various subgroups of $G = \widetilde{G}/Z(\widetilde{G})$ and construct some cohomology elements in $H^*(G,k)$ which we need for our investigation below.

2.1 Subgroups of $\widetilde{G}$

We first define subgroups of $\widetilde{G} = GL(p,q)$. Rows and columns are indexed by the set $GF(p) = \mathbb{Z}_p = \{0,\ 1,\ \cdots,\ p-1\}$. Let $K_0 \subset GF(q)^\times$ be the multiplicative subgroup of order p^n. Let fix an element $\zeta_0 \in K_0$ of order p^n and set $\zeta = \zeta_0^{p^{n-1}}$ so that ζ is of order p.

For $\alpha_i \in GF(q)^\times,\ 0 \leqq i \leqq p-1$, let $d(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{p-1})$ be the diagonal matrix with (i,i)-entry α_i. And set $c(\alpha) = d(\alpha, \alpha, \cdots, \alpha)$ for $\alpha \in GF(q)^\times$. Set

$$\widetilde{Q} = \{\ d(\alpha_0, \alpha_1, \cdots, \alpha_{p-1})\ ;\ \alpha_i \in K_0\ \}, \quad \widetilde{Z} = \{\ c(\alpha)\ ;\ \alpha \in K_0\ \}$$

and

$$\widetilde{D} = \{\ d(\beta_0, \beta_1, \cdots, \beta_{p-1})\ ;\ \beta_i \in GF(q)^\times,\ (|\beta_i|, p) = 1\ \}$$

Let Σ_p be the symmetric group on $\{0, 1, \cdots, p-1\} = \mathbb{Z}_p$. We identify each permutation with the correponding permutation matrix in $\widetilde{G}$.

Let $\tilde{a} \in \widetilde{G}$ be the permutation matrix corresponding to the cyclic permutation $(0\ 1\ \cdots\ p-1)$. Then

$$\begin{aligned} &\tilde{a}^{-1} \cdot d(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}) \cdot \tilde{a} = d(\alpha_{p-1}, \alpha_0, \alpha_1, \cdots, \alpha_{p-2}) \\ &\{\tilde{a} \cdot d(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{p-1})\}^p = c(\alpha) \ \text{ where } \ \alpha = \alpha_0\alpha_1 \cdots \alpha_{p-1} \end{aligned} \tag{2.1}$$

Set

$$\tilde{d} = d(\zeta_0, 1, \cdots, 1), \quad \tilde{b} ;= d(1, \zeta, \zeta^2, \cdots, \zeta^{p-1}), \quad \tilde{c} ;= c(\zeta), \quad \tilde{u} ;= d(1, \zeta_1, \zeta_1^4, \cdots, \zeta_1^{i^2}, \cdots, \zeta_1^{(p-1)^2})$$

where $\zeta_1 = \zeta^{\frac{1}{2}(p+1)}$ so that $\zeta_1^2 = \zeta$. Then we have the following equalities ;

$$(\tilde{a}\tilde{d})^p = c(\zeta_0), \qquad [\tilde{a}, \tilde{b}] = \tilde{c}, \qquad \tilde{u}^{-1}\tilde{a}\tilde{u} = \tilde{a}\tilde{b}\tilde{c}^{(p-1)/2}, \quad \tilde{u}^{-1}\tilde{b}\tilde{u} = \tilde{b} \tag{2.2}$$

Set $\widetilde{P} = \langle\ \widetilde{Q},\ \tilde{a}\ \rangle = \widetilde{Q} \rtimes \langle\ \tilde{a}\ \rangle$. Then $\widetilde{P}$ is a Sylow p-subgroup of $\widetilde{G}$.

For each $0 \neq s \in \mathbb{Z}_p = GF(p)$, consider the permutation $p(s)$ on $\mathbb{Z}_p$ defined by

$$p(s) = \begin{pmatrix} 0 & 1 & 2 & \cdots & i & \cdots & p-1 \\ 0 & s & 2s & \cdots & is & \cdots & (p-1)s \end{pmatrix}$$

And denote by $\tilde{h}(s)$ the corresponding permutation matrix to $p(s)$. Then the following equalities hold.

$$\begin{aligned} &\tilde{h}(s)d(\alpha_0, \alpha_1, \cdots, \alpha_i, \cdots, \alpha_{p-1})\tilde{h}(s)^{-1} = d(\alpha_0, \alpha_s, \cdots, \alpha_{is}, \cdots, \alpha_{(p-1)s}) \\ &\tilde{h}(s)^{-1}\tilde{a}\tilde{h}(s) = \tilde{a}^s, \qquad \tilde{h}(s)^{-1}\tilde{b}\tilde{h}(s) = \tilde{b}^{s^{-1}}, \qquad \tilde{h}(s)^{-1}\tilde{u}\tilde{h}(s) = \tilde{u}^{s^{-2}} \end{aligned} \tag{2.3}$$

2.2 Subgroups of G

Now we shall work in the group $G = \widetilde{G}/Z(\widetilde{G}) = PGL(p,q)$. We denote the images in G of elements and subgroups of $\widetilde{G}$ defined above by deleting $\tilde{}$ attached to them. Thus, for example, $P = Q \rtimes \langle\ a\ \rangle$ is a Sylow p-subgroup of G. We also denote by W the image in G of the subgroup Σ_p of $\widetilde{G}$. Let $W_0 \cong \Sigma_{p-1}$ be the subgroup of W corresponding to the stabilzer of the point $0 \in \{0, 1, \cdots, p-1\}$ and W_1 be the subgroup of W corresponding to the pointwise stabilzer of the set $\{0, 1\}$. Thus if we set $H = \{\ h(s)\ ;\ 0 \neq s \in \mathbb{Z}_p\ \}$, then $H \subset W_0$.

The results we shall describe are all due to the study by Alperin and Fong [2],

2.2.1 p-Local subgroups of G

By the equality (2.1), we see that $Z(P) = \langle\ b\ \rangle$. Again by the equality (2.1), we see that any element in aQ is of order p and is P-conjugate to ad^k for some k with $0 \leqq k \leqq p-1$. Set

$$E_k = \langle\ ad^k,\ b\ \rangle, \quad A_k = \langle\ ad^k\ \rangle, \quad 0 \leqq k \leqq p-1, \qquad B = \langle\ b\ \rangle, \quad U = \langle\ u\ \rangle$$

$$E_p = \text{the subgroup generated by elements of order } p \text{ in } Q$$

If $p = 3$, then $E_p = \langle\ u,\ b\ \rangle$.

By the equality (2.3), $(ad)^{h(s)} = a^{s^{-1}}d$ and $(a^{s^{-1}}d)^s$ is P-conjugate to ad^s by the equality (2.1). Thus the $p-1$ subgroups $A_k,\ 1 \leqq k \leqq p-1$ are $P \rtimes H$-conjugate.

Lemma 2.1 *The following statements hold.*

1. *The set* $\{\ E_i\ ;\ 0 \leqq i \leqq p\ \}$ *is a representatives set for the* P*-conjugacy classes of maximal elementary abelian* p*-subgroups of* P.
2. $E_i,\ 1 \leqq i \leqq p-1$ *are* $N_G(P)$*-conjuagte and the set* $\{\ E_0,\ E_1,\ E_p\ \}$ *is a representatives set for the* G*-conjugacy classes of maximal elementary abelian* p*-subgroups of* G.
3. E_0 *and* E_1 *are of rank* 2. E_p *is of rank* $p-1$.
4. *Let* $G_0 \subset G$ *with* $G_0 \cong PSL(p,q) \subset PGL(p,q)$. *Then* G_0 *is a normal subgroup of* G *of index* p *and* $E_1 \nsubseteq G_0$.

We are mainly concerned with the subgroup $E_0 = \langle\ a,\ b\ \rangle$ so that we set $E = E_0$. We can write $N_G(E) = E \rtimes L$ with $L \cong SL(2,p)$. We may assume that $U \rtimes H \subset L$ where the corresponding matrices in $SL(2,p) \cong L$ of the elements $u,\ h(s)$ are given as follows (see (2.2), 2.3));

$$u \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad h(s) \mapsto \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \tag{2.4}$$

Lemma 2.2 *The followig statements hold.*

1. $O_{p'}(N_G(Q)) = D$ *and* $N_G(Q) = D \rtimes (Q \rtimes W)$.
2. $A \rtimes H \subset W$. *And* $W = W_1(A \rtimes H), \quad W_1 \cap (A \rtimes H) = 1$.
3. $N_G(P) = Q \rtimes (A \rtimes H)$.
4. $N_G(E) = E \rtimes L$.
5. *The fusion in* P *is controlled by* $N_G(Q)$ *and* $N_G(E)$ (and $N_G(P)$).

We end this subsection with the following lemma.

Lemma 2.3 *Set* $\mathcal{Q}_0 = \{\ N_G(P) \cap {}^gQ\ ;\ g \in G\ \}$. *Then any subgroup* $R \in \mathcal{Q}_0$ *is* $N_G(P)$*-conjugate to a subgroup* Q *or is conjugate to* A.

2.3 Some elements in $H^*(G,k)$

In this section, we shall construct some cohomology elements in $H^*(G,k)$, especially, the element $\rho \in H^{2p(p-1)}(G,k)$ such that $\mathrm{res}_E\,\rho = \rho_0$ where $\rho_0 \in H^{2p(p-1)}(E,k)$ is the element given in Section 1.5. The results we shall see may be known. However, we can not find appropriate literature and for the sake of completeness, we do. The study by Sasaki [14] is useful for our investigation.

2.3.1 $H^*(N_G(Q),k)$

We first consider $H^*(N_G(Q),k)$. As $N_G(Q) = O_{p'}(N_G(Q)) \rtimes (Q \rtimes W)$ by Lemma 2.2, we may work in $N(Q)\ ;= Q \rtimes W$. Set

$$\widetilde{Q}_1 = \{\ d(1,1,\alpha_2 \cdots, \alpha_{p-1})\ ;\ \alpha_i \in K_0\ \} \subset \widetilde{Q} \subset \widetilde{G}, \quad \tilde{x}_1 = d(1,\zeta_0,1,\cdots,1) \in \widetilde{Q}$$

and $Q_1 \subset G$ and $x_1 \in G$ be the images in G of $\widetilde{Q}_1$ and $\tilde{x}_1$, respectively. Then $Q = \langle\ x_1\ \rangle \times Q_1$.

We can write $\Omega k_Q = \langle\ (x_1 - 1),\ J(kQ_1)\ \rangle_{kQ} = J(kQ)$ and consider the element in $H^1(Q,k) = \mathrm{Hom}_{kQ}(\Omega k_Q, k)$ satisfying

$$(x_1 - 1) \mapsto 1, \quad J(kQ_1) \mapsto 0$$

Let $\mu \in H^2(Q,k)$ be the Bockstein of the element. $W_1 \subset W$ normalizes Q_1 and centralizes x_1 because we defined $W_1 \cong \Sigma_{p-2}$ to be the pointwise stablilizer of $\{0,1\}$. Thus if we set $N_1 = Q \rtimes W_1 \subset N(Q)$, then μ is canoniacally extended to the element in $H^2(N_1,k)$ which we denote by the same symbol μ.

Let $\lambda_0,\ \mu_0 \in H^2(E,k)$ be the elements given in Section 1.5. And recall that we set $B = \langle\ b\ \rangle \subset E$.

Lemma 2.4 *The following equalities hold.*

$$\mathrm{res}_E\, \mathrm{norm}_{N_1}^{N(Q)}(\mu) = -(\mu_0^p - \mu_0\lambda_0^{p-1})^{p-1} = -\left\{\prod\nolimits_{y\in GF(p)}(\mu_0 - y\lambda_0)\right\}^{p-1}$$

Proof. Set $\mu_1 = \mathrm{res}_B\,\mu$. We have $bQ_1 = x_1^{p^{n-1}}Q_1$ and we see that $\mu_1 \in H^2(B,k)$ is the Bockstein of the element in $H^1(B,k) = \mathrm{Hom}_{kB}(\Omega k_B, k)$ satisfying $(b-1) \mapsto 1$. By the equality (2.3), we have $b^{h(s)} = b^{s^{-1}}$. Thus $\mu_1^{h(s)} = s \cdot \mu_1$. Then by the Mackey formula, we have

$$\mathrm{res}_E\, \mathrm{norm}_{N_1}^{N(Q)}\, \mu = \mathrm{norm}_B^E(\prod\nolimits_{0\neq s\in \mathbb{Z}_p} \mu_1^{h(s)}) = \mathrm{norm}_B^E((p-1)! \cdot \mu_1^{p-1}) = -\,\mathrm{norm}_B^E(\mu_1)$$

Thus the first equality holds by Proposition 4.1.4 [3]. The second equality is easy to see.

The trivial $k\Sigma_p$-module is periodic of period $2(p-1)$ and we have a $2(p-1)$-fold self extension of $k_{N(Q)}$ of the form

$$0 \to k_{N(Q)} \to U_{2(p-1)} \to \cdots \to U_1 \to k_{N(Q)} \to 0$$

which is the first $2(p-1)$ terms of a projective resolution of $k_{N(Q)}$ as $kN(Q)/Q$-module. Let $\chi \in H^{2(p-1)}(N(Q),k)$ be the cohomolgy element corresponding to the sequence. Then we may assume that

$$\mathrm{res}_E\, \chi = \lambda_0^{p-1} \tag{2.5}$$

Lemma 2.5 *Set*

$$\rho_1 = \chi^p - \mathrm{norm}_{N_1}^{N(Q)}(\mu) \in H^{2p(p-1)}(N_G(Q),k) \quad \textit{and} \quad \sigma_1 = \chi \cdot \mathrm{norm}_{N_1}^{N(Q)}(\mu) \in H^{2(p^2-1)}(N_G(Q),k)$$

Then

$$\mathrm{res}_E\, \rho_1 = \prod_{x\in GF(p^2)-GF(p)} (\mu_0 - x\lambda_0), \quad \mathrm{res}_E\, \sigma_1 = \left\{\lambda_0 \prod_{y\in GF(p)} (\mu_0 - y\lambda_0)\right\}^{p-1}$$

Proof. A proof of Lemma 4.2 [14] works well by the equlaity (2.5) and Lemma 2.4.

2.3.2 $\rho \in H^{2p(p-1)}(G,k)$ with $\mathrm{res}_E\, \rho = \rho_0$

By Lemma 2.5, $\mathrm{res}_E\, \rho_1 \in H^{2p(p-1)}(E,k)$ and $\mathrm{res}_E\, \sigma_1 \in H^{2(p^2-1)}(E,k)$ are invariant under the action of $GL(E) = AutE$. They are the so called Dickson invariants (see Section 8.1, 8.2 [4]). In particular, they are $N_G(E)$-invariant. Thus by Lemma 2.2.5, we have elements $\rho \in H^{2p(p-1)}(G,k)$ and $\sigma \in H^{2(p^2-1)}(G,k)$ such that

$$\mathrm{res}_E\, \rho = \mathrm{res}_E\, \rho_1, \qquad \mathrm{res}_E\, \sigma = \mathrm{res}_E\, \sigma_1 \tag{2.6}$$

Thus $\mathrm{res}_E\, \rho = \rho_0$.

2.4 Some endotrivial kG-module

Let L_ρ be the Carlson module of ρ. We see that $\mathrm{res}_B\,\rho \in H^{2p(p-1)}(B,k) = H^{2p(p-1)}(Z(P),k)$ is not nilpotent and we can apply the method of constructing endotrivial modules (see [8], [10]). All the results below are due to Carlson [8].

We are concerned with the group E. The variety $V_G(L_\rho)$ decomposes as

$$V_G(L_\rho) = V_0 \cup V_0' \;\text{ with }\; V_0 \cap V_0' = \{0\} \;\text{ where }\; V_0 = \mathrm{res}^*_E(V_E(L_\rho)) = \mathrm{res}^*_E(V_E(L_{\rho_0}))$$

Here L_{ρ_0} is the Carlson module of ρ_0. Then L_ρ decomposes as

$$L_\rho = L_0 \oplus L_0' \quad\text{where}\quad V_G(L_0) = V_0, \quad V_G(L_0') = V_0'$$

Now set $Y = \Omega^{2p(p-1)}k_G/L_0'$. Y is an endotrivial kG-module which appears as a pushout in the following diagram ;

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L_0' & = & L_0' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L_\rho & \longrightarrow & \Omega^{2p(p-1)}k_G & \longrightarrow & k_G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L_0 & \longrightarrow & Y & \longrightarrow & k_G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

For our discussion, the dual $X = Y^*$ of Y is convenient to use. Set $M_0 = L_0^*$ so that we have an exact sequence of the form

$$0 \to k_G \to X \to M_0 \to 0 \quad\text{and}\quad V_G(M_0) = \mathrm{res}^*_E(V_E(L_{\rho_0})) \tag{2.7}$$

The endotrivial module X satisfies the following ;

$$X\downarrow_E = \Omega^{-2p(p-1)}k_E \oplus \text{ proj}, \qquad X\downarrow_{E_i} = k_{E_i} \oplus \text{ proj}, \;\text{ for }\; 1 \leqq i \leqq p \tag{2.8}$$

By the construction of Y, if we set $N_p = B \rtimes H$, then $Y\downarrow_{N_p} = k_{N_p} \oplus$ proj. N_p and $N_0 = A \rtimes H$ are conjugate in $N_G(E)$ (see (2.4)). Thus we have

$$X\downarrow_{N_0} = k_{N_0} \oplus \text{ proj} \tag{2.9}$$

3 Proofs of Theorem 1.1 and 1.4

In this section, we shall give proofs of theorems in Section 1.

3.1 A proof of Theorem 1.1

3.1.1 $\Omega_Q^{2(p-1)}k_G$

First we shall construct the first $p-1$ terms of a Q-projective resolutions of $\Omega_Q^{p-1}k_G = \begin{matrix} S \\ S_{p-2} \end{matrix}$. Simple kG-modules S_i's and S are all self-dual. Taking the dual of the Q-projective resolution (1.6) of k_G, we have a Q-injective resolution

$$0 \to k_G \xrightarrow{\pi_0^*} P_Q(0) \xrightarrow{\pi_1^*} P_Q(1) \to \cdots \xrightarrow{\pi_{p-3}^*} P_Q(p-3) \xrightarrow{\pi_{p-2}^*} P_Q(p-2) \xrightarrow{g} \begin{matrix} S_{p-2} \\ S \end{matrix} \to 0 \qquad (3.1)$$

which is a Q-projective resolution of $(\Omega_Q^{p-1} k_G)^* = \begin{matrix} S_{p-2} \\ S \end{matrix}$. As $S\downarrow_Q$ is projective, we see that the exact sequence $0 \to S \to \begin{matrix} S_{p-2} \\ S \end{matrix} \to S_{p-2} \to 0$ is Q-split and a Q-projective resolution of S is a usual projective resolution. Thus by the sequence (3.1) the first $p-1$ terms of a Q-projective resolution of S_{p-2} has the form

$$0 \to \Omega_Q^{p-1} S_{p-2} \xrightarrow{g_{p-1}} P'_Q(0) \xrightarrow{g_{p-2}} P'_Q(1) \to \cdots \to P'_Q(p-3) \xrightarrow{g_1} P_Q(p-2) \xrightarrow{g_0} S_{p-2} \to 0 \qquad (3.2)$$

where $P'_Q(i) = P_Q(i) \oplus \text{proj}$. Furthermore, we have an exact sequence of the following form ;

$$0 \to \Omega^{p-1} S \to k_G \oplus \text{proj} \to \Omega_Q^{p-1} S_{p-2} \to 0 \qquad (3.3)$$

We also have the Q-split exact sequence $0 \to S_{p-2} \to \Omega_Q^{p-1} k_G \to S \to 0$ and by the same argument as above, we have the followings. By the sequence (3.2), the first $p-1$ terms of a Q-projective resolution of $\Omega_Q^{p-1} k_G$ has the form

$$0 \to \Omega_Q^{2(p-1)} k_G \xrightarrow{f_{p-1}} P''_Q(0) \xrightarrow{f_{p-2}} P''_Q(1) \to \cdots \to P''_Q(p-3) \xrightarrow{f_1} P''_Q(p-2) \xrightarrow{f_0} \Omega_Q^{p-1} k_G \to 0 \qquad (3.4)$$

where $P''_Q(i) = P_Q(i) \oplus \text{proj}$. Furthermore, we have an exact sequence of the following form ;

$$0 \to \Omega_Q^{p-1} S_{p-2} \to \Omega_Q^{2(p-1)} k_G \oplus \text{proj} \to \Omega^{p-1} S \to 0 \qquad (3.5)$$

By (3.3), we have an exact sequence of the form $0 \to k_G \to \Omega_Q^{p-1} S_{p-2} \to \Omega^{p-2} S \to 0$. And there exists a kG-module M such that we have exact sequences of the form ;

$$0 \to k_G \to \Omega_Q^{2(p-1)} k_G \to M \to 0, \quad 0 \to \Omega^{p-2} S \to M \oplus \text{ proj } \to \Omega^{p-1} S \to 0$$

Thus the second statement in Theorem 1.1 follows.

3.1.2 $\Omega_Q^{2(p-1)} k_G \downarrow_{N_G(P)}$

In this section, we investigate the restriction of $\Omega_Q^{2(p-1)} k_G$ to the subgroup $N_G(P)$. We refer the aricle by Bouc [6] for general results of relative syzygies.

Set $N = N_G(P)$ and $\mathcal{Q}_0 = \{\ N \cap {}^g Q\ ;\ g \in G\ \}$. And set $\mathcal{Q}'_0 = \{Q,\ A\}$. By Lemma 2.3, for kN-modules, $\mathcal{Q}_0$-projective covers coincide with $\mathcal{Q}'_0$-projective covers.

$N = P \rtimes H$ and $N/Q \cong A \rtimes H$. Let $N \to N/P = H$ be the canonical group surjection. The map $H \to GF(p)^\times$, $h(s) \mapsto s$ is a group homomophism (actually, isomorphism). Let $\varphi_N : N \to GF(p)^\times$ be the composite of these tow maps and we denote by the same symbol φ_N the corresponding one dimensional kN-module. Then by the equality (2.3), we can see that

$$\Omega_Q^2 k_N = \varphi_N \qquad (3.6)$$

Taking relative sygyzies is compatible with the restriction to subgroups and the followings hold ;

$$\Omega_Q k_G \downarrow_N \equiv \Omega_{\mathcal{Q}_0} k_N = \Omega_{\mathcal{Q}'_0} k_N \pmod{\mathcal{Q}_0}$$

By the fact that $Q \cap A = 1$, and by a result of Thévenaz and Bouc (Lemma 5.2.1 [6], see also an argument by Alperin [1]), we have

$$\Omega \Omega_{\mathcal{Q}'_0} k_N = \Omega_Q \Omega_A k_N$$

Thus by the commutativity of taking relative syzygies,

$$\Omega_Q k_G \downarrow_N \equiv \Omega^{-1}\Omega_Q\Omega_A k_N \pmod{\mathcal{Q}_0}, \qquad \Omega_Q^2 k_G \downarrow_N \equiv \Omega^{-2}\Omega_A^2 \varphi_N \pmod{\mathcal{Q}_0}$$

where we used the equality (3.6).

Thus for any even integer $2m$, a Green correspondent of $\Omega_Q^{2m} k_G$ is $\Omega^{-2m}\Omega_A^{2m}\varphi_N^m$ and is an endotrivial kN-module (Proposition 4.2 [10]). For $m = \frac{1}{2}(p-1)$, the dimension of $\Omega_Q^{2m} k_G = \Omega_Q^{p-1} k_G$ is $q^{\frac{1}{2}p(p-1)}$, the degree of the Steinbeg character. We see that $q^{\frac{1}{2}p(p-1)} - 1$ is divisible by p^{n+1} but is not divisible by p^{n+2}. Thus $\Omega_Q^{p-1} k_G$ itself is not endotrivial and Remark 1.2 follows.

A Green correspondent of $\Omega_Q^{2(p-1)} k_G$ is $\Omega^{-2(p-1)}\Omega_A^{2(p-1)}\varphi_N^{p-1} = \Omega^{-2(p-1)}\Omega_A^{2(p-1)} k_N$. The sequences (1.6) and (3.4) are Q-split and therefore we have

$$\Omega_Q^{2(p-1)} k_G + \sum_{i=0}^{p-2} (-1)^{p-2-i} P_Q(i)'' = \Omega_Q^{p-1} k_G = k_G + \sum_{i=0}^{p-2} (-1)^{p-2-i} P_Q(i) \tag{3.7}$$

in the Green ring (the representation ring) of kQ-modules. As the sequences are sequences of kG-modules, the equality (3.7) holds in the Green ring of kR-modules for any $R \in \mathcal{Q}_0$. Thus $\Omega_Q^{2(p-1)} k_G \downarrow_R = k_R \oplus \mathrm{proj}$. We can write as

$$\Omega_Q^{2(p-1)} k_G \downarrow_N = \Omega^{-2(p-1)}\Omega_A^{2(p-1)} k_N \ \oplus\ V$$

where V is a $\mathcal{Q}_0$-projective kN-module. And then we can conclude that V is projective and a proof of Theorem 1.1 is completed.

3.2 A proof of Theorem 1.4

3.2.1 $\Omega_Q^{2(p-1)} k_G = X$

We shall show that $\Omega_Q^{2(p-1)} k_G \cong X$ where X is the endotrivial kG-module given in Section 2.4.

We saw that $\Omega_Q^{2(p-1)} k_G$ is endotrivial. In the group $N = N_G(P)$, any conjugate of A intersects trivially with E_i, $i \neq 0$. Thus as endotrivial kN-modules, $\Omega_Q^{2(p-1)} k_G$ and the module X have the same "type" by the equality (2.8). We can see that the equality (3.7) holds in the Green ring of kN_0 where $N_0 = A \rtimes H$ because $A \in \mathcal{Q}_0$. Thus $\Omega_Q^{2(p-1)} k_G \downarrow_{N_0} = k_{N_0} \oplus$ proj. Then by the equality (2.9), we see that Green correspondents of $\Omega_Q^{2(p-1)} k_G$ and X are isomorphic and the result follows.

3.2.2 $V_G(S)$

We refer to Benson's book [3] for the support variety of modules.

Let $0 \to k_G \xrightarrow{f} \Omega_Q^{2(p-1)} k_G \to M \to 0$ be the first exact sequence given in (1.8). By the second exact sequence in (1.8), M is a periodic module. Thus $f \downarrow_E$ is a not projective map because if it were, then $M \downarrow_E$ would have a direct summand isomorphic to $\Omega^{-1} k_E$, a contradiction.

Consider the restriction of the sequence to E. We saw that $\Omega_Q^{2(p-1)} k_G \cong X$ and therefore we have $\Omega_Q^{2(p-1)} k_G \downarrow_E = \Omega^{-2p(p-1)} k_E$. Thus the exact sequence which we consider has the form ;

$$0 \to k_E \xrightarrow{f_0} \Omega^{-2p(p-1)} k_E \to M' \to 0 \tag{3.8}$$

where M' is a direct summand of $M \downarrow_E$.

We have an isomorphism $\mathrm{Hom}_{kE}(k_E, \Omega^{-2p(p-1)} k_E) \cong H^{2p(p-1)}(E,k) = k[\lambda_0, \mu_0] + \sqrt{0}$, where λ_0, $\mu_0 \in H^2(E,k)$ is the cohomology elements given in Section 1.5. The corresponding elements $\nu \in H^{2p(p-1)}(E,k)$ to f_0 under the isomorphism is $N_G(E)$-invariant. We see that $H^*(E,k)^{N_G(E)} = k[\lambda_0, \mu_0]^{N_G(E)} + \sqrt{0}^{N_G(E)}$. By that fact that $N_G(E)/E \cong SL(2,p)$, $k[\lambda_0, \mu_0]^{N_G(E)}$ is generated by $\rho_0 \in H^{2p(p-1)}(E,k)$ and $\sigma_0' = \lambda_0^p \mu_0 - \lambda_0 \mu_0^p \in H^{2(p+1)}(E,k)$ (see Section 8.2 [4]). Thus $\nu \equiv \rho_0 \pmod{\sqrt{0}}$. $M' = L_\nu^*$ where L_ν is the Carlson module of ν. Thus $V_E(M') = V_E(\nu) = V_E(\rho_0)$.

Again by the second exact sequence in (1.8), $V_E(\rho_0) = V_E(M') \subset V_E(S)$. As S is E-projective and periodic, we can conclude that $V_G(S) = \mathrm{res}_E^*(V_E(\rho_0))$ and a proof of Theorem 1.4 is completed.

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