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ON QUADRATIC NONLINEAR KLEIN-GORDON EQUATIONS

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ABSTRACT. We report our recent results on nonlinear Klein-Gordon equations with quadratic interactions.

1. INTRODUCTION

In this article, we survey recent progress on asymptotic behavior of solutions to the nonlinear Klein-Gordon equations

$$\partial_t^2 v - \Delta v + m^2 v = v^2, (t,x) \in \mathbf{R} \times \mathbf{R}^n \tag{1.1}$$

with quadratic self interaction term for $n=1$ or $n=2$, where v is a real-valued function, $m>0$ is the mass of particle, and Δ is the Laplacian. We also consider a system of quadratic nonlinear Klein-Gordon equations

$$\left\{ \begin{array}{c} \partial_t^2 v_1 - \Delta v_1 + m_1^2 v_1 = v_1 v_2, \\ \partial_t^2 v_2 - \Delta v_2 + m_2^2 v_2 = v_1^2 \end{array} \right. (t,x) \in \mathbf{R} \times \mathbf{R}^2, \tag{1.2}$$

where $m_1, m_2 > 0$ are the masses of particles. Quadratic nonlinearities do not include resonance terms for (1.1) and for (1.2) if $2m_1 \neq m_2$. There are two different ways to treat quadratic nonlinear Klein-Gordon equations. One of them is the method of algebraic normal forms due to Kosecki [26] for (1.1) and Sunagawa [32] for (1.2) under the non resonance mass condition $2m_1 \neq m_2$. The method of algebraic normal forms does not yield any nonlocal nonlinearity, however it leads to a derivative loss difficulty. Another way is the method of the normal forms of Shatah [31], which yields a complicated nonlocal nonlinear problem, but the derivative loss difficulty is avoided. When nonlinearity contains derivatives of the unknown function, we again encounter the derivative loss difficulty, so it seems that the method of Shatah also does not work well for this case.

2. *Quadratic nonlinear Klein-Gordon equations in 1d*

When $n=1$, in [19] the large time asymptotic profile of small solutions to the Cauchy problem (1.1) was obtained without the restriction of a compact support on the initial data which was assumed in [3]. One of important tools of [3] was based on the transformation by hyperbolic polar coordinates following [25]. The application of hyperbolic polar coordinates implies the restriction to the interior of the light cone and therefore requires compactness of the initial data.

In order to state the result in [19], we change $u = \frac{1}{2}\left(v + i \langle i\nabla\rangle^{-1} v_t\right)$ in (1.1) with $m=1$, where $\langle i\nabla\rangle = \sqrt{1-\Delta}$, then u satisfies the following Cauchy problem

$$\left\{ \begin{array}{c} \mathcal{L}u = i\frac{1}{2} \langle i\nabla\rangle^{-1} (u+\overline{u})^2, (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^n; \end{array} \right. \tag{2.1}$$

where $\mathcal{L} = \partial_t + i\langle i\nabla\rangle$, $u_0 = \frac{1}{2}\left(v_0 + i\langle i\nabla\rangle^{-1} v_1\right)$. We denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathcal{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, with the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int |\phi(x)|^p dx\right)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbf{R}^n} |\phi(x)|$ if $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{m,s} = \{\phi \in \mathbf{L}^p; \|\langle x\rangle^s \langle i\nabla\rangle^m \phi\|_{\mathbf{L}^p} < \infty\},$$

for $m, s \in \mathbf{R}$, $1 \le p \le \infty$, where $\langle x\rangle = \sqrt{1+|x|^2}$. For simplicity we write $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$. We usually omit the index zero if this does not cause confusion, e.g. we write $\mathbf{H}^m$ instead of $\mathbf{H}^{m,0}$. We denote the Fourier transform of the function ϕ by

$$\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{-ix\cdot\xi}\phi(x)\,dx.$$

Then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{ix\cdot\xi}\phi(\xi)d\xi.$$

Our main result in [19] is the following.

Theorem 2.1. *Let $m = 1$, $u_0 \in \mathbf{H}^{2,1}$ and the norm $\|u_0\|_{\mathbf{H}^{2,1}} = \varepsilon$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the Cauchy problem (2.1) has a unique global solution $u(t) \in \mathbf{C}([0,\infty); \mathbf{H}^{2,1})$ satisfying the time decay estimate*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon(1+t)^{-\frac{1}{2}}.$$

Furthermore there exists a unique final state $\widehat{u}_+ \in \mathbf{L}^\infty$ such that

$$\left\|\mathcal{F}e^{i\langle i\partial_x\rangle t}u(t) - \widehat{u}_+ e^{-i\Omega|u_+|^2\log t}\right\|_{\mathbf{L}^\infty} \le Ct^{\delta-\frac{1}{8}},$$

and the large time asymptotics

$$u(t) = \frac{1}{\sqrt{it}}e^{-i\sqrt{t^2-x^2}}\frac{\theta\left(\frac{x}{t}\right)}{\left\langle i\frac{x}{t}\right\rangle^{\frac{3}{2}}}\widehat{u}_+\left(\frac{x}{\sqrt{t^2-x^2}}\right)$$
$$\times\exp\left(i\Omega\left(\frac{x}{\sqrt{t^2-x^2}}\right)\left|\widehat{u}_+\left(\frac{x}{\sqrt{t^2-x^2}}\right)\right|^2\log t\right) + O\left(\varepsilon t^{\delta-\frac{5}{8}}\right)$$

is valid uniformly with respect to $x \in \mathbf{R}$, where $\theta(x) = 1$ for $|x| < 1$ and $\theta(x) = 0$ for $|x| \ge 1$, δ is a small positive constant and

$$\Omega(\xi) = \frac{\lambda^2}{2}\langle\xi\rangle^3\left(\frac{1}{3\langle 2\xi\rangle} + \frac{5}{2\langle\xi\rangle}\right).$$

There are some works devoted to the study of the cubic nonlinear Klein-Gordon equation

$$\begin{cases} v_{tt} + v - v_{xx} = \lambda v^3, & (t,x) \in \mathbf{R}\times\mathbf{R}, \\ v(0,x) = v_0(x), v_t(0,x) = v_1(x), & x \in \mathbf{R} \end{cases} \tag{2.2}$$

with $\lambda \in \mathbf{R}$. When $\lambda < 0$, the global existence of solutions to (2.2) can be easily obtained in the energy space, however which is not sufficient to imply the large time asymptotic behavior of solutions. The sharp $\mathbf{L}^\infty$ - time decay estimates of solutions and non existence of the usual scattering states for equation (2.2) were shown in [8] under the condition that the initial data are sufficiently regular and have a compact support.

Some sufficient conditions on quadratic or cubic nonlinearities were given in [3], which allow us to prove global existence and find sharp asymptotics of small solutions to the Cauchy problem with small and regular initial data having a compact support. Moreover it was proved that the asymptotic profile of solutions differs from that of the linear Klein-Gordon equation. Compactness condition on the data was removed in [11] in the case of a real-valued solution. When the initial data are complex-valued, the global existence and $\mathbf{L}^{\infty}$ - time decay estimates of small solutions to the Klein-Gordon equation with cubic nonlinearity $|v|^2 v$ were obtained in [33] under some conditions of regularity and a compact support of the initial data. As far as we know the problem of finding the large time asymptotics is still open for the case of the cubic nonlinearity v^3 with the complex-valued initial data.

Existence of scattering operators in the neighborhood of the origin in the space $\mathbf{H}^{1+\frac{n}{2},1} \cap \mathbf{H}^{\frac{n}{2},1}$ for the super critical nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + u = \mu |u|^{p-1} u, \ \ (t,x) \in \mathbf{R} \times \mathbf{R}^n,$$

was proved in [13], where $p > 1 + \frac{2}{n}$, $\mu \in \mathbf{C}$, $n = 1, 2$ (see [6] in which the case $n = 3, p > 1 + \frac{2}{n}$ was treated). The same method is useful for a system of equations

$$\begin{cases} \partial_t^2 v_1 - \Delta v_1 + m_1^2 v_1 = |v_2|^{p-2} v_2 v_1, \\ \partial_t^2 v_2 - \Delta v_2 + m_2^2 v_2 = |v_2|^{p-2} v_1^2. \end{cases}$$

Note that the mass condition is not needed in the super critical case $p > 1 + \frac{2}{n}$. The regularity of order $1 + \frac{n}{2}$ was required for the above problem to obtain the sharp $\mathbf{L}^{\infty}$ - time decay estimates. Non existence of usual scattering states was studied in [9], [28] for the case of sub-critical and critical nonlinearities $|v|^{p-1} v$ with $p \leq 1 + \frac{2}{n}$ and space dimension $n \geq 2$. However non existence problem is still open for $n = 1$ and $1 < p < 3$.

In [15], we used the method of normal forms of Shatah to obtain a sharp asymptotic behavior of small solutions to the Cauchy problem for the quadratic nonlinear Klein-Gordon equation $\mathcal{L}u = i \langle i\partial_x \rangle^{-1} \overline{u}^2$ without a condition of a compact support of the initial data. In [15], we have used the fact that the bilinear Fourier multiplier operator

$$\int_{\mathbf{R}^2} e^{ix(\xi+\eta)} \frac{\widehat{\phi}(\xi) \widehat{\psi}(\eta)}{\langle \xi + \eta \rangle + \langle \xi \rangle + \langle \eta \rangle} d\xi d\eta$$

has the Hölder type estimate $\|\mathcal{T}(\phi, \psi)\|_{\mathbf{L}^p} \leq C \|\phi\|_{\mathbf{L}^q} \|\psi\|_{\mathbf{L}^r}$, where $1 \leq p \leq q, r \leq \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. However the case of the general nonlinearity $i \langle i\partial_x \rangle^{-1} (u + \overline{u})^2$ was an open problem since we need to estimate the bilinear Fourier multiplier operators

$$\int_{\mathbf{R}^2} e^{ix(\xi+\eta)} \frac{\widehat{\phi}(\xi) \widehat{\psi}(\eta)}{\langle \xi + \eta \rangle - \langle \xi \rangle - \langle \eta \rangle} d\xi d\eta$$

and

$$\int_{\mathbf{R}^2} e^{ix(\xi+\eta)} \frac{\widehat{\phi}(\xi) \widehat{\psi}(\eta)}{\langle \xi + \eta \rangle - \langle \xi \rangle + \langle \eta \rangle} d\xi d\eta.$$

In [19] we proved the desired estimates for these bilinear operators. In order to remove the quadratic term from (2.1), we multiply both sides of (2.1) by the free Klein-Gordon evolution group $\mathcal{F}\mathcal{U}(-t) = \mathcal{F}e^{it\langle i\partial_x \rangle} = e^{it\langle \xi \rangle}\mathcal{F}$ and put $\varphi_u(t, \xi) =$

$e^{it\langle\xi\rangle}\widehat{u}(t,\xi)$ to get

$$\partial_t\varphi_u(t,\xi) = i\frac{1}{2}\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\mathcal{F}\left((u+\overline{u})^2\right) = \sum_{j=1}^{3} I_j, \tag{2.3}$$

where

$$I_1 = \frac{i}{2\sqrt{2\pi}\langle\xi\rangle}\int_{\mathbf{R}} e^{it(\langle\xi\rangle+\langle\xi-\eta\rangle+\langle\eta\rangle)}\overline{\varphi_u(t,\eta-\xi)\varphi_u(t,-\eta)}d\eta,$$
$$I_2 = \frac{i}{2\sqrt{2\pi}\langle\xi\rangle}\int_{\mathbf{R}} e^{it(\langle\xi\rangle-\langle\xi-\eta\rangle-\langle\eta\rangle)}\varphi_u(t,\xi-\eta)\varphi_u(t,\eta)\,d\eta,$$
$$I_3 = \frac{i}{\sqrt{2\pi}\langle\xi\rangle}\int_{\mathbf{R}} e^{it(\langle\xi\rangle-\langle\xi-\eta\rangle+\langle\eta\rangle)}\varphi_u(t,\xi-\eta)\overline{\varphi_u(t,-\eta)}d\eta.$$

Define the bilinear operators

$$\mathcal{T}_j(\phi,\psi)(x) = \frac{1}{2\pi}\iint_{\mathbf{R}^2} e^{ix(\xi+\eta)}m_j(\xi,\eta)\widehat{\phi}(\xi)\widehat{\psi}(\eta)\,d\xi d\eta$$

with symbols

$$m_1(\xi,\eta) = \frac{\lambda}{2\langle\xi+\eta\rangle(\langle\xi+\eta\rangle+\langle\xi\rangle+\langle\eta\rangle)},$$
$$m_2(\xi,\eta) = \frac{\lambda}{2\langle\xi+\eta\rangle(\langle\xi+\eta\rangle-\langle\xi\rangle-\langle\eta\rangle)},$$
$$m_3(\xi,\eta) = \frac{\lambda}{2\langle\xi+\eta\rangle(\langle\xi+\eta\rangle-\langle\xi\rangle+\langle\eta\rangle)}.$$

Then we have

$$I_1 = \partial_t e^{it\langle\xi\rangle}\mathcal{F}\mathcal{T}_1(\overline{u},\overline{u}) - 2e^{it\langle\xi\rangle}\mathcal{F}\mathcal{T}_1\left(\overline{u},\overline{\mathcal{L}u}\right),$$

$$I_2 = \partial_t e^{it\langle\xi\rangle}\mathcal{F}\mathcal{T}_2(u,u) - 2e^{it\langle\xi\rangle}\mathcal{F}\mathcal{T}_2(u,\mathcal{L}u),$$

and

$$I_3 = \partial_t e^{it\langle\xi\rangle}\mathcal{F}\mathcal{T}_3(u,\overline{u}) - e^{it\langle\xi\rangle}\mathcal{F}\mathcal{T}_3\left(u,\overline{\mathcal{L}u}\right) - e^{it\langle\xi\rangle}\mathcal{F}\mathcal{T}_3(\mathcal{L}u,\overline{u}).$$

Therefore returning to the function $u(t,x) = \mathcal{U}(t)\mathcal{F}^{-1}_{\xi\to x}\varphi_u$, we get from (2.3)

$$\begin{aligned}&\mathcal{L}(u-\mathcal{T}_1(\overline{u},\overline{u})-\mathcal{T}_2(u,u)-\mathcal{T}_3(u,\overline{u}))\\ &= -2\mathcal{T}_1\left(\overline{u},\overline{\mathcal{L}u}\right)-2\mathcal{T}_2(u,\mathcal{L}u)-\mathcal{T}_3\left(u,\overline{\mathcal{L}u}\right)-\mathcal{T}_3(\mathcal{L}u,\overline{u}).\end{aligned} \tag{2.4}$$

Denote the symbols

$$m_4(\xi,\eta) = i\frac{\lambda}{2}(2m_1(\xi,\eta)-m_3(\eta,\xi))\langle\eta\rangle^{-1},$$
$$m_5(\xi,\eta) = -i\frac{\lambda}{2}(2m_2(\xi,\eta)-m_3(\xi,\eta))\langle\eta\rangle^{-1}$$

and the corresponding bilinear Fourier multiplier operators by $\mathcal{T}_4$ and $\mathcal{T}_5$. In view of (2.1) we find

$$\begin{aligned}&\mathcal{L}(u-\mathcal{T}_1(\overline{u},\overline{u})-\mathcal{T}_2(u,u)-\mathcal{T}_3(u,\overline{u}))\\ &= \mathcal{T}_4\left(\overline{u},(\overline{u}+u)^2\right)+\mathcal{T}_5\left(u,(\overline{u}+u)^2\right).\end{aligned} \tag{2.5}$$

Thus we consider the cubic nonlinear nonlocal problem. This is the target equation which we study. We note that the nonlocal nonlinearities of the right hand sides of

(2.5) include the resonance nonlinearities which are not removable by the method of the normal form.

3. *Bilinear operators and their estimates in 1d*

We write the bilinear operators $\mathcal{T}_j$ as

$$\mathcal{T}_j(\phi,\psi)=\frac{1}{2\pi}\iint_{\mathbf{R}^2}dydz\mathbb{K}_j^{\mu,\nu}(y,z)\langle i\partial_x\rangle^{\mu}\phi(x-y)\langle i\partial_x\rangle^{\nu}\psi(x-z),$$

where the kernels $\mathbb{K}_j^{\mu,\nu}(y,z)=\mathcal{F}^{-1}_{\xi\to y}\mathcal{F}^{-1}_{\eta\to z}\left(\langle\xi\rangle^{-\mu}\langle\eta\rangle^{-\nu}m_j(\xi,\eta)\right)$, $1\le j\le 5$.

In the next lemma we state the estimate of the kernels $\mathbb{K}_j^{\mu,\nu}(y,z)$ without a proof (see [19]). Define $\sigma_j=0$ for $j=1,2,3$, and $\sigma_j=1$ for $j=4,5$.

Lemma 3.1. *Let $\mu>-1$, $\nu+\sigma_j>-1$, $\mu+\nu+\sigma_j>0$. Then the estimate*

$$\left|\mathbb{K}_j^{\mu,\nu}(y,z)\right|\le C\langle y\rangle^{-4}\langle z\rangle^{-4}|y|^{\gamma-1}|z|^{\gamma-1}$$

is true for all $y,z\in\mathbf{R}\setminus\{0\}$, where $0<\gamma<\frac{1}{4}\min(\mu+1,\nu+\sigma_j+1,\mu+\nu+\sigma_j)$.

Next we give the estimate of the bilinear Fourier multiplier operators $\mathcal{T}(\phi,\psi)$ defined by the multiplier $m(\xi,\eta)$

$$\begin{aligned}\mathcal{T}(\phi,\psi)&=\frac{1}{2\pi}\iint_{\mathbf{R}^2}e^{ix(\xi+\eta)}m(\xi,\eta)\widehat{\phi}(\xi)\widehat{\psi}(\eta)\,d\xi d\eta\\&=\frac{1}{2\pi}\iint_{\mathbf{R}^2}dydz\mathbb{K}(y,z)\phi(x-y)\psi(x-z),\end{aligned}$$

where $\mathbb{K}(y,z)=\mathcal{F}^{-1}_{\xi\to y}\mathcal{F}^{-1}_{\eta\to z}m(\xi,\eta)$.

Lemma 3.2. *Suppose that a kernel $\mathbb{K}(y,z)$ obeys the estimate*

$$|\mathbb{K}(y,z)|\le C\langle y\rangle^{-4}\langle z\rangle^{-4}|y|^{\gamma-1}|z|^{\gamma-1}$$

for all $y,z\in\mathbf{R}\setminus\{0\}$, where $\gamma\in(0,1)$. Then the following estimates are valid

$$\|\mathcal{T}(\phi,\psi)\|_{\mathbf{L}^p}\le C\|\phi\|_{\mathbf{L}^q}\|\psi\|_{\mathbf{L}^r},$$

$$\|x\mathcal{T}(\phi,\psi)\|_{\mathbf{L}^p}\le C\left(\|x\phi\|_{\mathbf{L}^q}+\|\phi\|_{\mathbf{L}^q}\right)\|\psi\|_{\mathbf{L}^r}$$

and

$$\begin{aligned}\|\mathcal{P}\mathcal{T}(\phi,\psi)\|_{\mathbf{L}^p}&\le C\left(\|\mathcal{P}\phi\|_{\mathbf{L}^q}+\|\partial_t\phi\|_{\mathbf{L}^q}\right)\|\psi\|_{\mathbf{L}^r}\\&\quad+C\|\phi\|_{\mathbf{L}^{q'}}\left(\|\mathcal{P}\psi\|_{\mathbf{L}^{r'}}+\|\partial_t\psi\|_{\mathbf{L}^{r'}}\right)\end{aligned}$$

for

$$1\le p\le q,r,q',r'\le\infty,\ \frac{1}{p}=\frac{1}{q}+\frac{1}{r}=\frac{1}{q'}+\frac{1}{r'},$$

provided that the right-hand sides are finite, where $\mathcal{P}=t\partial_x+x\partial_t$.

For the proof of Lemma 3.2, see [19]. Application of Lemma 3.2 to the bilinear Fourier multiplier operators

$$\mathcal{T}_j(\phi,\psi)=\frac{1}{2\pi}\iint_{\mathbf{R}^2}dydz\mathbb{K}_j^{\mu,\nu}(y,z)\langle i\partial_x\rangle^{\mu}\phi(x-y)\langle i\partial_x\rangle^{\nu}\psi(x-z)$$

yields the following result.

Lemma 3.3. *Let $\mu > -1$, $\nu + \sigma_j > -1$, $\mu + \nu + \sigma_j > 0$, $\mu' > -1$, $\nu' + \sigma_j > -1$, $\mu' + \nu' + \sigma_j > 0$, $\sigma_j = 0$ for $j = 1, 2, 3$, and $\sigma_j = 1$ for $j = 4, 5$. Then the following estimates are valid*

$$\|\mathcal{T}_j(\phi,\psi)\|_{\mathbf{L}^p} \le C\|\phi\|_{\mathbf{H}^{\mu}_q}\|\psi\|_{\mathbf{H}^{\nu}_r},$$

$$\|x\mathcal{T}_j(\phi,\psi)\|_{\mathbf{L}^p} \le C\|\phi\|_{\mathbf{H}^{\mu,1}_q}\|\psi\|_{\mathbf{H}^{\nu}_r}$$

and

$$\begin{aligned}\|\mathcal{P}\mathcal{T}_j(\phi,\psi)\|_{\mathbf{L}^p} &\le C\left(\|\mathcal{P}\phi\|_{\mathbf{H}^{\mu}_q} + \|\partial_t\phi\|_{\mathbf{H}^{\mu}_q}\right)\|\psi\|_{\mathbf{H}^{\nu}_r} \\ &\quad + C\|\phi\|_{\mathbf{H}^{\mu'}_{q'}}\left(\|\mathcal{P}\psi\|_{\mathbf{H}^{\nu'}_{r'}} + \|\partial_t\psi\|_{\mathbf{H}^{\nu'}_{r'}}\right),\end{aligned}$$

for $1 \le j \le 5$, where

$$1 \le p \le q, r, q', r' \le \infty, \frac{1}{p} = \frac{1}{q} + \frac{1}{r} = \frac{1}{q'} + \frac{1}{r'},$$

provided that the right-hand sides are finite.

For the proof of Lemma 3.3, see [19].

4. Decomposition of the free Klein-Gordon evolution group

We decompose the free Klein-Gordon evolution group $\mathcal{U}(t) = e^{-i\langle i\partial_x\rangle t} = \mathcal{F}^{-1}E(t)\mathcal{F}$, where $E(t) = e^{-it\langle\xi\rangle}$ similarly to the factorization of the free Schrödinger evolution group. We denote the dilation operator by

$$\mathcal{D}_\omega\phi = \frac{1}{\sqrt{i\omega}}\phi\left(\frac{x}{\omega}\right), \quad (\mathcal{D}_\omega)^{-1} = i\mathcal{D}_{\frac{1}{\omega}}.$$

Define the multiplication factor $M(t) = e^{-it\langle ix\rangle\theta(x)}$, where $\theta(x) = 1$ for $|x| < 1$ and $\theta(x) = 0$ for $|x| \ge 1$. We introduce the operator

$$(\mathcal{B}\phi(\xi))(x) = \frac{\theta(x)}{\langle ix\rangle^{\frac{3}{2}}}\phi\left(\frac{x}{\langle ix\rangle}\right).$$

The inverse operator $\mathcal{B}^{-1}$ acts on the functions $\phi(x)$ defined on $(-1,1)$ as follows

$$\left(\mathcal{B}^{-1}\phi(x)\right)(\xi) = \frac{1}{\langle\xi\rangle^{\frac{3}{2}}}\phi\left(\frac{\xi}{\langle\xi\rangle}\right)$$

for all $\xi \in \mathbf{R}$. This follows by setting $\xi = \frac{x}{\langle ix\rangle} \in \mathbf{R}$ and deducing $x = \frac{\xi}{\langle\xi\rangle} \in (-1,1)$. We now introduce the operators

$$\mathcal{V}(t) = \mathcal{B}^{-1}\overline{M}(t)\mathcal{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$$

and

$$\mathcal{W}(t) = (1-\theta)\mathcal{D}_t^{-1}\mathcal{F}^{-1}e^{-it\langle\xi\rangle}$$

so that we have the representation for the free Klein-Gordon evolution group

$$\begin{aligned}\mathcal{U}(t)\mathcal{F}^{-1} &= e^{-it\langle i\partial_x\rangle}\mathcal{F}^{-1} = \mathcal{F}^{-1}e^{-it\langle\xi\rangle} = \mathcal{D}_tM(t)(\mathcal{B}\mathcal{V}(t) + \mathcal{W}(t)) \\ &= \mathcal{D}_tM(t)\mathcal{B} + \mathcal{D}_tM(t)\mathcal{B}(\mathcal{V}(t) - 1) + \mathcal{D}_tM(t)\mathcal{W}(t).\end{aligned} \tag{4.1}$$

The first term $\mathcal{D}_tM(t)\mathcal{B}\phi$ of the right-hand side of (4.1) describes the well-known leading term of the large time asymptotics of solutions of the linear Klein-Gordon equation $\mathcal{L}u = 0$ with initial data ϕ and is in the inside of the light cone. The second term of the right-hand side of (4.1) is considered as a remainder term which is in the inside of the light cone, whereas the last term represents the large time asymptotics in the outside of the light cone which decays more rapidly in time. We also have

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t) &= \mathcal{F}e^{it\langle i\partial_x\rangle} = e^{it\langle\xi\rangle}\mathcal{F} = \mathcal{V}^{-1}(t)\,\mathcal{B}^{-1}\overline{M}(t)\,\mathcal{D}_t^{-1} + \mathcal{W}^{-1}(t)\,\mathcal{D}_t^{-1}\\ &= \mathcal{B}^{-1}\overline{M}(t)\,\mathcal{D}_t^{-1} + \left(\mathcal{V}^{-1}(t) - 1\right)\mathcal{B}^{-1}\overline{M}(t)\,\mathcal{D}_t^{-1} + \mathcal{W}^{-1}(t)\,\mathcal{D}_t^{-1},\end{aligned}$$

where the right-inverse operators

$$\mathcal{V}^{-1}(t) = \overline{E}(t)\,\mathcal{F}\mathcal{D}_t M(t)\,\mathcal{B}$$

and

$$\mathcal{W}^{-1}(t) = \overline{E}(t)\,\mathcal{F}\mathcal{D}_t(1-\theta),$$

where $E(t) = e^{-it\langle\xi\rangle}$.

5. Outline of Proof of Theorem 2.1

Application of the operator $\mathcal{F}\mathcal{U}(-t)$ to equation (2.5), factorization of free Klein-Gordon operator and the estimates of bilinear operators by Lemma 3.3 yield the ordinary differential equation

$$\partial_t\mathcal{F}\mathcal{U}(-t)\,u = it^{-1}\Omega(\xi)\left(\mathcal{F}\mathcal{U}(-t)\,u\right)\left|\mathcal{F}\mathcal{U}(-t)\,u\right|^2 + R$$

for all $t \geq 1$ uniformly with respect to $\xi \in \mathbf{R}$, where R is a remainder term. By changing the dependent variable

$$\psi(t) = \left(\mathcal{F}\mathcal{U}(-t)\,u\right)e^{-i\Omega\int_1^t|\mathcal{F}\mathcal{U}(-\tau)u(\tau)|^2\frac{d\tau}{\tau}}$$

to get the desired a-priori estimate $\|\mathcal{F}\mathcal{U}(-t)\,u\|_{\mathbf{L}^\infty} \leq C$. Main problem is to prove R is the remainder term in the function space

$$\mathbf{X}_T = \left\{u \in \mathbf{C}\left([0,T];\mathbf{L}^2\right); \|u\|_{\mathbf{X}_T} < \infty\right\},$$

where

$$\begin{aligned}&\|u\|_{\mathbf{X}_T}\\ = &\sup_{t\in[0,T]}\left(\langle t\rangle^{-\gamma}\|u(t)\|_{\mathbf{H}^2} + \langle t\rangle^{-3\gamma}\left\|\langle i\nabla\rangle\mathcal{U}(t)\,x\mathcal{U}(-t)\,w(t)\right\|_{\mathbf{H}^1} + \langle t\rangle^{\frac{1}{2}}\|u(t)\|_{\mathbf{L}^\infty}\right),\end{aligned}$$

and $\gamma > 0$ is small. We have used the operator $\mathcal{J}$ since $\mathcal{J} = \langle i\nabla\rangle\mathcal{U}(t)\,x\mathcal{U}(-t)$. For the details of the proof, see [19].

6. *A system of quadratic nonlinear Klein-Gordon equations in 2d*

We consider a system of equations (1.2). In the same way as in the derivation of (2.1), by changing the dependent variables $u_j = \frac{1}{2}\left(v_j + i\langle i\nabla\rangle_{m_j}^{-1}\partial_t v_j\right)$, we find that u_1 and u_2 satisfy the following system of equations

$$\left\{\begin{aligned}\mathcal{L}_{m_1}u_1 &= 2i\langle i\nabla\rangle_{m_1}^{-1}(\mathrm{Re}u_1)(\mathrm{Re}u_2),\\ \mathcal{L}_{m_2}u_2 &= 2i\langle i\nabla\rangle_{m_2}^{-1}(\mathrm{Re}u_1)^2,\end{aligned}\right. \tag{6.1}$$

where $\mathcal{L}_m = \partial_t + i\langle i\nabla\rangle_m$, $\langle i\nabla\rangle_m = \sqrt{m^2-\Delta}$. To state our results in [20] we introduce the function space

$$\mathbf{X}_\infty = \left\{\phi = (\phi_1,\phi_2) \in \mathbf{C}\left([0,\infty);\mathbf{L}^2\right); \|\phi\|_{\mathbf{X}_\infty} < \infty\right\},$$

where the norm

$$\|\phi\|_{\mathbf{X}_\infty} = \sum_{j=1}^{2} \sup_{t\in[0,\infty)} \|\phi_j(t)\|_{\mathbf{Y}}$$

and the norm $\mathbf{Y}$ is defined as follows

$$\|\phi(t)\|_{\mathbf{Y}} = \sup_{2\le q\le \frac{4}{2-\sigma}} \langle t\rangle^{1-\frac{2}{q}} \|\phi(t)\|_{\mathbf{H}_q^{\mu-2\left(1-\frac{2}{q}\right)}}$$

with $1<\sigma<2$, $\mu\ge\sigma$. We also denote

$$\mathbf{X}_{\infty,\rho} = \left\{\phi\in\mathbf{C}\left([0,\infty);\mathbf{L}^2\right); \|\phi\|_{\mathbf{X}_\infty}\le\rho\right\}.$$

Our first result in [20] is the existence of solutions to the Cauchy problem (6.1) with the initial data $u_0=(u_{1,0},u_{2,0})$.

Theorem 6.1. *Let $m_2<2m_1$. Assume that initial data $u_0\in\mathbf{H}^{\mu}_{\frac{4}{2+\sigma}}\cap\mathbf{H}^{\mu}$, $\mu\ge\sigma$, $\sigma\in\left(1,\frac{8}{7}\right]$, with a norm $\|u_0\|_{\mathbf{H}^{\mu}_{\frac{4}{2+\sigma}}\cap\mathbf{H}^{\mu}}\le\varepsilon$. Then there exists $\varepsilon>0$ such that the Cauchy problem (6.1) with the initial data u_0 has a unique global solution $u=(u_1,u_2)\in\mathbf{C}([0,\infty);\mathbf{H}^{\mu})$ satisfying the estimate $\|u\|_{\mathbf{X}_\infty}\le C\varepsilon^{\frac{2}{3}}$. Furthermore for any small $u_0\in\mathbf{H}^{\mu}_{\frac{4}{2+\sigma}}\cap\mathbf{H}^{\mu}$ there exists a unique scattering state $u_+=(u_{1,+},u_{2,+})\in\mathbf{H}^{\mu}$ such that*

$$\lim_{t\to\infty}\sum_{j=1}^{2}\left\|u_j(t)-e^{-it\langle i\nabla\rangle_{m_j}}u_{j,+}\right\|_{\mathbf{H}^{\mu}}=0.$$

We next consider the final state problem. We suppose a final value $u_+\in\mathbf{H}^{\mu}_{\frac{4}{2+\sigma}}\cap\mathbf{H}^{\mu}$ and solve equation (6.1) in the functional space $\mathbf{X}_\infty$ under the final state condition

$$\sum_{j=1}^{2}\left\|u_j(t)-e^{-it\langle i\nabla\rangle_{m_j}}u_{j,+}\right\|_{\mathbf{L}^2}\to 0 \tag{6.2}$$

as $t\to\infty$. The last estimate means that we look for solutions of (6.1) in the neighborhood of a free solution in $\mathbf{L}^2$ - sense.

Theorem 6.2. *Let $m_2<2m_1$. Assume that the final value $u_+=(u_{1,+},u_{2,+})\in\mathbf{H}^{\mu}_{\frac{4}{2+\sigma}}\cap\mathbf{H}^{\mu}$, $\mu\ge\sigma$, $\sigma\in\left(1,\frac{8}{7}\right]$, with a norm $\|u_0\|_{\mathbf{H}^{\mu}_{\frac{4}{2+\sigma}}\cap\mathbf{H}^{\mu}}\le\varepsilon$. Then there exists $\varepsilon>0$ such that equation (6.1) has a unique global solution $u=(u_1,u_2)\in\mathbf{C}([0,\infty);\mathbf{H}^{\mu})$ satisfying the estimate $\|u\|_{\mathbf{X}_\infty}\le C\varepsilon^{\frac{2}{3}}$ and condition (6.2).*

We next state the existence of the scattering operators. We introduce the function space $\widetilde{\mathbf{X}}_\infty=\left\{\phi=(\phi_1,\phi_2)\in\mathbf{C}\left([0,\infty);\mathbf{L}^2\right);\|\phi\|_{\mathbf{X}_\infty}<\infty\right\}$, where the norm

$$\begin{aligned}\|\phi\|_{\mathbf{X}_\infty} &= \sum_{j=1}^{2}\sup_{t\in[0,\infty)}\Big(\|\phi_j(t)\|_{\mathbf{H}^{\sigma}}+\|\mathcal{J}_{m_j}\phi_j(t)\|_{\mathbf{H}^{\sigma-1}}\\ &\quad+\|\partial_t\phi_j(t)\|_{\mathbf{H}^{\sigma-1}}+\|\mathcal{P}\phi_j(t)\|_{\mathbf{H}^{\sigma-1}}\Big).\end{aligned}$$

We also define

$$\widetilde{\mathbf{X}}_{\infty,\rho}=\left\{\phi\in\mathbf{C}\left([0,\infty);\mathbf{L}^2\right);\|\phi\|_{\mathbf{X}_\infty}\le\rho\right\},$$

where the operator

$$\mathcal{J}_m = \langle i\nabla\rangle_m e^{-i\langle i\nabla\rangle_m t} x e^{i\langle i\nabla\rangle_m t} = \langle i\nabla\rangle_m x + it\nabla$$

is analogous to the operator $x + it\nabla = e^{-\frac{it}{2}\Delta} x e^{\frac{it}{2}\Delta}$ in the case of the nonlinear Schrödinger equation (see [10]) and commutes with $\mathcal{L}_m$: $[\mathcal{L}_m, \mathcal{J}_m] = \mathcal{L}_m \mathcal{J}_m - \mathcal{J}_m \mathcal{L}_m = 0$. However $\mathcal{J}_m$ is not a purely differential operator, so it is apparently difficult to calculate its action on the nonlinearities. We also use the first order differential operator $\mathcal{P} = t\nabla + x\partial_t$ which is closely related to $\mathcal{J}_m$ by the identity $\mathcal{P} = \mathcal{L}_m x - i\mathcal{J}_m$. It acts easily on the nonlinearities and almost commutes with $\mathcal{L}_m$: $[\mathcal{L}_m, \mathcal{P}] = -i\langle i\nabla\rangle^{-1} \nabla\mathcal{L}_m$, where we applied the commutator $\left[x, \langle i\nabla\rangle_m^{\beta}\right] = \beta\langle i\nabla\rangle_m^{\beta-2}\nabla$.

Theorem 6.3. *Let $m_2 < 2m_1$. Assume that initial data $u_0 \in \mathbf{H}^{\sigma,1}$, $\sigma \in \left(1, \frac{8}{7}\right]$, with a norm $\|u_0\|_{\mathbf{H}^{\sigma,1}} \leq \varepsilon$. Then there exists $\varepsilon > 0$ such that the Cauchy problem (6.1) with the initial data u_0 has a unique global solution $u = (u_1, u_2) \in \mathbf{C}\left([0,\infty); \mathbf{H}^{\sigma,1}\right)$ satisfying the estimate $\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon^{\frac{2}{3}}$. Furthermore for any small $u_0 \in \mathbf{H}^{\sigma,1}$, there exists a unique scattering state $u_+ = (u_{1,+}, u_{2,+}) \in \mathbf{H}^{\sigma,1}$ such that*

$$\sum_{j=1}^{2} \left\| e^{it\langle i\nabla\rangle_{m_j}} u_j(t) - u_{j,+} \right\|_{\mathbf{H}^{\sigma,1}} \to 0$$

as $t \to \infty$.

Finally we consider the final state problem for equation (6.1) in the functional space $\widetilde{\mathbf{X}}_\infty$ under the final state condition (6.2) with a final value $u_+ \in \mathbf{H}^{\sigma,1}$.

Theorem 6.4. *Let $m_2 < 2m_1$. Assume that the final value $u_+ \in \mathbf{H}^{\sigma,1}, \sigma \in \left(1, \frac{8}{7}\right]$, with a norm $\|u_+\|_{\mathbf{H}^{\sigma,1}} \leq \varepsilon$. Then there exists $\varepsilon > 0$ such that equation (6.1) has a unique global solution $u = (u_1, u_2) \in \mathbf{C}\left([0,\infty); \mathbf{H}^{\sigma,1}\right)$ satisfying the estimate $\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon^{\frac{2}{3}}$ and condition (6.2).*

Remark 6.1. *We denote by*

$$\mathbf{H}_{p,\rho}^{k,s} = \left\{ \phi = (\phi_1, \phi_2) \in \mathbf{H}_p^{k,s}; \|\phi\|_{\mathbf{H}_p^{k,s}} = \sum_{j=1}^{2} \|\phi_j\|_{\mathbf{H}_p^{k,s}} \leq \rho \right\}.$$

By Theorem 6.4, there exists the wave operator

$$W_+ : u_+ \in \mathbf{H}_{2,\varepsilon}^{\sigma,1} \to u(0) \in \mathbf{H}_{2,\varepsilon^{\frac{2}{3}}}^{\sigma,1}. \tag{6.3}$$

Theorem 6.3 is valid for the negative time and we find that there exists a unique global solution $u = (u_1, u_2) \in \mathbf{C}\left((-\infty, 0]; \mathbf{H}^{\sigma,1}\right)$ of (6.1) with the initial data $u(0) \in \mathbf{H}_{2,\varepsilon^{\frac{2}{3}}}^{\sigma,1}$. Furthermore for any small $u(0) \in \mathbf{H}_{2,\varepsilon^{\frac{2}{3}}}^{\sigma,1}$, there exists the unique scattering state $u_- = (u_{1,-}, u_{2,-}) \in \mathbf{H}_{2,\varepsilon^{\frac{4}{9}}}^{\sigma,1}$. Thus we have the inverse wave operator

$$W_-^{-1} : u(0) \in \mathbf{H}_{2,\varepsilon^{\frac{2}{3}}}^{\sigma,1} \to u_- \in \mathbf{H}_{2,\varepsilon^{\frac{4}{9}}}^{\sigma,1}. \tag{6.4}$$

Then by (6.3) and (6.4) we can define the scattering operator

$$S = W_-^{-1} W_+ : u_+ \in \mathbf{H}_{2,\varepsilon}^{\sigma,1} \to u_- \in \mathbf{H}_{2,\varepsilon^{\frac{4}{9}}}^{\sigma,1}.$$

The inverse scattering operator S^{-1} is also defined from $\mathbf{H}_{2,\varepsilon^{\frac{4}{9}}}^{\sigma,1}$ to $\mathbf{H}_{2,\varepsilon^{\frac{16}{81}}}^{\sigma,1}$.

Under the mass condition $2m_1 > m_2$ in Proposition 7.1 below we state the estimates of the bilinear operators associated with the system (6.1) and apply them to derive a-priori estimates of solutions to (6.1) as in the previous works [16], [17], [18] where a single equation was treated. Our conditions are more natural ones on the data comparing with the previous papers (see, [26], [29], [30] for a single equation and [32] for a system of equations, where the higher order derivatives for the data were assumed). Our result on the existence of the scattering operator in $\mathbf{H}^{1+\delta,1}$, $\delta > 0$, is new even for a single equation, see [16], [17], [18]. We note that global existence in time of solutions for (1.1) was obtained in an almost energy $\mathbf{H}^{1+\delta}$ class recently for $n = 2$ in [7], where $\delta > 0$.

It was shown in [32] that a small solution (v_1, v_2) to the Cauchy problem for system (1.2) exists globally and is asymptotically free under the non resonance mass condition $2m_1 \neq m_2$ and rather strong hypotheses on the initial data. See [14] in which the final value problem of (1.1) was considered with the final data which are in $\mathbf{H}^{4,1} \cap \mathbf{H}^{3,1}$ by using the method of algebraic normal forms by Sunagawa [32]. This method works well for (1.2) under the mass condition $2m_1 \neq m_2$. Global existence and time decay of small solutions were obtained in [24] for the resonance case $2m_1 = m_2$, under some regularity and compactness conditions on the initial data, see also [23] for another resonance case, whereas the large time asymptotic profile is not well known for the case of $2m_1 = m_2$ for (1.2).

Under the mass condition $2m_1 > m_2$, we give a positive answer to the scattering problem in an almost natural weighted Sobolev space $\mathbf{H}^{1+\delta,1}$ with $\delta > 0$. Another point of our theorems is to say existence of the scattering states and wave operators in the lower order Sobolev space $\mathbf{H}^{1+\delta}_{\frac{4}{3}-\delta} \cap \mathbf{H}^{1+\delta}$. It seems that the method of algebraic normal forms by [32] does not work well for the construction of the scattering operator even if we consider the problem in higher order Sobolev spaces. On the other hand, the method of algebraic normal forms works well for a proof of global existence of solutions in the case of $2m_1 < m_2$. However our proof depends on Proposition 7.1, and so does not work for this case. Thus the existence of the scattering operator is an open problem for the case of $2m_1 \leq m_2$.

Since the Klein-Gordon equation is a relativistic version of the Schrödinger equation, it is interesting to compare our results with those concerning the system of nonlinear Schrödinger equations in two space dimensions

$$\left\{ \begin{array}{l} i\partial_t u_1 + \frac{1}{2m_1}\Delta u_1 = \overline{u_1} u_2, \\ i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = u_1^2. \end{array} \right. \tag{6.5}$$

In [21], time decay of small solutions of the Cauchy problem (6.5) and the non existence of the usual scattering states were studied under the resonance mass condition $2m_1 = m_2$. However the existence of the modified scattering states is not known. For the non resonance case $2m_1 \neq m_2$ and $m_1 \neq m_2$, there are no results for the global existence of solutions to the Cauchy problem for the system of nonlinear Schrödinger equations (6.5). On the other hand, the final value problem was studied in [22] and the wave operators were constructed as follows. Define the homogeneous Sobolev semi-norm by

$$\|f\|_{\dot{\mathbf{H}}^m} = \left\| (-\Delta)^{\frac{m}{2}} f \right\|_{\mathbf{L}^2}.$$

Proposition 6.5. *Let $2m_1 \neq m_2$, $m_1 \neq m_2$. Assume that $\phi_{1+} \in \mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{-2b}$, $\phi_{2+} \in \mathbf{H}^{0,2}$. Then there exists $\varepsilon > 0$ such that for any (ϕ_{1+}, ϕ_{2+}) with the norm*

$\|\phi_{1+}\|_{\mathbf{H}^{0,2}\cap\dot{\mathbf{H}}^{-2b}} + \|\phi_{2+}\|_{\mathbf{H}^{0,2}} \leq \varepsilon$, *the system (6.5) has a unique global solution* $u = (u_1, u_2) \in \mathbf{C}([1,\infty);\mathbf{L}^2)$, *such that the following estimate*

$$\sum_{j=1}^{2} \left\| u_j(t) - e^{i\frac{t}{2m_j}\Delta}\phi_{j+} \right\|_{\mathbf{L}^2} \leq Ct^{-b} \tag{6.6}$$

holds for all $t \geq 1$, *where* $\frac{1}{2} < b < 1$.

Proposition 6.6. *Let* $m_1 = m_2$. *Assume that* $\phi_{1+} \in \mathbf{H}^{0,2}\cap\dot{\mathbf{H}}^{-2b}$, $\phi_{2+} \in \mathbf{H}^{\frac{1}{2},2}$ *and the intersection of support of* $\widehat{\phi_{1+}}$ *and support of* $\widehat{\phi_{2+}}$ *is empty. Then there exists* $\varepsilon > 0$ *such that for any* (ϕ_{1+}, ϕ_{2+}) *with the norm* $\|\phi_{1+}\|_{\mathbf{H}^{0,2}\cap\dot{\mathbf{H}}^{-2b}} + \|\phi_{2+}\|_{\mathbf{H}^{0,2}} \leq \varepsilon$, *the system (6.5) has a unique global solution* $u = (u_1, u_2) \in \mathbf{C}([1,\infty);\mathbf{L}^2)$ *such that (6.6) holds with* $\frac{1}{2} < b < \frac{3}{4}$.

Propositions 6.5 and 6.6 correspond to Theorem 6.4, though the final data conditions and non resonance mass condition are different.

In order to remove the critical nonlinearities, we use the method of the normal forms of Shatah [31] which requires us to estimate the bilinear operators depending on the nonlinearities and the Klein-Gordon evolution group $\mathcal{U}_m(t) = e^{-it\langle i\nabla\rangle_m} = \mathcal{F}^{-1}e^{-it\langle\xi\rangle_m}\mathcal{F}$.

We define the bilinear operators $\mathcal{T}_{a,b,c}$ by

$$\mathcal{T}_{a,b,c}(f,g)(x) = \int_{\mathbf{R}^4} e^{ix\cdot(\xi+\eta)} L_{a,b,c}(\xi,\eta)\,\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi d\eta$$

with $a, b, c \in \mathbf{R}$ and the symbols

$$L_{a,b,c}(\xi,\eta) = \frac{1}{4\pi^2\left(\langle\xi+\eta\rangle_a + \langle\xi\rangle_b \operatorname{sign} b + \langle\eta\rangle_c \operatorname{sign} c\right)},$$

where $\langle x\rangle_a = \sqrt{a^2+|x|^2}$. A direct calculation yields

$$\begin{aligned} &\mathcal{U}_a(-t)\overline{u_b(t)u_c(t)} = \mathcal{U}_a(-t)\left(e^{it\langle i\nabla\rangle_b}\overline{w_b(t)}\right)\left(e^{it\langle i\nabla\rangle_c}\overline{w_c(t)}\right) \\ = \;& \frac{1}{2\pi}\mathcal{F}^{-1}\int_{\mathbf{R}^2} e^{it(\langle\xi\rangle_a+\langle\xi-\eta\rangle_b+\langle\eta\rangle_c)}\widehat{\overline{w_b}}(t,\xi-\eta)\,\widehat{\overline{w_c}}(t,\eta)\,d\eta \end{aligned}$$

with $u_b(t) = \mathcal{U}_b(t)w_b(t)$ and $u_c(t) = \mathcal{U}_c(t)w_c(t)$. Then we obtain the identity

$$\begin{aligned} &\mathcal{U}_a(-t)\overline{u_b(t)u_c(t)} \\ = \;& -i\partial_t\mathcal{U}_a(-t)\mathcal{T}_{a,b,c}(\overline{u_b},\overline{u_c}) \\ & +i\mathcal{U}_a(-t)\left(\mathcal{T}_{a,b,c}\left(\overline{u_b},\overline{\mathcal{L}_c u_c}\right) + \mathcal{T}_{a,b,c}\left(\overline{\mathcal{L}_b u_b},\overline{u_c}\right)\right), \end{aligned} \tag{6.7}$$

where $\mathcal{L}_b = \partial_t + i\langle i\nabla\rangle_b$. Next we get

$$\begin{aligned} &\mathcal{U}_a(-t)u_b(t)u_c(t) = \mathcal{F}^{-1}e^{it\langle\xi\rangle_a}\mathcal{F}\left(e^{-it\langle i\nabla\rangle_b}w_b\right)\left(e^{-it\langle i\nabla\rangle_c}w_c\right) \\ = \;& -i\partial_t\mathcal{U}_a(-t)\mathcal{T}_{a,-b,-c}(u_b,u_c) \\ & +i\mathcal{U}_a(-t)\left(\mathcal{T}_{a,-b,-c}(u_b,\mathcal{L}_c u_c) + \mathcal{T}_{a,-b,-c}(\mathcal{L}_b u_b,u_c)\right). \end{aligned} \tag{6.8}$$

Finally we obtain

$$\begin{aligned}
&\mathcal{U}_a(-t)\,\overline{u_b(t)}u_c(t) = \mathcal{F}^{-1}e^{it\langle\xi\rangle_a}\mathcal{F}\left(e^{it\langle i\nabla\rangle_b}\overline{w_b}\right)\left(e^{-it\langle i\nabla\rangle_c}w_c\right)\\
=\ & -i\partial_t\mathcal{U}_a(-t)\,\mathcal{T}_{a,b,-c}(\overline{u_b},u_c)\\
&+i\mathcal{U}_a(-t)\left(\mathcal{T}_{a,b,-c}\left(\overline{\mathcal{L}_b u_b},u_c\right)+\mathcal{T}_{a,b,-c}(\overline{u_b},\mathcal{L}_c u_c)\right). \qquad (6.9)
\end{aligned}$$

We now apply (6.7)-(6.9) to (6.1) to remove quadratic nonlinearities from the right hand sides of (6.1). Multiplying both sides of (6.1) by $\mathcal{U}_{m_1}(-t)$ and $\mathcal{U}_{m_2}(-t)$, respectively, we obtain

$$\partial_t\mathcal{U}_{m_1}(-t)\,u_1 = \frac{i}{2}\langle i\nabla\rangle_{m_1}^{-1}\mathcal{U}_{m_1}(-t)\,(u_1+\overline{u_1})\,(u_2+\overline{u_2})\,, \qquad (6.10)$$

$$\partial_t\mathcal{U}_{m_2}(-t)\,u_2 = \frac{i}{2}\langle i\nabla\rangle_{m_2}^{-1}\mathcal{U}_{m_2}(-t)\,(u_1+\overline{u_1})^2. \qquad (6.11)$$

By virtue of (6.7)-(6.9) we find

$$\begin{aligned}
&\mathcal{U}_{m_1}(-t)\,(u_1+\overline{u_1})\,(u_2+\overline{u_2})\\
=\ & -i\partial_t\mathcal{U}_{m_1}(-t)\left(\mathcal{T}_{m_1,m_1,m_2}(\overline{u_1},\overline{u_2})+\mathcal{T}_{m_1,-m_1,-m_2}(u_1,u_2)\right.\\
&\left.+\mathcal{T}_{m_1,-m_1,m_2}(u_1,\overline{u_2})+\mathcal{T}_{m_1,m_1,-m_2}(\overline{u_1},u_2)\right)\\
&+i\mathcal{U}_{m_1}(-t)\left(\mathcal{T}_{m_1,m_1,m_2}\left(\overline{u_1},\overline{\mathcal{L}_{m_2}u_2}\right)+\mathcal{T}_{m_1,m_1,m_2}\left(\overline{\mathcal{L}_{m_1}u_1},\overline{u_2}\right)\right.\\
&+\mathcal{T}_{m_1,-m_1,-m_2}(u_1,\mathcal{L}_{m_2}u_2)+\mathcal{T}_{m_1,-m_1,-m_2}(\mathcal{L}_{m_1}u_1,u_2)\\
&+\mathcal{T}_{m_1,-m_1,m_2}\left(u_1,\overline{\mathcal{L}_{m_2}u_2}\right)+\mathcal{T}_{m_1,-m_1,m_2}(\mathcal{L}_{m_1}u_1,\overline{u_2})\\
&\left.+\mathcal{T}_{m_1,m_1,-m_2}(\overline{u_1},\mathcal{L}_{m_2}u_2)+\mathcal{T}_{m_1,m_1,-m_2}\left(\overline{\mathcal{L}_{m_1}u_1},u_2\right)\right) \qquad (6.12)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{U}_{m_2}(-t)\,(u_1+\overline{u_1})^2\\
=\ & -i\partial_t\mathcal{U}_{m_2}(-t)\left(\mathcal{T}_{m_2,m_1,m_1}(\overline{u_1},\overline{u_1})+\mathcal{T}_{m_2,-m_1,-m_1}(u_1,u_1)\right.\\
&\left.+\mathcal{T}_{m_2,-m_1,m_1}(u_1,\overline{u_1})+\mathcal{T}_{m_2,m_1,-m_1}(\overline{u}_1,u_1)\right)\\
&+i\mathcal{U}_{m_2}(-t)\left(\mathcal{T}_{m_2,m_1,m_1}\left(\overline{u_1},\overline{\mathcal{L}_{m_1}u_1}\right)+\mathcal{T}_{m_2,m_1,m_1}\left(\overline{\mathcal{L}_{m_1}u_1},\overline{u_1}\right)\right.\\
&+\mathcal{T}_{m_2,-m_1,-m_1}(u_1,\mathcal{L}_{m_1}u_1)+\mathcal{T}_{m_2,-m_1,-m_1}(\mathcal{L}_{m_1}u_1,u_1)\\
&+\mathcal{T}_{m_2,-m_1,m_1}\left(u_1,\overline{\mathcal{L}_{m_1}u_1}\right)+\mathcal{T}_{m_2,-m_1,m_1}(\mathcal{L}_{m_1}u_1,\overline{u_1})\\
&\left.+\mathcal{T}_{m_2,m_1,-m_1}(\overline{u_1},\mathcal{L}_{m_1}u_1)+\mathcal{T}_{m_2,m_1,-m_1}\left(\overline{\mathcal{L}_{m_1}u_1},u_1\right)\right). \qquad (6.13)
\end{aligned}$$

We substitute (6.12) and (6.13) into (6.10) and (6.11), respectively to get

$$\begin{aligned}
\mathcal{L}_{m_1}(u_1+Q_1(u_1,u_2)) &= \mathcal{C}_1(u_1,u_2)\,,\\
\mathcal{L}_{m_2}(u_2+Q_2(u_1)) &= \mathcal{C}_2(u_1,u_2)\,, \qquad (6.14)
\end{aligned}$$

where

$$\begin{aligned}
Q_1(u_1,u_2) &= -\frac{1}{2}\langle i\nabla\rangle_{m_1}^{-1}\left(\mathcal{T}_{m_1,m_1,m_2}(\overline{u_1},\overline{u_2})+\mathcal{T}_{m_1,-m_1,-m_2}(u_1,u_2)\right.\\
&\quad\left.+\mathcal{T}_{m_1,-m_1,m_2}(u_1,\overline{u_2})+\mathcal{T}_{m_1,m_1,-m_2}(\overline{u_1},u_2)\right),
\end{aligned}$$

$$\begin{aligned}
Q_2(u_1) &= -\frac{1}{2}\langle i\nabla\rangle_{m_2}^{-1}\left(\mathcal{T}_{m_2,m_1,m_1}(\overline{u_1},\overline{u_1})+\mathcal{T}_{m_2,-m_1,-m_1}(u_1,u_1)\right.\\
&\quad\left.+\mathcal{T}_{m_2,-m_1,m_1}(u_1,\overline{u_1})+\mathcal{T}_{m_2,m_1,-m_1}(\overline{u}_1,u_1)\right),
\end{aligned}$$

$$\begin{aligned}\mathcal{C}_1(u_1,u_2) &= -\frac{1}{2}\langle i\nabla\rangle_{m_1}^{-1}\left(\mathcal{T}_{m_1,m_1,m_2}\left(\overline{u_1},\overline{\mathcal{L}_{m_2}u_2}\right)+\mathcal{T}_{m_1,m_1,m_2}\left(\overline{\mathcal{L}_{m_1}u_1},\overline{u_2}\right)\right.\\ &+\mathcal{T}_{m_1,-m_1,-m_2}\left(u_1,\mathcal{L}_{m_2}u_2\right)+\mathcal{T}_{m_1,-m_1,-m_2}\left(\mathcal{L}_{m_1}u_1,u_2\right)\\ &+\mathcal{T}_{m_1,-m_1,m_2}\left(u_1,\overline{\mathcal{L}_{m_2}u_2}\right)+\mathcal{T}_{m_1,-m_1,m_2}\left(\mathcal{L}_{m_1}u_1,\overline{u_2}\right)\\ &\left.+\mathcal{T}_{m_1,m_1,-m_2}\left(\overline{u_1},\mathcal{L}_{m_2}u_2\right)+\mathcal{T}_{m_1,m_1,-m_2}\left(\overline{\mathcal{L}_{m_1}u_1},u_2\right)\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_2(u_1,u_2) &= -\frac{1}{2}\langle i\nabla\rangle_{m_2}^{-1}\left(\mathcal{T}_{m_2,m_1,m_1}\left(\overline{u_1},\overline{\mathcal{L}_{m_1}u_1}\right)+\mathcal{T}_{m_2,m_1,m_1}\left(\overline{\mathcal{L}_{m_1}u_1},\overline{u_1}\right)\right.\\ &+\mathcal{T}_{m_2,-m_1,-m_1}\left(u_1,\mathcal{L}_{m_1}u_1\right)+\mathcal{T}_{m_2,-m_1,-m_1}\left(\mathcal{L}_{m_1}u_1,u_1\right)\\ &+\mathcal{T}_{m_2,-m_1,m_1}\left(u_1,\overline{\mathcal{L}_{m_1}u_1}\right)+\mathcal{T}_{m_2,-m_1,m_1}\left(\mathcal{L}_{m_1}u_1,\overline{u_1}\right)\\ &\left.+\mathcal{T}_{m_2,m_1,-m_1}\left(\overline{u_1},\mathcal{L}_{m_1}u_1\right)+\mathcal{T}_{m_2,m_1,-m_1}\left(\overline{\mathcal{L}_{m_1}u_1},u_1\right)\right).\end{aligned}$$

Note that we substitute equations (6.1) in the definitions of $\mathcal{C}_1(u_1,u_2)$ and $\mathcal{C}_2(u_1,u_2)$, so that Q_j and $\mathcal{C}_j$ are quadratic and cubic nonlinearities, respectively. Thus we can transform the original system to the cubic nonlinear problem (6.14). However the estimates of the bilinear operators $\mathcal{T}_{a,b,c}$ have a small order derivative loss (see Proposition 7.1 below), which does not allow us to apply directly the Hölder inequality, the $\mathbf{L}^p-\mathbf{L}^q$ time decay estimates and the vector fields method. In order to compensate the derivative loss in the bilinear operators $\mathcal{T}_{a,b,c}$, we use the splitting argument as in the previous papers [18], [16], [17]

$$1=\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}-\langle t\rangle^{2\nu-2}\Delta\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2},$$

where the first term has a gain of regularity and the second one has a better time decay. Then we find from (6.1)

$$\begin{aligned}\mathcal{L}_{m_1}u_1 &= \frac{i}{2}\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}\langle i\nabla\rangle_{m_1}^{-1}(u_1+\overline{u}_1)(u_2+\overline{u}_2)\\ &-\frac{i}{2}\langle t\rangle^{2\nu-2}\Delta\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}\langle i\nabla\rangle_{m_1}^{-1}(u_1+\overline{u}_1)(u_2+\overline{u}_2),\end{aligned}\tag{6.15}$$

$$\begin{aligned}\mathcal{L}_{m_2}u_2 &= \frac{i}{2}\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}\langle i\nabla\rangle_{m_2}^{-1}(u_1+\overline{u}_1)^2\\ &-\frac{i}{2}\langle t\rangle^{2\nu-2}\Delta\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}\langle i\nabla\rangle_{m_2}^{-1}(u_1+\overline{u}_1)^2.\end{aligned}\tag{6.16}$$

We apply the method of normal forms to remove the first terms in the right-hand sides of (6.15) and (6.16)

$$\begin{cases}\mathcal{L}_{m_1}\left(u_1+\widetilde{Q}_1(t,u_1,u_2)\right)=\sum_{k=1,2}Q_{2k+1}(t,u_1,u_2)+\widetilde{\mathcal{C}}_1(t,u_1,u_2),\\ \mathcal{L}_{m_2}\left(u_2+\widetilde{Q}_2(t,u_1)\right)=\sum_{k=1,2}Q_{2k+2}(t,u_1)+\widetilde{\mathcal{C}}_2(t,u_1,u_2),\end{cases}\tag{6.17}$$

where

$$\widetilde{Q}_1(t,u_1,u_2)=\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}Q_1(u_1,u_2),\ \widetilde{Q}_2(t,u_1)=\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}Q_2(u_1),$$

$$\begin{aligned}Q_3(t,u_1,u_2) &= -\frac{i}{2}\langle t\rangle^{2\nu-2}\Delta\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}\langle i\nabla\rangle_{m_1}^{-1}(u_1+\overline{u}_1)(u_2+\overline{u}_2),\\ Q_4(t,u_1) &= -\frac{i}{2}\langle t\rangle^{2\nu-2}\Delta\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}\langle i\nabla\rangle_{m_2}^{-1}(u_1+\overline{u}_1)^2,\end{aligned}$$

$$\begin{aligned} Q_5(t,u_1,u_2) &= 2(\nu-1)t\langle t\rangle^{2\nu-4}\Delta\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}\widetilde{Q}_1(t,u_1,u_2),\\ Q_6(t,u_1) &= 2(\nu-1)t\langle t\rangle^{2\nu-4}\Delta\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}\widetilde{Q}_2(t,u_1),\end{aligned}$$

$$\widetilde{C}_1(t,u_1,u_2)=\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}C_1(u_1,u_2),\ \widetilde{C}_2(t,u_1,u_2)=\left\langle i\langle t\rangle^{\nu-1}\nabla\right\rangle^{-2}C_2(u_1,u_2).$$

The first and second terms in the right-hand side of equations (6.17) are the quadratic nonlinearities with an explicit additional time decay, whereas the third terms are cubic nonlocal nonlinearities. System (6.17) is our target equation.

7. *Bilinear operators and their estimates in 2d*

We consider the bilinear operators $T_{a,\pm b,\pm c}$ defined by the multipliers $L_{a,\pm b,\pm c}(\xi,\eta)$. By a simple calculation we find for $a,b,c>0$

$$\begin{aligned} -4\pi^2L_{a,-b,-c}(\xi,\eta) &= \frac{1}{\langle\xi\rangle_b+\langle\eta\rangle_c-\langle\xi+\eta\rangle_a}\\ &= \frac{\langle\xi\rangle_b+\langle\eta\rangle_c+\langle\xi+\eta\rangle_a}{M+2\langle\xi\rangle_b\langle\eta\rangle_c-2(\xi\cdot\eta)}=\frac{g(\xi,\eta)}{h(\xi,\eta)},\end{aligned}$$

where

$$\begin{aligned} g(\xi,\eta) &= (\langle\xi\rangle_b+\langle\eta\rangle_c+\langle\xi+\eta\rangle_a)(M+2\langle\xi\rangle_b\langle\eta\rangle_c+2(\xi\cdot\eta)),\\ h(\xi,\eta) &= (M+2\langle\xi\rangle_b\langle\eta\rangle_c)^2-4(\xi\cdot\eta)^2,\end{aligned}$$

and $M=b^2+c^2-a^2$. We write $\langle x\rangle_a=\sqrt{a^2+|x|^2}$ and also $\langle x\rangle=\sqrt{1+|x|^2}$. It is easy to check the identity

$$\begin{aligned} h(\xi,\eta) &= (M+2bc)^2+4(b\langle\eta\rangle_c-c\langle\xi\rangle_b)^2\\ &+4(M+2bc)(\langle\xi\rangle_b\langle\eta\rangle_c-bc)+4\left(|\xi|^2|\eta|^2-(\xi\cdot\eta)^2\right).\end{aligned}$$

Also we have $|\xi|^2|\eta|^2-(\xi\cdot\eta)^2=(\xi_1\eta_2-\xi_2\eta_1)^2$. Therefore

$$\begin{aligned} h(\xi,\eta) &= (M+2bc)^2+4(b\langle\eta\rangle_c-c\langle\xi\rangle_b)^2\\ &+4(M+2bc)(\langle\xi\rangle_b\langle\eta\rangle_c-bc)+4(\xi_1\eta_2-\xi_2\eta_1)^2.\end{aligned}$$

In the polar coordinates

$$\xi=(|\xi|\cos\phi_\xi,|\xi|\sin\phi_\xi),\eta=(|\eta|\cos\phi_\eta,|\eta|\sin\phi_\eta).$$

we have the identity

$$(\xi_1\eta_2-\xi_2\eta_1)^2=|\xi|^2|\eta|^2-(\xi\cdot\eta)^2=|\xi|^2|\eta|^2\sin^2(\phi_\xi-\phi_\eta).$$

Therefore we also obtain

$$\begin{aligned} h(\xi,\eta) &= (M+2bc)^2+4(b\langle\eta\rangle_c-c\langle\xi\rangle_b)^2\\ &+4(M+2bc)(\langle\xi\rangle_b\langle\eta\rangle_c-bc)+4|\xi|^2|\eta|^2\sin^2(\phi_\xi-\phi_\eta).\end{aligned}$$

We assume that $a,b,c>0$, and $b+c>a$, then we find $M+2bc>0$. Hence

$$h(\xi,\eta)\geq C\langle\xi\rangle^2+C\langle\eta\rangle^2+4|\xi|^2|\eta|^2\sin^2(\phi_\xi-\phi_\eta) \tag{7.1}$$

for all $\xi,\eta\in\mathbf{R}^2$. The condition $b+c>a$ requires us the mass condition.

We can easily see that the asymptotic behavior of the derivatives $\partial_\xi^\alpha \partial_\eta^\beta L_{a,-b,-c}(\xi,\eta)$ is determined by the term $(\xi_1\eta_2 - \xi_2\eta_1)^2$. Hence a multiplier theorem of Coifman-Mayer type [1], [2] can not be expected. However the radial derivatives of $L_{a,-b,-c}$ yield a decay property with respect to $|\xi|$ and $|\eta|$ which enable us to consider the problem in lower order Sobolev norms.

Our key bilinear estimate is given by

Proposition 7.1. *Assume that conditions*

$$a,b,c>0, a+c>b,\ b+c>a,\ a+b>c \tag{7.2}$$

are true. Let $\alpha,\beta_1,\beta_2 \geq 0$ *be such that at least one of the inequalities fulfills* $\alpha>1$, $\beta_1+\beta_2>1$, *or* $\beta_1>1$, $\alpha+\beta_2>1$, *or* $\beta_2>1$, $\alpha+\beta_1>1$. *Then the bilinear operators* $\mathcal{T}_{a,\pm b,\pm c}$, *are bounded from* $\mathbf{H}_{s_1}^{\beta_1}\times\mathbf{H}_{s_2}^{\beta_2}$ *to* $\mathbf{H}_p^{-\alpha}$, *i.e.*

$$\|\mathcal{T}_{a,\pm b,\pm c}(f,g)\|_{\mathbf{H}_p^{-\alpha}} \leq C\|f\|_{\mathbf{H}_{s_1}^{\beta_1}}\|g\|_{\mathbf{H}_{s_2}^{\beta_2}}$$

where $1\leq p\leq s_2\leq\infty$, $\frac{1}{s_1}+\frac{1}{s_2}=1+\frac{1}{p}-\frac{1}{l}$, $1\leq l\leq 2$.

We have by the following time decay estimate for the free evolution group $e^{-it\langle i\nabla\rangle_m}$ from [27].

Lemma 7.2. *The estimate is true*

$$\left\|e^{-it\langle i\nabla\rangle_m}\phi\right\|_{\mathbf{L}^p} \leq Ct^{\frac{2}{p}-1}\left\|\langle i\nabla\rangle^{2-\frac{4}{p}}\phi\right\|_{\mathbf{L}^{\frac{p}{p-1}}}$$

for $p\in[2,\infty]$.

We state a time decay estimate from [13].

Lemma 7.3. *The estimate is valid*

$$\|\phi\|_{\mathbf{L}^q} \leq C\langle t\rangle^{\frac{2}{q}-1}\left(\|\phi\|_{\mathbf{H}^{2-\frac{4}{q}}}+\|\mathcal{J}_m\phi\|_{\mathbf{H}^{1-\frac{4}{q}}}\right)$$

for all $t>0$, *where* $2<q<\infty$, *provided that the right-hand side is finite.*

8. Outline of Proof of Theorem 6.1

We start with the linearized version of (6.17) written as

$$\begin{cases} \mathcal{L}_{m_1}\left(u_1+\widetilde{Q}_1(t,v_1,v_2)\right) = \sum_{k=1,2} Q_{2k+1}(t,v_1,v_2)+\widetilde{C}_1(t,v_1,v_2), \\ \mathcal{L}_{m_2}\left(u_2+\widetilde{Q}_2(t,v_1)\right) = \sum_{k=1,2} Q_{2k+2}(t,v_1)+\widetilde{C}_2(t,v_1,v_2), \end{cases} \tag{8.1}$$

with the initial data $u(0)=u_0$, and a given function $v=(v_1,v_2)$, such that $v(0)=u_0$ and $v\in\mathbf{X}_{\infty,\rho}$, where $\rho=\varepsilon^{\frac{2}{3}}$, $\varepsilon>0$. The Cauchy problem for (8.1) defines the mapping $u=\mathcal{M}(v)$. We have global existence in time of small solutions if we prove

$$\|\mathcal{M}(v)\|_{\mathbf{X}_\infty} \leq C\varepsilon \leq \rho.$$

and

$$\|\mathcal{M}(v)-\mathcal{M}(w)\|_{\mathbf{X}_\infty} \leq \frac{1}{2}\|v-w\|_{\mathbf{X}_\infty}.$$

These estimates are obtained by the estimates of the bilinear operators 7.1 and time decay estimates Lemma 7.2. By the contraction mapping principle, we find that there exists a unique solution u of (6.17) such that $\|u\|_{\mathbf{X}_\infty} \le \rho$. By the integral equation associated with (6.17) we obtain

$$\begin{aligned} & e^{it\langle i\nabla\rangle_{m_1}}\left(u_1(t) + \widetilde{Q}_1(t,u_1,u_2)\right) - e^{is\langle i\nabla\rangle_{m_1}}\left(u_1(s) + \widetilde{Q}_1(s,u_1,u_2)\right) \\ = & \int_s^t e^{i\tau\langle i\nabla\rangle_{m_1}}\left(\sum_{k=1,2} Q_{2k+1}(\tau,u_1,u_2) + \widetilde{C}_1(\tau,u_1,u_2)\right)d\tau \end{aligned}$$

and

$$\begin{aligned} & e^{it\langle i\nabla\rangle_{m_2}}\left(u_2(t) + \widetilde{Q}_2(t,u_1)\right) - e^{is\langle i\nabla\rangle_{m_2}}\left(u_2(s) + \widetilde{Q}_2(s,u_1)\right) \\ = & \int_s^t e^{i\tau\langle i\nabla\rangle_{m_2}}\left(\sum_{k=1,2} Q_{2k+2}(\tau,u_1) + \widetilde{C}_2(\tau,u_1,u_2)\right)d\tau. \end{aligned}$$

Existence of scattering states is obtained by showing

$$\left\|e^{it\langle i\nabla\rangle_{m_1}}u_1(t) - e^{is\langle i\nabla\rangle_{m_1}}u_1(s)\right\|_{\mathbf{H}^\sigma} \le C\langle s\rangle^{-\gamma}$$

for all $t > s > 0$. From this estimate we have a unique scattering state $u_{1,+} \in \mathbf{H}^\sigma$ such that

$$\left\|u_1(t) - e^{-it\langle i\nabla\rangle_{m_1}}u_{1,+}\right\|_{\mathbf{H}^\sigma} \le C\langle t\rangle^{-\gamma}$$

as $t \to \infty$. In the same way, there exists a unique scattering state $u_{2,+} \in \mathbf{H}^\sigma$ such that

$$\left\|u_2(t) - e^{-it\langle i\nabla\rangle_{m_2}}u_{2,+}\right\|_{\mathbf{H}^\sigma} \le C\langle t\rangle^{-\gamma}.$$

This completes the proof of Theorem 6.1.

9. Outline of Proof of Theorem 6.2

Multiplying equations of system (6.17) by $e^{it\langle i\nabla\rangle_{m_1}}$ and $e^{it\langle i\nabla\rangle_{m_2}}$, respectively we write

$$\begin{aligned} \partial_t e^{it\langle i\nabla\rangle_{m_1}}\left(u_1 + \widetilde{Q}_1(t,u_1,u_2)\right) &= e^{it\langle i\nabla\rangle_{m_1}}\left(\sum_{k=1,2} Q_{2k+1}(t,u_1,u_2) + \widetilde{C}_1(t,u_1,u_2)\right), \\ \partial_t e^{it\langle i\nabla\rangle_{m_2}}\left(u_2 + \widetilde{Q}_2(t,u_1)\right) &= e^{it\langle i\nabla\rangle_{m_2}}\left(\sum_{k=1,2} Q_{2k+2}(t,u_1) + \widetilde{C}_2(t,u_1,u_2)\right). \end{aligned}$$

By the condition $u \in \mathbf{X}_\infty$ and Lemma 7.2 we have the estimate

$$\|u\|_{\mathbf{H}^{\sigma'}_{r(\sigma')}} \le C\langle t\rangle^{-\frac{\sigma-\sigma'}{2}}\|u\|_{\mathbf{X}_\infty} \tag{9.1}$$

for $0 \le \sigma' \le \sigma$, where $r(\sigma') = \frac{4}{2-\sigma+\sigma'}$. Then in view of Proposition 7.1 and estimate (9.1) we obtain

$$\left\|\widetilde{Q}_1(t,u_1,u_2)\right\|_{\mathbf{H}^{\sigma'}_{r(\sigma')}} + \left\|\widetilde{Q}_2(t,u_1)\right\|_{\mathbf{H}^{\sigma'}_{r(\sigma')}} \le C\|u\|^2_{\mathbf{X}_\infty}\langle t\rangle^{-\frac{\sigma-\sigma'}{2}-\gamma}.$$

Therefore we have

$$\left\|\widetilde{Q}_1(t,u_1,u_2)\right\|_{\mathbf{Y}} + \left\|\widetilde{Q}_2(t,u_1)\right\|_{\mathbf{Y}} \to 0 \tag{9.2}$$

as $t \to \infty$. Hence the integral equation associated with the final state problem for (6.17) can be written as

$$\begin{aligned} u_1(t) &= -\widetilde{Q}_1(t, u_1, u_2) + e^{-it\langle i\nabla\rangle_{m_1}} u_{1,+} \\ &- \int_t^\infty e^{-i(t-\tau)\langle i\nabla\rangle_{m_1}} \left(\sum_{k=1,2} Q_{2k+1}(\tau, u_1, u\ ) + \widetilde{C}_1(\tau, u_1, u_2) \right) d\tau \end{aligned} \tag{9.3}$$

and

$$\begin{aligned} u_2(t) &= -\widetilde{Q}_2(t, u_1) + e^{-it\langle i\nabla\rangle_{m_2}} u_{2,+} \\ &- \int_t^\infty e^{-i(t-\tau)\langle i\nabla\rangle_{m_2}} \left(\sum_{k=1,2} Q_{2k+2}(\tau, u_1) + \widetilde{C}_2(\tau, u_1, u_2) \right) d\tau. \end{aligned} \tag{9.4}$$

We next assume that $v \in \mathbf{X}_{\infty,\rho}$, $\rho = \varepsilon^{\frac{2}{3}}$ and consider the linearized version of (9.3) and (9.4)

$$\begin{aligned} u_1(t) &= -\widetilde{Q}_1(t, v_1, v_2) + e^{-i\langle i\nabla\rangle_{m_1} t} u_{1,+} \\ &- \int_t^\infty e^{-i\langle i\nabla\rangle_{m_1}(t-\tau)} \left(\sum_{k=1,2} Q_{2k+1}(t, v_1, v\ ) + \widetilde{C}_1(t, v_1, v_2) \right) d\tau \end{aligned} \tag{9.5}$$

and

$$\begin{aligned} u_2(t) &= -\widetilde{Q}_2(t, v_1) + e^{-i\langle i\nabla\rangle_{m_2} t} u_{2,+} \\ &- \int_t^\infty e^{-i\langle i\nabla\rangle_{m_2}(t-\tau)} \left(\sum_{k=1,2} Q_{2k+2}(t, v_1) + \widetilde{C}_2(t, v_1, v_2) \right) d\tau, \end{aligned} \tag{9.6}$$

which defines the mapping $u = \mathcal{M}(v)$. Theorem 6.2 comes from

$$\|\mathcal{M}(v)\|_{\mathbf{X}_\infty} \le C\varepsilon^{\frac{2}{3}}$$

and

$$\|\mathcal{M}(v) - \mathcal{M}(w)\|_{\mathbf{X}_\infty} \le \frac{1}{2} \|v - w\|_{\mathbf{X}_\infty}.$$

10. Outline of Proof of Theorem 6.3

From Proposition 7.1 we obtain

Lemma 10.1. *Assume that condition (7.2) is true. Let* $\alpha \in [0,1]$, $\alpha+\beta_1 > 1, \beta_2 > 1$, *or* $\beta_1 > 1$, $\alpha + \beta_2 > 1$, *and* $\alpha + \beta_3 > 1, \beta_4 > 1$, *or* $\beta_3 > 1$, $\alpha + \beta_4 > 1$, *and* $\frac{1}{s_1} + \frac{1}{s_2} = \frac{3}{2} - \frac{1}{l_1}$, $\frac{1}{s_3} + \frac{1}{s_4} = \frac{3}{2} - \frac{1}{l_2}$, $1 \le l_1, l_2 \le 2$. *Then the following estimates are valid*

$$\|x\mathcal{T}_{a,\pm b,\pm c}(\phi, \psi)\|_{\mathbf{H}^{-\alpha}} \le C \|\phi\|_{\mathbf{H}^{\beta_1,1}_{s_1}} \|\psi\|_{\mathbf{H}^{\beta_2}_{s_2}}$$

and

$$\begin{aligned} \|\mathcal{P}\mathcal{T}_{a,\pm b,\pm c}(\phi, \psi)\|_{\mathbf{H}^{-\alpha}} &\le C \left(\|\mathcal{P}\phi\|_{\mathbf{H}^{\beta_1}_{s_1}} + \|\partial_t\phi\|_{\mathbf{H}^{\beta_1}_{s_1}} \right) \|\psi\|_{\mathbf{H}^{\beta_2}_{s_2}} \\ &+ C \|\phi\|_{\mathbf{H}^{\beta_3}_{s_3}} \left(\|\mathcal{P}\psi\|_{\mathbf{H}^{\beta_4}_{s_4}} + \|\partial_t\psi\|_{\mathbf{H}^{\beta_4}_{s_4}} \right), \end{aligned}$$

provided that the right-hand sides are finite.

10.1. **Outline of Proof of Theorem 6.3.** We apply the contraction mapping principle in the function space $\widetilde{\mathbf{X}}_{\infty}$. As in the proof of Theorem 6.1, we consider the mapping $u = \mathcal{M}v$ defined by (8.1) with $v \in \widetilde{\mathbf{X}}_{\infty,\rho} = \left\{\phi \in \widetilde{\mathbf{X}}_{\infty}; \|\phi\|_{\mathbf{X}_{\infty}} \leq \rho\right\}$, where $\rho = \varepsilon^{\frac{2}{3}}$, $\varepsilon > 0$. Note that u also satisfies the linearized version of system (6.1)

$$\left\{\begin{array}{c} \mathcal{L}_{m_1} u_1 = 2i \langle i\nabla\rangle_{m_1}^{-1} (\mathrm{Re} v_1)(\mathrm{Re} v_2), \\ \mathcal{L}_{m_2} u_2 = 2i \langle i\nabla\rangle_{m_2}^{-1} (\mathrm{Re} v_1)^2. \end{array}\right. \tag{10.1}$$

The desired a priori estimate

$$\|u\|_{\mathbf{X}_{\infty}} \leq C\varepsilon + C\rho^2 \tag{10.2}$$

comes from the estimate

$$\sum_{j=1}^{2} \left\|\mathcal{J}_{m_j} u_j\right\|_{\mathbf{H}^{\nu-1}} \leq C\varepsilon + C\rho^2 \tag{10.3}$$

which requires the estimate

$$\sum_{j=1}^{2} \|\mathcal{P}u_j\|_{\mathbf{H}^{\nu}} \leq C\varepsilon + C\rho^2 \langle t\rangle^{-\gamma}. \tag{10.4}$$

For the proof of (10.4), see [20]. In the same way as in the proof of (10.2), we have

$$\|\mathcal{M}v - \mathcal{M}w\|_{\mathbf{X}_{\infty}} \leq \frac{1}{2}\|v - w\|_{\mathbf{X}_{\infty}}. \tag{10.5}$$

By (10.2) and (10.5) we find that there exists a unique solution of the integral equation associated with (6.17) in $\widetilde{\mathbf{X}}_{\infty}$. Tme decay estimate of Theorem 6.3 follows from (10.3).

10.2. **Existence of scattering states.** We consider the existence of scattering states by the integral equations associated with (6.17). In the same way as in the proof of Theorem 6.1 we have

$$\left\|e^{it\langle i\nabla\rangle_{m_j}} u_j(t) - e^{is\langle i\nabla\rangle_{m_j}} u_j(s)\right\|_{\mathbf{H}^{\nu}} \leq C\rho^2 \langle s\rangle^{-\gamma} \tag{10.6}$$

for $t > s > 0$. By (10.4) we find that

$$\left\|e^{it\langle i\nabla\rangle_{m_j}} (\mathcal{P}u)(t) - e^{is\langle i\nabla\rangle_{m_j}} (\mathcal{P}u)(s)\right\|_{\mathbf{H}^{\sigma-1}} \leq C\rho^2 \langle s\rangle^{-\gamma} \tag{10.7}$$

for $t > s > 0$. We again use the identity $\mathcal{J}_{m_j} = i\mathcal{P} - i\mathcal{L}_{m_j} x$ to obtain

$$\begin{aligned} \mathcal{J}_{m_1} u_1 &= i\mathcal{P}u_1 - i[\mathcal{L}_{m_1}, x] u_1 - ix\mathcal{L}_{m_1} u_1 \\ &= i\mathcal{P}u_1 - i[\mathcal{L}_{m_1}, x] u_1 + 2x\langle i\nabla\rangle_{m_1}^{-1} (\mathrm{Re} u_1)(\mathrm{Re} u_2). \end{aligned} \tag{10.8}$$

Changing $x = \langle i\nabla\rangle_{m_j}^{-1} \mathcal{J}_{m_j} - it\nabla\langle i\nabla\rangle_{m_j}^{-1}$ and applying the estimate $\|\phi\|_{\mathbf{H}^{\nu-1}} \leq C\|\phi\|_{\mathbf{L}^{\frac{2}{2-\sigma}}}$ we find

$$\begin{aligned} &\|xu_1(t)\, u_2(t)\|_{\mathbf{H}^{\nu-1}} \\ \leq\; &\left\|u_2 \langle i\nabla\rangle_{m_1}^{-1} \mathcal{J}_{m_1} u_1\right\|_{\mathbf{H}^{\nu-1}} + Ct\left\|u_2 \nabla\langle i\nabla\rangle_{m_j}^{-1} u_1\right\|_{\mathbf{H}^{\nu-1}} \\ \leq\; &C\|u\|_{\mathbf{L}^{\frac{4}{2-\sigma}}} \left\|\langle i\nabla\rangle_{m_1}^{-1} \mathcal{J}_{m_1} u_1\right\|_{\mathbf{L}^{\frac{4}{2-\sigma}}} + Ct\|u\|^2_{\mathbf{L}^{\frac{4}{2-\sigma}}} \\ \leq\; &C\|u\|_{\mathbf{L}^{\frac{4}{2-\sigma}}} \|\mathcal{J}_{m_1} u_1\|_{\mathbf{H}^{\nu}} + C\rho^2 \langle t\rangle^{1-\sigma} \leq C\rho^2 \langle t\rangle^{-\nu}. \end{aligned}$$

Therefore we get

$$\begin{aligned}&\left\|x\left(e^{i\langle i\nabla\rangle_{m_1}t}u_1(t)-e^{i\langle i\nabla\rangle_{m_1}s}u_1(s)\right)\right\|_{\mathbf{H}^{\sigma}}\\ = &\left\|e^{i\langle i\nabla\rangle_{m_1}t}\mathcal{J}_{m_1}u_1(t)-e^{i\langle i\nabla\rangle_{m_1}s}\mathcal{J}_{m_1}u_1(s)\right\|_{\mathbf{H}^{\nu}}\leq C\rho^2\langle s\rangle^{-\gamma}.\end{aligned}$$

This completes the proof of Theorem 6.3.

11. Outline of Proof of Theorem 6.4

By Lemma 7.3 we get the estimate

$$\|u_j(t)\|_{\mathbf{H}_q^{\sigma'}}\leq C\langle t\rangle^{\frac{2}{q}-1}\left(\|u_j\|_{\mathbf{H}^{\sigma}}+\|\mathcal{J}_{m_j}u_j\|_{\mathbf{H}^{\sigma-1}}\right)\leq C\|u\|_{\mathbf{X}_\infty}\langle t\rangle^{-\frac{\sigma-\sigma'}{2}}$$

for all $t>0$, where $2\leq q\leq\frac{4}{2-\sigma}$, $\sigma'=\sigma-2+\frac{4}{q}$. Therefore as in the proof of Theorem 6.2 we can write the integral equations (9.3) and (9.4) associated with the final state problem for (6.17). Then to apply the contraction mapping principle we assume that $v\in\widetilde{\mathbf{X}}_{\infty,\rho}$, $\rho=\varepsilon^{\frac{2}{3}}$ and define the mapping $u=\mathcal{M}(v)$ via the linearized equations (9.5) and (9.6). In the same way as in the proof of Theorem 6.3 we have

$$\|\mathcal{M}(v)\|_{\mathbf{X}_\infty}\leq C\varepsilon+C\rho^2,$$

$$\|\mathcal{M}(v)-\mathcal{M}(w)\|_{\mathbf{X}_\infty}\leq\frac{1}{2}\|v-w\|_{\mathbf{X}_\infty}$$

and

$$\left\|u_j(t)-e^{-it\langle i\nabla\rangle_{m_j}}u_{j,+}\right\|_{\mathbf{L}^2}\leq C\rho^2t^{-\gamma}$$

for all $t>0$. Thus we show that there exists a unique global solution $u\in\mathbf{C}([0,\infty);\mathbf{H}^{\sigma})$ of (6.1) satisfying the estimate $\|u\|_{\mathbf{X}_\infty}\leq C\varepsilon^{\frac{2}{3}}$ and condition (6.2). This completes the proof of Theorem 6.4.

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