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TORUS MANIFOLDS AND FACE RINGS OF BUCHSBAUM POSETS

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ABSTRACT. The paper aims to review the structure of the cohomology, equivariant cohomology, and the spectral sequence of the orbit type filtration of manifolds with locally standard torus actions. Certain restrictions are imposed on such manifolds, in particular it will be assumed that all proper faces of the orbit space are acyclic. In this case the simplicial poset dual to the orbit space is a homology manifold. The questions under consideration are closely related to socles of Buchsbaum simplicial posets, the theory in commutative algebra and combinatorics introduced recently by Novik and Swartz.

1. INTRODUCTION

A simplicial poset is a combinatorial notion corresponding to the familiar topological notion of simplicial cell complex, i.e. a regular cell complex all of whose cells are simplices. Let S be a simplicial cell subdivision of a given topological space R, and let f_j denote the number of j-dimensional simplices in S. The task traditionally raised in combinatorics is to find the relations on the numbers f_j for a given space R (e.g. a sphere, or a manifold). One of the greatest achievements in combinatorics was the invention of the face rings. Every simplicial poset S determines a graded ring $\Bbbk[S]$, called the face ring, whose Hilbert–Poincare series contains all the information about f-numbers. It was noted that topological properties of $R=|S|$, the geometrical realization of S, are in nice correspondence with algebraical properties of its face ring. For example, if R is a sphere, the face ring $\Bbbk[S]$ is Gorenstein (in particular, Cohen–Macaulay), and if R is a manifold, then $\Bbbk[S]$ is a Buchsbaum ring. These observations allowed to formulate combinatorial problems in the language of commutative algebra, and solve many of them.

If S is a simplicial cell subdivision of a sphere, then $\Bbbk[S]$ is Gorenstein and, therefore, the quotient of $\Bbbk[S]$ by a linear system of parameters $\theta_1, \cdots, \theta_n$ is a 0-dimensional Gorenstein algebra. This means that the quotient $\Bbbk[S]/(\theta_1, \cdots, \theta_n)$ is a Poincare duality algebra. A natural question is: can we find a manifold whose cohomology algebra is $\Bbbk[S]/(\theta_1, \cdots, \theta_n)$? The first example is well-known: any complete smooth toric variety has cohomology algebra exactly of this form. The idea to use projective toric varieties in the study of convex simplicial spheres lead Stanley [14] to the proof of the famous g-theorem (the necessity part of the theorem).

Certainly, not every ring $\Bbbk[S]/(\theta_1, \cdots, \theta_n)$ can be modeled by a toric variety even if $\Bbbk=\mathbb{Z}$. In the seminal paper [8] Davis and Januszkiewicz introduced the concept of what is now called a quasitoric manifold. A slight generalization of their construction can be used to produce a closed homology manifold X such that $H^*(X;\mathbb{Z}) \cong \mathbb{Z}[S]/(\theta_1, \cdots, \theta_n)$ for a given homology sphere S and a sequence of linear elements $\theta_1, \cdots, \theta_n$ which is a linear system of parameters over any field.

If S is a simplicial cell subdivision of a manifold rather than just a sphere, the corresponding combinatorial theory becomes more complicated. Schenzel [13] computed the dimensions of graded components of the algebra $\Bbbk[S]/(\theta_1, \cdots, \theta_n)$, which are now called the h'-numbers of S. They depend on f-numbers and Betti numbers of S. In more recent works [11, 12] Novik and Swartz considered a distinguished submodule $\tilde{I}_{NS} \subset \Bbbk[S]/(\theta_1, \cdots, \theta_n)$ and computed its rank. They showed that, whenever the geometrical realization $|S|$ of a simplicial poset S is an orientable homology manifold, the double quotient $(\Bbbk[S]/(\theta_1, \cdots, \theta_n))/\tilde{I}_{NS}$ is a Poincare duality algebra. The dimensions of its graded components are called the h''-numbers of S.

The problem of certain interest is to construct topological models for the algebras $\Bbbk[S]/(\theta_1, \cdots, \theta_n)$ and $(\Bbbk[S]/(\theta_1, \cdots, \theta_n))/\tilde{I}_{NS}$ in case $|S|$ is a homology manifold. It is quiet natural to suspect that such a model would support a half-dimensional torus action as in the spherical case. Of course, the first idea is to find a closed topological manifold X such that $H^*(X) \cong (\mathbb{Z}[S]/(\theta_1, \cdots, \theta_n))/\tilde{I}_{NS}$. However, it seems that this approach fails, since in this case $H^*(X)$ is only concentrated in even degrees which implies that the underlying combinatorial structure S of X is a sphere (not a general manifold as required) [10].

However, we may take the natural candidates for X: the manifolds with locally standard actions. Under some restrictions on X we explicitly computed their cohomology and equivariant cohomology rings. They are not isomorphic to $\mathbb{Z}[S]$ or $\mathbb{Z}[S]/(\theta_1, \cdots, \theta_n)$ or $(\mathbb{Z}[S]/(\theta_1, \cdots, \theta_n))/\tilde{I}_{NS}$ but there exist interesting relations between these objects.

This paper is a review of the author's results proved previously in [2],[3]. The goal of the paper is twofold. First, we want to develop a topological approach to study the face rings of simplicial manifolds. Second, we want to study the topology of manifolds with locally standard actions. Note, that there are several recent constructions in differential geometry providing non-trivial examples of such manifolds. Examples include toric origami manifolds [7] and toric log symplectic manifolds [9]. Both are the generalizations of symplectic toric manifolds but unlike symplectic toric case such manifolds may have nontrivial topology of the orbit space.

2. Commutative algebra preliminaries

Fix a ground ring $\Bbbk$ (which is a field or $\mathbb{Z}$) and consider a simplicial complex K with the vertex set $[m] = \{1, \cdots, m\}$. Let $\Bbbk[m] := \Bbbk[v_1, \cdots, v_m]$, denote the graded ring of polynomials, where we set $\deg v_i = 2$. Recall that the algebra

$$\Bbbk[K] := \Bbbk[m]/(v_{i_1} \cdot \ldots \cdot v_{i_k} \mid \{i_1, \cdots, i_k\} \notin K)$$

is called the *face ring* (or the Stanley–Reisner algebra) of a simplicial complex K. A face ring is graded in even degrees and becomes a $\Bbbk[m]$-module via the natural projection map $\Bbbk[m] \to \Bbbk[K]$.

This construction has a well-known generalization to simplicial cell complexes (otherwise called simplicial posets). Recall that a finite partially ordered set (poset for short) S is called simplicial if (1) there exists a minimal element $\hat{0} \in S$; (2) for any $I \in S$, the order ideal $S_{\leqslant I} := \{J \in S \mid J \leqslant I\}$ is isomorphic to a poset of faces of a k-dimensional simplex for some $k \geqslant 0$ (i.e. a boolean lattice of rank $k+1$). The number k is called the dimension of $I \in S$, and $k+1$ the rank of I. The elements of S are called simplices and the elements of rank 1 vertices. The rank of I is equal to the number of vertices of I (i.e. the number of vertices $i \in S$, $i < I$) and is denoted by $|I|$.

Let $I_1 \vee I_2$ denote the set of least upper bounds of simplices $I_1, I_2 \in S$, and $I_1 \cap I_2 \in S$ denote the intersection of simplices (it is well-defined and unique if $I_1 \vee I_2 \neq \varnothing$).

Definition 2.1. The *face ring* $\Bbbk[S]$ of a simplicial poset S is the quotient of the polynomial ring $\Bbbk[\{v_I\}]$, generated by variables $\{v_I \mid I \in S\}$, $\deg v_I = 2|I|$, by the

relations
$$v_{I_1} \cdot v_{I_2} = v_{I_1 \cap I_2} \cdot \sum_{J \in I_1 \vee I_2} v_J, \qquad v_{\hat{0}} = 1.$$
The sum over an empty set is assumed to be 0.

When S is a poset of simplices of a simplicial complex, this ring coincides with the one defined previously. In general, if $[m]$ denotes the set of vertices of S, we still have a ring homomorphism $\Bbbk[m] \to \Bbbk[S]$ which sends v_i to v_i, but this homomorphism may not be surjective. It defines the structure of a $\Bbbk[m]$-module on $\Bbbk[S]$.

In the following we assume that S is pure of dimension $n-1$, which means that all maximal simplices of S have n vertices. We call the map $\lambda\colon [m] \to \Bbbk^n$ a *characteristic function*, if the following so called $(*)$-condition holds: whenever $i_1, \cdots, i_n$ are the vertices of a maximal simplex, the corresponding values $\lambda(i_1), \cdots, \lambda(i_n)$ are the basis of $\Bbbk^n$. Let $(\lambda_{i,1}, \cdots, \lambda_{i,n})$ be the coordinates of the vector $\lambda(i)$ in a fixed basis of $\Bbbk^n$ for each $i \in [m]$.

For every characteristic function we can construct the linear elements of the face ring:
$$\theta_j = \sum_{i \in [m]} \lambda_{i,j} v_i \in \Bbbk[K]_2 \quad \text{for } j = 1, \dots, n.$$
It is known (see e.g. [5, Lm3.3.2]) that $\theta_1, \cdots, \theta_n \in \Bbbk[K]$ is a linear system of parameters which means that $\Bbbk[K]/(\theta_1, \cdots, \theta_n)$ is an algebra of Krull dimension 0 (i.e. a finite dimensional vector space). In the following we denote the ideal $(\theta_1, \cdots, \theta_n)$ by Θ.

Let f_j denote the number of j-dimensional simplices of a simplicial poset S. The h-numbers of S are defined by the relation $\sum_{j=0}^{n} h_j t^{n-j} = \sum_{j=0}^{n} f_{j-1}(t-1)^{n-j}$, where t is a formal variable. The Hilbert–Poincare series of the face ring is expressed in terms of the h-numbers:
$$\operatorname{Hilb}(\Bbbk[S]; t) = \frac{\sum_{j=0}^{n} h_j t^{2j}}{(1-t^2)^n}.$$

For a simplex $I \in S$ let $\operatorname{lk}_S I$ denote the poset $\{J \in S \mid J \geqslant I\}$. It is easily seen that $\operatorname{lk}_S I$ is a simplicial poset whose minimal element is I.

Definition 2.2. A simplicial poset S is called *Buchsbaum* (over $\Bbbk$) if it is pure, and $\widetilde{H}_r(\operatorname{lk}_S I; \Bbbk) = 0$ for any proper simplex $I \in S$, $I \neq \hat{0}$ and any $r < \dim \operatorname{lk}_S I$.

If, moreover, $\widetilde{H}_r(S; \Bbbk) = 0$ for $r < \dim S$, then S is called *Cohen–Macaulay.*

Here and henceforth the notation $H_*(S)$ stands for the homology of the geometrical realization $|S|$ of a poset S with coefficients in $\Bbbk$. By abuse of terminology we call a simplicial poset S a *homology sphere* (resp. *homology manifold*) if its geometrical realization is a homology sphere (resp. homology manifold). It can be easily proved (see [16]) that every homology sphere is Cohen–Macaulay. Similarly, every homology manifold is Buchsbaum.

The classical results of Stanley and Reisner [16, 15] state that S is Cohen–Macaulay over $\Bbbk$ if and only if $\Bbbk[S]$ is a Cohen–Macaulay ring (which means that every homogeneous system of parameters in this ring is a regular sequence). It follows that
$$\operatorname{Hilb}(\Bbbk[S]/\Theta; t) = \sum_{j=0}^{n} h_j t^{2j}$$
for Cohen–Macaulay simplicial posets. In particular, h-numbers of such posets are nonnegative. Moreover, if S is a homology sphere, then the algebra $\Bbbk[S]/\Theta$ is a Poincare duality algebra. This implies the well-known Dehn–Sommerville relations $h_j = h_{n-j}$ for homology spheres.

The corresponding theory for Buchsbaum posets and homology manifolds is more complicated. The study of Buchsbaum complexes was initiated by Schenzel [13] in 1981. Recently a big progress in this theory was made by Novik and Swartz [11, 12]. Schenzel proved that a simplicial complex K is Buchsbaum if and only if $\Bbbk[K]$ is Buchsbaum. In Buchsbaum case there holds

$$\operatorname{Hilb}(\Bbbk[K]/\Theta; t) = \sum_{j=0}^{n} h'_j t^{2j},$$

where

$$h'_j := h_j + \binom{n}{j}\left(\sum_{k=1}^{j}(-1)^{j-k-1}\operatorname{rk}\widetilde{H}_{k-1}(K;\Bbbk)\right).$$

Novik–Swartz extended these results to simplicial posets. Moreover, they proved the following statements for general Buchsbaum posets. First, recall that the socle of a $\Bbbk[m]$-module $\mathcal{M}$ is the subspace $\operatorname{Soc}\mathcal{M} = \{x \in \mathcal{M} \mid x \cdot \Bbbk[m]_+ = 0\}$, where $\Bbbk[m]_+$ is the part of the polynomial ring of the positive degree.

- There exists a distinguished graded subspace $I_{NS} \subset \operatorname{Soc}\Bbbk[S]/\Theta$.
- $(I_{NS})_{2j} \cong \binom{n}{j}\widetilde{H}^{j-1}(S)$ for $j = 0, \dots, n$
- If S is an orientable homology manifold, then $I_{NS} = \operatorname{Soc}\Bbbk[S]/\Theta$. Let $\tilde{I}_{NS}$ denote the subspace of I_{NS} which coincides with I_{NS} in degrees $< 2n$, and in degree $2n$ corresponds to the subspace of all cohomology classes in $H^{2n}(S) \cong I_{NS}$ which vanish on the fundamental class of S.
- Under the assumptions of the previous paragraph, the quotient ring $(\Bbbk[S]/\Theta)/\tilde{I}_{NS}$ is a Poincare duality algebra.

Let us define h''-numbers of S as follows:

$$h''_j = h'_j - \binom{n}{j}\operatorname{rk}\widetilde{H}^{j-1}(S) \text{ for } j < n,$$

and $h''_n = h'_n - (\operatorname{rk}H^{n-1}(S) - 1)$. It follows from the statements above that h''-numbers of any Buchsbaum poset are nonnegative. Moreover, for an orientable homology manifold S we have $\operatorname{Hilb}((\Bbbk[S]/\Theta)/\tilde{I}_{NS}; t) = \sum_{j=0}^{n} h''_j t^{2j}$. Poincare duality then implies the well-known generalized Dehn–Sommerville relations for homology manifolds: $h''_j = h''_{n-j}$.

Note that for Cohen–Macaulay posets (in particular for homology spheres) the numbers h''_j, h'_j, and h_j coincide.

3. Topological models in spherical case

When S is a homology sphere and the base ring is either $\mathbb{Z}$ or $\mathbb{Q}$, there exists a topological model for the algebra $\Bbbk[S]/\Theta$. More precisely, there exists a closed $\Bbbk$-homology $2n$-manifold X such that its cohomology algebra $H^*(X;\Bbbk)$ is isomorphic to $\Bbbk[S]/\Theta$ and equivariant cohomology is isomorphic to $\Bbbk[S]$. Existence of such objects gives a simple explanation for the Poincare duality in $\Bbbk[S]/\Theta$.

Note that any complete smooth toric variety X is an example of such topological model. Indeed, let Δ_X be the non-singular fan corresponding to X; K be the underlying simplicial complex of Δ_X, and $\lambda(i) = (\lambda_{i,1}, \cdots, \lambda_{i,n}) \in \mathbb{Z}^n$ be the primitive generator of the i-th ray of Δ_X for $i \in [m]$. Then Danilov–Jurkiewicz theorem states $H^*(X;\mathbb{Z}) \cong \mathbb{Z}[K]/\Theta$, where Θ is generated by the linear forms $\sum_{i\in[m]} \lambda_{i,j} v_i$ for $j = 1, \dots, n$.

In general, the topological model can be obtained using Davis–Januszkiewicz construction [8]. Let us identify the space $\mathbb{R}^n$ with the Lie algebra of a compact torus T^n. Then each nonzero rational vector $w \in \mathbb{Q}^n \subset \mathbb{R}^n$ determines a circle subgroup $\exp(w) \subset T^n$. If we are given a homology sphere S on the vertex-set $[m]$, and a characteristic function $\lambda\colon [m] \to \mathbb{Q}^n$, then we obtain the collection of circle subgroups $\{T_i := \exp(\lambda(i))$ for $i \in [m]\}$. Let T_I denote the product $T_{i_1} \times \cdots \times T_{i_k}$ for any simplex $I \in S$ with vertices $i_1, \cdots, i_k$. Consider the space $P = \operatorname{cone}|S|$, which is a homology ball. Its boundary has a simple face structure dual to S; we denote by G_I the face of P dual to $I \in S$. We have $\dim G_I = n - |I|$, and the vertices of S correspond to the facets of P. Now we can construct the space

$$X = (P \times T^n)/\sim$$

where $(x_1, t_1) \sim (x_2, t_2)$ if and only if x_1 coincides with x_2 and lies in the interior of G_I for some I, and $t_1 t_2^{-1} \in T_I$. Then X is a closed rational homology manifold which satisfies $H_T^*(X; \mathbb{Q}) \cong \mathbb{Q}[S]$ and $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[S]/\Theta$ (see [8], [10]). If λ is a characteristic function over $\mathbb{Z}$, then X is a $\mathbb{Z}$-homology manifold. Moreover, if S is PL-equivalent to a boundary of simplex, then X is a topological manifold, and if S is the boundary of a convex simplicial polytope, then X can be given a smooth structure as described in [6].

4. Manifolds with locally standard actions

Now let S be an orientable homology manifold. Our goal is to study reasonable spaces, which model the rings $\Bbbk[S]/\Theta$ and $(\Bbbk[S]/\Theta)/\tilde{I}_{NS}$, or at least reflect their properties.

Let us recall the notion of a manifold with locally standard action. Let X^{2n} be a smooth compact manifold (also assumed connected, orientable) with an effective smooth action of a half-dimensional torus $T^n \circlearrowleft X^{2n}$. The action is called locally standard if it is locally equivalent to the standard action

$$T^n \circlearrowleft \mathbb{C}^n \cong \mathbb{R}^{2n} \qquad (t_1, \cdots, t_n) \cdot (z_1, \cdots, z_n) = (t_1 z_1, \cdots, t_n z_n).$$

It means that there is an atlas of charts on M, each equivariantly diffeomorphic, up to automorphism of torus, to a T^n-invariant subset of $\mathbb{C}^n$. The orbit space of the standard action $\mathbb{C}^n/T^n$ is the nonnegative cone $\mathbb{R}^n_{\geqslant} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_j \geqslant 0\}$. It has a natural face stratification, and the faces correspond to different stabilizers of the action. Therefore, the orbit space of any locally standard action $Q := X/T^n$ has the natural structure of a manifold with corners.

Consider a facet $F_i \subset Q$. An orbit $x \in F_i^\circ$ has a 1-dimensional stabilizer $G_i \subset T^n$,

$$G_i = \exp(\langle \lambda_{i,1}, \cdots, \lambda_{i,n} \rangle)$$

for some primitive vector $(\lambda_{i,1}, \cdots, \lambda_{i,n}) \in \mathbb{Z}^n$. This construction associates a primitive vector to each facet of Q. These vectors are the analogues of primitive generators for the rays in the fan of a toric variety.

A manifold with corners Q is called nice, if any codimension k facet of Q lies in exactly k facets. It can be easily proved that the orbit spaces of locally standard actions are nice manifolds with corners. For a manifold with corners Q consider the poset of faces of Q ordered by reversed inclusion. Thus far Q becomes the minimal element of S_Q and whenever Q is nice S_Q is a simplicial poset. In order to distinguish between abstract simplices of S_Q and the faces of Q as topological spaces, we denote the former by $I, J \in S_Q$ as before and the corresponding faces of Q by F_I, F_J etc. Facets of Q correspond to vertices of S_Q (the set of vertices is denoted by $[m]$ as before). We have a map $\lambda\colon [m] \to \mathbb{Z}^n$ sending $i \in [m]$ to $\lambda(i) = (\lambda_{i,1}, \cdots, \lambda_{i,n})$, which can be shown to be a characteristic map (over $\mathbb{Z}$ thus over any $\Bbbk$).

For a manifold X with locally standard torus action the free part of action determines a principal torus bundle $X^{free} \to Q^\circ$, where Q° is the interior of Q. It can be extended over Q which gives a principal torus bundle $\eta\colon Y \to Q$. It is known, that up to equivariant homeomorphism any manifold with locally standard action is uniquely determined by the data $(Q, \eta\colon Y \to Q, \lambda\colon [m] \to \mathbb{Z}^n)$. More precisely, Yoshida [17] proved that X is equivariantly homeomorphic to the model space $Y/{\sim}$, where $y_1 \sim y_2$ if and only if $\eta(y_1) = \eta(y_2)$ lies in the interior of F_I, and y_1, y_2 lie in one T_I-orbit. We have a natural projection map $f\colon Y \to X$.

There are natural topological filtrations on Q, Y, and X:

$$Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q, \qquad Y_0 \subset Y_1 \subset \cdots \subset Y_n = Y,$$
$$X_0 \subset X_1 \subset \cdots \subset X_n = X,$$

where Q_j is the union of j-faces of Q, $Y_j = \eta^{-1}(Q_j)$, and X_j is the union of all j-dimensional orbits of X (this filtration on X is called the orbit type filtration). These filtrations are compatible with the maps $\eta\colon Y \to Q$, $f\colon Y \to X$, and the projection to the orbit space $X \to Q$.

Let $(E_Q)^*_{*,*} \Rightarrow H_*(Q)$, $(E_Y)^*_{*,*} \Rightarrow H_*(Y)$, and $(E_X)^*_{*,*} \Rightarrow H_*(X)$ be the homological spectral sequences associated with the above filtrations (all coefficients in $\Bbbk$). The map $f\colon Y \to X$ induces maps of the spectral sequences on each page $f^r_*\colon (E_Y)^r_{*,*} \to (E_X)^r_{*,*}$, $r \geqslant 1$.

5. Acyclic proper faces

Further on we impose two restrictions on X. First, we assume that Q is an orientable manifold with corners (equiv., X is orientable, see [4]), and all its proper faces are acyclic (over $\Bbbk$). Second, the principal torus bundle $Y \to Q$ is assumed trivial. Thus $X = (Q \times T^n)/{\sim}$. The following propositions were proved in [1, 2].

Proposition 5.1. *The poset S_Q is an orientable homology manifold (over $\Bbbk$). In particular, S_Q is a Buchsbaum simplicial poset.*

Proposition 5.2. *There exists a homological spectral sequence $(\dot{E}_Q)^r_{p,q} \Rightarrow H_{p+q}(Q)$, $(\dot{d}_Q)^r\colon (\dot{E}_Q)^r_{p,q} \to (\dot{E}_Q)^r_{p-r,q+r-1}$ with the properties:*

1. *$(\dot{E}_Q)^1 = H((E_Q)^1, d_Q^-)$, where the differential $d_Q^-\colon (E_Q)^1_{p,q} \to (E_Q)^1_{p-1,q}$ coincides with $(d_Q)^1$ for $p < n$, and vanishes otherwise.*
2. *The module $(\dot{E}_Q)^r_{*,*}$ coincides with $(E_Q)^r_{*,*}$ for $r \geqslant 2$.*
3. $(\dot{E}_Q)^1_{p,q} = \begin{cases} H_p(\partial Q), & \text{if } q = 0, p < n; \\ H_{q+n}(Q, \partial Q), & \text{if } p = n, q \leqslant 0; \\ 0, & \text{otherwise.} \end{cases}$
4. *Nontrivial differentials for $r \geqslant 1$ have the form $(\dot{d}_Q)^r\colon (\dot{E}_Q)^r_{n,1-r} \to (\dot{E}_Q)^r_{n-r,0}$ and coincide with the connecting homomorphisms $\delta_{n+1-r}\colon H_{n+1-r}(Q, \partial Q) \to H_{n-r}(\partial Q)$ in the long exact sequence of the pair $(Q, \partial Q)$.*

Let Λ_* denote the homology module of a torus: $\Lambda_* = \bigoplus_s \Lambda_s$, $\Lambda_s = H_s(T^n)$.

Proposition 5.3. *There exists a homological spectral sequence $(\dot{E}_Y)^r_{p,q} \Rightarrow H_{p+q}(Y)$ such that*

1. *$(\dot{E}_Y)^1 = H((E_Y)^1, d_Y^-)$, where the differential $d_Y^-\colon (E_Y)^1_{p,q} \to (E_Y)^1_{p-1,q}$ coincides with $(d_Y)^1$ for $p < n$, and vanishes otherwise.*
2. *$(\dot{E}_Y)^r = (E_Y)^r$ for $r \geqslant 2$.*
3. *$(\dot{E}_Y)^r_{p,q} = \bigoplus_{q_1+q_2=q} (\dot{E}_Q)^r_{p,q_1} \otimes \Lambda_{q_2}$ and $(\dot{d}_Y)^r = (\dot{d}_Q)^r \otimes \mathrm{id}_\Lambda$ for $r \geqslant 1$.*

Proposition 5.4. *There exists a homological spectral sequence $(\dot{E}_X)^r_{p,q} \Rightarrow H_{p+q}(X)$ and the morphism of spectral sequences $\dot{f}^r_*\colon (\dot{E}_Y)^r_{*,*} \to (\dot{E}_X)^r_{*,*}$ such that:*

1. *$(\dot{E}_X)^1 = H((E_X)^1, d_X^-)$ where the differential $d_X^-\colon (E_X)^1_{p,q} \to (E_X)^1_{p-1,q}$ coincides with $(d_X)^1$ for $p < n$, and vanishes otherwise. The map $\dot{f}^1_*$ is induced by $f^1_*\colon (E_Y)^1 \to (E_X)^1$.*

(2) $(\dot{E}_X)^r = (E_X)^r$ *and* $\dot{f}_*^r = f_*^r$ *for* $r \geqslant 2$.
(3) $(E_X)^1_{p,q} = (\dot{E}_X)^1_{p,q} = 0$ *for* $p < q$.
(4) $\dot{f}^1_*\colon (\dot{E}_Y)^1_{p,q} \to (\dot{E}_X)^1_{p,q}$ *is an isomorphism for* $p > q$ *and injective for* $p = q$.
(5) *As a consequence of previous items, for* $r \geqslant 1$*, the differentials* $(\dot{d}_X)^r$ *are either isomorphic to* $(\dot{d}_Y)^r$ *(when they hit the cells with* $p > q$*), or isomorphic to the composition of* $(\dot{d}_Y)^r$ *with* $\dot{f}_*^r$ *(when they hit the cells with* $p = q$*), or zero (otherwise).*
(6) *The ranks of diagonal terms at a second page are the* h'*-numbers of the poset* S_Q *dual to the orbit space:* $\mathrm{rk}(\dot{E}_X)^2_{q,q} = \mathrm{rk}(E_X)^2_{q,q} = h'_{n-q}(S_Q)$.
(7) *The cokernel of the injective map* $\dot{f}^1_*\colon (\dot{E}_Y)^1_{q,q} \to (\dot{E}_X)^1_{q,q}$ *has rank* $h''_{n-q}(S_Q)$ *if* $q < n$.

6. Cohomology and equivariant cohomology of X

Under the same assumptions of orientability, proper face acyclicity, and triviality of $\eta\colon Y \to Q$, there holds

Theorem 6.1 ([4]). *There is an isomorphism of rings (and* $\Bbbk[m]$*-modules)*

$$H^*_T(X) \cong \Bbbk[S_Q] \oplus H^*(Q),$$

where the 0*-degree components are identified.*

The expression for the ordinary cohomology $H^*(X)$ can be extracted from the calculations of spectral sequences in the previous section and Poincare duality on X. It appears to be more complicated comparing to equivariant cohomology. Let $H^*_T(X) \to H^*(X)$ be the ring homomorphism induced by the inclusion of a fiber in the Borel fibration

$$X \hookrightarrow X \times_T ET \xrightarrow{\pi} BT. \tag{6.1}$$

There is a face ring inside $H^*_T(M)$. Thus we have a composed map $\sigma\colon \Bbbk[S_Q] \hookrightarrow H^*_T(X) \to H^*(X)$. This map factors through $\Bbbk[S_Q]/\Theta$, since Θ maps to $\pi^*(H^+(BT))$ under the first map and $\pi^*(H^+(BT))$ vanishes in ordinary cohomology according to (6.1). We have the diagram of ring homomorphisms

$$\begin{array}{ccc} \Bbbk[S_Q] & \longrightarrow & \Bbbk[S_Q]/\Theta \\ \big\downarrow & \overset{\sigma}{\searrow} & \big\downarrow \rho \\ H^*_T(X) & \longrightarrow & H^*(X) \end{array}$$

The ring homomorphism ρ has a clear geometrical meaning: the element $v_I \in \Bbbk[S_Q]/\Theta$ maps to the cohomology class Poincare dual to face submanifold $X_I \subset X$ lying over the face $F_I \subset Q$. In general ρ is neither injective nor surjective.

This homomorphism has the following properties.

Theorem 6.2 ([3]).

- $\ker\rho \subseteq \tilde{I}_{NS} \subseteq \mathrm{Soc}(\Bbbk[S_Q]/\Theta)$. *Recall that* $(\tilde{I}_{NS})_{2j} \cong \binom{n}{j}\widetilde{H}^{j-1}(S_Q)$ *for* $j < n$*, and* $(\tilde{I}_{NS})_{2n} \cong \{a \in \widetilde{H}^{n-1}(S_Q) \mid a[S_Q] = 0\}$. *By Poincare duality we have* $(\tilde{I}_{NS})_{2j} \cong \binom{n}{j}(\widetilde{H}_{n-j}(\partial Q)/\langle[\partial Q]\rangle)$. *Here we need to quotient out the fundamental class of* ∂Q *since we have reduced cohomology on the left.*
- $(\ker\rho)_{2j} \cong \binom{n}{j}\ker(H_{n-j}(\partial Q) \to H_{n-j}(Q))$, *for* $j > 0$.
- $\rho((\Bbbk[S_Q]/\Theta)_+)$ *is an ideal in* $H^*(X)$*;*

- $H^*(X)/\rho((\Bbbk[S_Q]/\Theta)_+) = \bigoplus_{j=0}^{2n} A^j$, *where*

$$A^j \cong \bigoplus_{\substack{p+q=j\\ p<q}} \binom{n}{q} H^p(Q, \partial Q) \oplus \bigoplus_{\substack{p+q=j\\ p\geqslant q}} \binom{n}{q} H^p(Q). \tag{6.2}$$

- *The homomorphism* $\Bbbk[S_Q] \oplus H^*(Q) \cong H_T^*(X) \to H^*(X)$ *maps* $H^*(Q)$ *isomorphically to the summands in* (6.2) *having* $q = 0$.

Corollary 6.3. *Betti numbers of X depend only on Q but not on the characteristic function λ.*

Proof. The ranks of the graded components of $\Bbbk[S_Q]/\Theta$ are the h'-numbers which do not depend on Θ (hence λ) by Schenzel's result. On the other hand, the ranks of the graded components of the kernel and cokernel of the map $\Bbbk[S_Q]/\Theta \to H^*(X)$ are expressed only in terms of Q. ☐

To state things more shortly, let $\mathcal{F}^*(X)$ denote the image of $\Bbbk[S_Q]/\Theta$ in $H^*(X)$, i.e. a subalgebra spanned by the classes of X Poincare dual to face submanifolds. We call $\mathcal{F}^*(X)$ the face part of the cohomology ring. Then we have a diagram of graded ring homomorphisms

$$\begin{array}{ccccc} \Bbbk[S_Q]/\Theta & \twoheadrightarrow & \mathcal{F}^*(X) & \twoheadrightarrow & (\Bbbk[S_Q]/\Theta)/\tilde{I}_{NS} \\ & & \big\downarrow & & \\ & & H^*(X) & & \end{array}$$

which means that the face part of cohomology is clamped between $\Bbbk[S_Q]/\Theta$ and $(\Bbbk[S_Q]/\Theta)\tilde{I}_{NS}$.

Corollary 6.4. *The Betti numbers of X in even degrees are bounded below by the h''-numbers of S_Q:*

$$\operatorname{rk} H^{2j}(X) \geqslant h''_j.$$

Finally, let us mention that the independence of Betti numbers from the characteristic function does not hold for general manifolds with locally standard actions.

Example 6.5. Two manifolds $M_1 = S^3 \times S^1$ and $M_2 = S^2 \times S^1 \times S^1$ can be given a locally standard actions of T^2 such that the orbit space in both cases is $Q = S^1 \times [0,1]$, the product of a circle and an interval. Surely, M_1 and M_2 have different Betti numbers. The results shown above do not apply in this case, since proper faces of Q are not acyclic. See details in [2].

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