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On the spectral properties of operators describing normal oscillations in 3-dimensional rotating stratified fluid

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Abstract: - We consider mathematical properties of the three-dimensional compressible rotating fluid in a homogeneous gravity field, which may find an application in the study of the Atmosphere and the Ocean. In particular, we investigate the structure and localization of the spectrum of internal oscillations for differential operators generated by such flows. This spectrum may be very useful for studying the stability of the flows, since it is closely related to the non-uniqueness of the limit amplitude of the stabilized flow. Also, it is important in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal oscillations. We consider both inviscid and viscous fluid for various boundary conditions. The novelty of this research is to consider simultaneously the effects of rotation and stratification, which has been studied separately in previous works.

Key-Words: - Partial Differential Equations, Sobolev Spaces, Compressible Fluid, Rotational Fluid, Stratified Fluid, Essential Spectrum, Internal Waves.

1 Preliminaries

Let us consider a bounded domain $\Omega \subset R^3$ with the boundary $\partial\Omega$ of the class C^1 and the following system of fluid dynamics

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \nu\Delta u_1 - \omega u_2 - \nu\beta\dfrac{\partial}{\partial x_1}\left(\operatorname{div}\vec{u}\right) + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} - + \omega u_1 - \nu\Delta u_2 - \nu\beta\dfrac{\partial}{\partial x_2}\left(\operatorname{div}\vec{u}\right) + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} - \nu\Delta u_3 - \nu\beta\dfrac{\partial}{\partial x_3}\left(\operatorname{div}\vec{u}\right) + \rho + \dfrac{\partial p}{\partial x_3} = 0 \\ \dfrac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \alpha^2\dfrac{\partial p}{\partial t} + \operatorname{div}\vec{u} = 0 \qquad x \in \Omega, \quad t \ge 0. \end{cases} \qquad (1)$$

Here $\vec{u} = (u_1, u_2, u_3)$ is a velocity field, $p(x,t)$ is the scalar field of the dynamic pressure and $\rho(x,t)$ is the dynamic density. In this model, the stationary distribution of density is described by the function e^{-Nx_3}, so N is a positive constant. For the compressibility coefficient α, the kinematic viscosity coefficient ν, and the volume (bulk) viscosity coefficient β we assume $\alpha > 0$, $\nu > 0$, $\beta \ge 0$. We also suppose that ω is a positive constant so that system (1) describes linear motions of compressible stratified barotropic viscous fluid which is rotating over the vertical axis with a constant angular velocity $\vec{\omega} = (0,0,\omega)$.

We consider as well the inviscid case of the model decribed by (1):

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \omega u_2 + \dfrac{\partial p}{\partial x_1} = 0 \\ \dfrac{\partial u_2}{\partial t} + \omega u_1 + \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial u_3}{\partial t} + \rho + \dfrac{\partial p}{\partial x_3} = 0 \\ \dfrac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \alpha^2\dfrac{\partial p}{\partial t} + \operatorname{div}\vec{u} = 0 \qquad x \in \Omega, \quad t \ge 0. \end{cases} \qquad (2)$$

For inviscid case, the equations (2) are deduced in [1]-[3]. For viscous compressible fluid, the system (1) is deduced, for example, in [4].

The mathematical properties of rotational inviscid fluid were studied in various works of S. Sobolev, starting from the famous paper [5]. The studying of qualitative properties of solutions of PDE systems modeling rotational compressible flows was started by V. Maslennikova in [6] and was developed later in her future works. We may observe that, despite an extensive study of stratified flows from the physical point of view (see, for example, [7]-[11]), there have been relatively few works considering the mathema-tical aspect of the problem, some results may be found in [12]-[16]. Particularly, for $\nu = 0$ and $\beta = 0$, for the case of compressible fluid ($\alpha > 0$), in [16] it was proved that the essential spectrum of operator of normal vibrations is the interval of the imaginary axis $[-iN, iN]$. For rotational inviscid fluid, the corresponding result was proved in [17], [24]. And, finally, the spectral properties of stratified compressible viscous fluid were studied in [18], [19]. However, the case of rotating stratified (either inviscid or viscious) fluid has not been considered previously. The novelty of this problem, the explicit relationship between the parameters of rotation and stratification in the description of the spectral properties and its possible applications to the dynamics of the Atmosphere and the Ocean was the motivation of this paper.

Let us consider first the system (2) with the boundary condition

$$\vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0, \qquad (3)$$

where $\vec{n}$ is an external normal vector for the boundary $\partial\Omega$. We consider the following problem of normal oscillations

$$\begin{aligned} \vec{u}(x,t) &= \vec{v}(x)e^{-\lambda t} \\ \rho(x,t) &= Nv_4(x)e^{-\lambda t} \\ p(x,t) &= \frac{1}{\alpha}v_5(x)e^{-\lambda t} \quad , \quad \lambda \in C. \end{aligned} \qquad (4)$$

We denote $\tilde{v} = (\vec{v}, v_4, v_5)$ and write (2) as

$$L\tilde{v} = 0 \qquad (5)$$

where $L = M - \lambda I$ and

$$M = \begin{pmatrix} 0 & -\omega & 0 & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_1} \\ \omega & 0 & 0 & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_2} \\ 0 & 0 & 0 & N & \frac{1}{\alpha}\frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{1}{\alpha}\frac{\partial}{\partial x_1} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}. \qquad (6)$$

Let us denote as M_1 the differential operator (6) corresponding to the boundary conditions (3).

We define the domain of the differential operator M_1 as follows.

$D(M_1) = \left\{ \begin{array}{l} \vec{v} \in (L_2(\Omega))^3 \mid \exists f \in L_2(\Omega): \\ (\vec{v}, \nabla\varphi) = (f, \varphi) \ \forall \varphi \in W_2^1(\Omega) \end{array} \right\} \times W_2^1(\Omega) \times W_2^1(\Omega)$, where $W_2^1(\Omega)$ is a Sobolev functional space with the norm

$$\|f\| = \left(\int_\Omega \left[|\nabla f|^2 + f^2 \right] dx \right)^{1/2}. \qquad (7)$$

On the other hand, we will consider the system (1) with the boundary conditions

$$\vec{u}\big|_{\partial\Omega} = 0. \qquad (8)$$

For system (1) we apply the separation of variables (4)-(5), and thus the matrix M will take the form

$$M=\begin{pmatrix} -\nu\Delta-\nu\beta\frac{\partial^2}{\partial x_1^2} & -\nu\beta\frac{\partial^2}{\partial x_1\partial x_2}-\omega & -\nu\beta\frac{\partial^2}{\partial x_1\partial x_3} & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_1} \\ -\nu\beta\frac{\partial^2}{\partial x_1\partial x_2}+\omega & -\nu\Delta-\nu\beta\frac{\partial^2}{\partial x_2^2} & -\nu\beta\frac{\partial^2}{\partial x_2\partial x_3} & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_2} \\ -\nu\beta\frac{\partial^2}{\partial x_1\partial x_3} & -\nu\beta\frac{\partial^2}{\partial x_2\partial x_3} & -\nu\Delta-\nu\beta\frac{\partial^2}{\partial x_3^2} & N & \frac{1}{\alpha}\frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{1}{\alpha}\frac{\partial}{\partial x_1} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}. \quad (9)$$

We denote as M_2 the differential operator (9) associated with the boundary conditions (8). In this way, the domain of operator M_2 can be defined as follows.

$$D(M_2)=\left\{ \begin{matrix} \vec{v}\in\left(\overset{0}{W_2^1}(\Omega)\right)^3, v_4\in L_2(\Omega), v_5\in L_2(\Omega): \\ M\vec{v}\in\left(L_2(\Omega)\right)^5 \end{matrix} \right\},$$

where $\overset{0}{W_2^1}(\Omega)$ is a closure of the functional space $C_0^\infty(\Omega)$ in the norm (7).

From the physical point of view, the separation of variables (4) serves as a tool to establish the possibility to represent every non-stationary process described by (1), (2), as a linear superposition of the normal oscillations. The knowledge of the spectrum of normal vibrations may be very useful for studying the stability of the flows. Also, the spectrum of operators M_1, M_2 is important in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations. Our aim is to study the spectrum of operators M_1, M_2.

2 Statement of the Problem

We observe first that the operators M_1, M_2 are closed operators, and their domains are dense in $\left(L_2(\Omega)\right)^5$.

Let us denote by $\sigma_{ess}(M)$ the essential spectrum of a closed linear operator M. We recall that the essential spectrum

$\sigma_{ess}(M)=\{\lambda\in C:(M-\lambda I)\text{ is not of Fredholm type}\}$, is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity (see [20] ,[21]).

In this way, every spectral point which does not belong to the essential spectrum, is an eigenvalue of finite multiplicity. To find the essential spectrum of the operator M, we will use the following property (see [22]):

$$\sigma_{ess}(M) = Q \cup S,$$

where

$$Q = \begin{Bmatrix} \lambda \in C : (M - \lambda I) \text{is not elliptic} \\ \text{in sense of Douglis-Nirenberg} \end{Bmatrix}$$

and

$$S = \begin{Bmatrix} \lambda \in C \setminus Q : \text{the boundary conditions of } (M - \lambda I) \\ \text{do not satisfy Lopatinski conditions} \end{Bmatrix}.$$

We will use the definition and following properties of ellipticity in sense of Douglis-Nirenberg from [23], and the definition of the Lopatinski conditions, from [22].

We also will use the following criterion which is attributed to Weyl ([20],[21]): a necessary and sufficient condition that a real finite value λ be a point of the essential spectrum of a self-adjoint operator M is that there exist a sequence of elements $v_n \in D(M)$ such that

$$\|v_n\| = 1, \quad v_n \rightharpoonup 0, \quad \|(M - \lambda I)v_n\| \to 0. \qquad (10)$$

We will find the essential spectrum of the operators M_1, M_2. For that, we will use the concepts of Lopatinski conditions and ellipticity in sense of Douglis-Nirenberg. Additionally, for the operator M_1 we will prove the property of skew-selfadjointness and, for all the values of the spectral parameter belonging to the essential spectrum, we will construct an explicit Weyl secuence (10), which is the main result of this work. For the operator M_2 we will localize the sector of the complex plane to which all the eigenvalues belong.

Finally, we will compare the obtained spectral results for stratified rotating fluid with the previous analogous results considering separately the cases of rotation and stratification, either for viscous or for inviscid fluid.

3 The solution of the problem

Theorem 1.

The operator M_1 is skew-selfadjoint.

Proof.

We observe that M_1 can be represented as

$$M_1 = M_0 + B_\omega + B_N, \qquad (11)$$

where

$$B_\omega = \begin{pmatrix} 0 & -\omega & 0 & 0 & 0 \\ \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N & 0 \\ 0 & 0 & -N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since B_ω, B_N are anti-symmetric bounded operators, then it is sufficient to prove the skew-selfadjointness for the operator M_0 with the domain

$$D(M_0) = D(M_1).$$

Let $\tilde{u}, \tilde{v} \in D(M_0)$. Integrating by parts, we obtain

$$(M_0\tilde{u}, \tilde{v}) = -(\tilde{u}, M_0\tilde{v}).$$

Now, let $\tilde{v} \in D(M_0^*)$. Thus, $\tilde{v} \in L_2(\Omega)$ and there exists $\tilde{f} \in L_2(\Omega)$ such that

$$(M_0\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{f}) \text{ for all } \tilde{u} \in D(M_0).$$

Take $\tilde{u} = (0,0,0,0,u_5)$, $u_5 \in W_2^1(\Omega)$. Then, we will have

$$(\nabla u_5, \tilde{v}) = (u_5, f_5).$$

For $\tilde{u} = (u_1, u_2, u_3, 0, 0)$ we obtain

$$(div\ \tilde{u}, v_5) = (\tilde{u}, \tilde{f}).$$

From the last two relations we conclude that v_5 has a weak gradient from $L_2(\Omega)$ and $v_5 \in W_2^1(\Omega)$. Since M_0 is not acting on the fourth component of the vector $\tilde{u}$, we may consider $u_4 = v_4 = f_4 = 0$. In this way, we have verified that

$$D(M_0^*) \subset D(M_0).$$

The reciprocal inclusion can be proved analogously and thus the theorem is proved.

Theorem 2.

Let $a = \min\{\omega, N\}$, $A = \max\{\omega, N\}$. Then, the essential spectrum of M_1 is the following symmetrical set of the imaginary axis:

$$\{0\} \cup [-iA, -ia] \cup [ia, iA].$$

Proof 1.

According to [23], [24], for operator M in (6), we can choose the numbers $s_i = t_j = 0$ for $i, j = 1,2,3,4$ and $s_5 = t_5 = 1$. In this way, the main symbol $\tilde{L}(\xi)$ takes the following form:

$$\tilde{L}(\xi) = \begin{pmatrix} -\lambda & -\omega & 0 & 0 & \frac{1}{\alpha}\xi_1 \\ -\omega & -\lambda & 0 & 0 & \frac{1}{\alpha}\xi_2 \\ 0 & 0 & -\lambda & N-\lambda & \frac{1}{\alpha}\xi_3 \\ 0 & 0 & -N & 0 & 0 \\ \frac{1}{\alpha}\xi_1 & \frac{1}{\alpha}\xi_2 & \frac{1}{\alpha}\xi_3 & 0 & 0 \end{pmatrix}$$

and thus

$$\det \tilde{L}(\xi) = \frac{\lambda}{\alpha^2}\left[(\lambda^2 + N^2)(\xi_1^2 + \xi_2^2) + (\lambda^2 + \omega^2)\xi_3^2\right]. \quad (12)$$

We can see from (12) that if

$$\lambda \notin \left[\{0\} \cup (-iA, -ia) \cup (ia, iA)\right],$$

then the operator L is elliptic in sense of Douglis-Nirenberg. Now, let us prove that the boundary condition (3) satisfies Lopatinski conditions.
If we write the conditions (3) in form

$$G\tilde{u}\big|_{\partial\Omega} = 0,$$

we obtain immediately that

$$G = (n_1, n_2, n_3, 0, 0)$$

and G is a vector row. It can be easily seen that $\hat{L}(\tilde{\xi}, \tau)$ is a matrix whose size is 5x5, and that $G\hat{L}$ is a non-zero row with five components. In other terms, the Lopatinski condition is satisfied, which completes the proof.

Proof 2. (construction of an explicit Weyl sequence)

From theorem 1 we know that the spectrum of the operator M_1 belongs to the imaginary axis. Taking into account (12), we consider $\lambda_0 \in \pm(ia, iA)\setminus\{0\}$ and choose a vector $\xi \neq 0$ such that

$$\left(\lambda_0^2 + N^2\right)\left(\xi_1^2 + \xi_2^2\right) + \left(\lambda_0^2 + \omega^2\right)\xi_3^2 = 0.$$

Therefore, there exist the vector η such that

$$\tilde{L}(\xi)\eta = 0. \qquad (13)$$

Solving (13) with respect to η , we obtain one of possible solutions:

$$\begin{cases} \eta_1 = \dfrac{\lambda_0\xi_1 - \omega\xi_2}{\alpha\left(\lambda_0^2 + \omega^2\right)}, \quad \eta_2 = \dfrac{\lambda_0\xi_2 + \omega\xi_1}{\alpha\left(\lambda_0^2 + \omega^2\right)}, \\ \eta_3 = \dfrac{\lambda_0\xi_3}{\lambda_0^2 + N^2}, \quad \eta_4 = \dfrac{-N\xi_3}{\alpha\left(\lambda_0^2 + N^2\right)}, \quad \eta_5 = 1. \end{cases}$$

We observe that $\eta_i \neq 0$, $i = 1,2,3,4,5$. Now, let us choose a function $\psi_0 \in C_0^\infty(\Omega)$, $\int\limits_{\|x\|\le 1} \psi_0^2(x)dx = 1$.

We fix $x_0 \in \Omega$ and define

$$\psi_k(x) = k^{\frac{3}{2}}\psi_0\left(k\left(x - x_0\right)\right), \; k = 1,2,\ldots$$

We define the Weyl sequence $\tilde{v}^k$ as follows:

$$\begin{cases} v_j^k(x) = \eta_j e^{ik^3\langle x,\xi\rangle}\left(\psi_k + \dfrac{i}{k^3\xi_j}\dfrac{\partial\psi_k}{\partial x_j}\right), \; j = 1,2,3 \\ v_4^k(x) = \eta_4\psi_k e^{ik^3\langle x,\xi\rangle} \\ v_5^k(x) = -\dfrac{i}{k^3}\psi_k e^{ik^3\langle x,\xi\rangle} \\ \langle x,\xi\rangle = x_1\xi_1 + x_2\xi_2 + x_3\xi_3, \qquad k = 1,2,\ldots \end{cases} \qquad (14)$$

It can be easily seen that the sequence (14) satisfies the conditions (10) and thus the Theorem is proved.

We note that the limit points $\pm ia, \pm iA$ belong to the essential spectrum due to the fact that an essential spectrum is a closed set. We would like to observe as well that the sequence (14), being an explicit solution of the system (5) for λ belonging to the essential spectrum, serves as an example of non-uniqueness of the solution, due to the arbitrary election of the function ψ_0.

Theorem 3

The essential spectrum of the operator M_2 is composed of three real isolated points

$$\sigma_{ess}\left(M^2\right)=\left\{0,\frac{1}{\nu\alpha^2(\beta+1)},\frac{1}{\nu\alpha^2(\beta+2)}\right\}.$$

Proof.

We observe that, due to [23], [24], we can choose

$$s_1=s_2=s_3=0,\ s_4=s_5=-1,$$
$$t_1=t_2=t_3=2,\ t_4=t_5=1,$$

so that the main symbol of the operator (5)-(9) $L=M_2-\lambda I$, will be expressed as:

$$\tilde{L}(\xi)=\begin{pmatrix} -\nu|\xi|^2-\nu\beta\xi_1^2 & -\nu\beta\xi_1\xi_2 & -\nu\beta\xi_1\xi_3 & 0 & \frac{1}{\alpha}\xi_1 \\ -\nu\beta\xi_1\xi_2 & -\nu|\xi|^2-\nu\beta\xi_2^2 & -\nu\beta\xi_2\xi_3 & 0 & \frac{1}{\alpha}\xi_2 \\ -\nu\beta\xi_1\xi_3 & -\nu\beta\xi_2\xi_3 & -\nu|\xi|^2-\nu\beta\xi_3^2 & 0 & \frac{1}{\alpha}\xi_3 \\ 0 & 0 & 0 & -\lambda & 0 \\ \frac{1}{\alpha}\xi_1 & \frac{1}{\alpha}\xi_2 & \frac{1}{\alpha}\xi_3 & 0 & -\lambda \end{pmatrix}$$

We calculate the determinant of the last matrix:

$$\det\left(\widetilde{M_2-\lambda I}\right)(\xi)=\frac{\lambda\nu^2}{\alpha^2}|\xi|^6\left(\nu\lambda\alpha^2(\beta+1)-1\right),$$

and thus we can see that for two points

$$\lambda=0 \quad \text{and} \quad \lambda=\frac{1}{\nu\alpha^2(\beta+1)}$$

the operator $L=M_2-\lambda I$ is not elliptic in sense of Douglis-Nirenberg. It is easy to see, additionally, that for the point $\lambda = 1/\nu\alpha^2(\beta+2)$ the condition of Lopatinski is not satisfied, which concludes the proof of the Theorem.

Theorem 4.

Let $A=\max\{\omega,N\}$. Then, the spectrum of operator M_2 is symmetrical with respect to the real axis, and all the eigenvalues of operator M_2 are in the following sector of the complex plane:

$$Z=\left\{\lambda\in C:\ \operatorname{Re}\lambda\geq 0,\ |\operatorname{Im}\lambda|\leq A+\frac{(\operatorname{Re}\lambda)}{\nu\alpha^2\beta A}\right\}.$$

Proof.

Let us denote $v^* = (v_1, v_2, v_3, v_4)$ and take notations for the matrices B_ω , B_N from (11).

Then, the system $(M_2 - \lambda I)\{v^*, v_5\} = 0$ can be written in the form

$$\begin{cases} -\lambda v^* + B_\omega v^* + B_N v^* - \nu\Delta\vec{v} - \nu\beta \operatorname{div}\vec{v} + \dfrac{1}{\alpha}\nabla v_5 = 0 \\ -\lambda v_5 + \dfrac{1}{\alpha}\operatorname{div}\vec{v} = 0 \end{cases} .$$

Now we multiply the last system by $\overline{\{v^*, v_5\}}$ and then integrate by parts in Ω. In this way, we obtain the following equations:

$$-\lambda\|v^*\|^2 + (B_\omega v^*, v^*) + (B_N v^*, v^*) +$$
$$+\nu\sum_{k=1}^{3}\|\nabla v_k\|^2 + \nu\beta\|\operatorname{div}\vec{v}\|^2 - \frac{1}{\alpha}(v_5, \operatorname{div}\vec{v}) = 0$$
$$-\lambda\|v_5\|^2 + \frac{1}{\alpha}(\operatorname{div}\vec{v}, v_5) = 0 .$$

We sum up these two equations and then separate the real and the imaginary parts, keeping in mind the fact that for a skew-symmetric matrix B the expression (Bv^*, v^*) is imaginary.

$$\operatorname{Re}\lambda = \frac{\nu\sum_{k=1}^{3}\|\nabla v_k\|^2 + \nu\beta\|\operatorname{div}\vec{v}\|^2}{\|v^*\|^2 + \|v_5\|^2} \ge 0 ,$$

$$|\operatorname{Im}\lambda| = -i\frac{(B_\omega v^*, v^*) + (B_N v^*, v^*) + \frac{1}{\alpha}\left[(\operatorname{div}\vec{v}, v_5) - (v_5, \operatorname{div}\vec{v})\right]}{\|v^*\|^2 + \|v_5\|^2} ,$$

from which we have $|\operatorname{Im}\lambda| \le A + \dfrac{(\operatorname{Re}\lambda)}{\nu\alpha^2\beta A}$.

It remains to prove now that the spectrum is symmetrical with respect to the real axis. For that purpose, we apply the complex-conjugation to the original system of $M_2 - \lambda I = 0$:

$$\begin{cases} -\bar{\lambda}\bar{v}^* + B_\omega \bar{v}^* + B_N \bar{v}^* - \nu\Delta\bar{\vec{v}} - \nu\beta \operatorname{div}\bar{\vec{v}} + \dfrac{1}{\alpha}\nabla \bar{v}_5 = 0 \\ -\bar{\lambda}\bar{v}_5 + \dfrac{1}{\alpha}\operatorname{div}\bar{\vec{v}} = 0 \end{cases}$$

and thus we can see that, if λ is an eigenvalue of M_2, then $\bar{\lambda}$ is also an eigenvalue of operator M_2, which concludes the proof of the theorem.

4 Conclusions and discussion

For the inviscid case of compressible rotating stratified fluid, as we have seen, the essential spectrum of inner oscillations is the symmetrical bounded set of the imaginary axis

$$\{0\}\cup[-iA,-ia]\cup[ia,iA].$$

Comparing these results with the compressible viscous case, we can conclude that the considered problems and the results of Theorems 2 and 3, are remarkable and interesting due to the special property that, for the viscous fluid, the two points of the essential spectrum

$$\frac{1}{\nu\alpha^2(\beta+1)},\frac{1}{\nu\alpha^2(\beta+2)}$$

move to infinity for $\nu,\beta\to 0$; while the essential spectrum of the inviscid fluid contains an interval of the imaginary axis.

Additionally, we can see, that the results obtained for the inviscid fluid in theorem 2 correspond, in a certain way, to the statement of theorem 3 if we put

$$\operatorname{Re}\lambda=0:\ \left(\operatorname{Re}\lambda=0,\ |\operatorname{Im}\lambda|\le A\right).$$

Finally, we would like to observe that, if we put, for example, $N=0$ in (2), then, according to theorem 2, the essential spectrum will be the interval of the imaginary axis $[-i\omega,i\omega]$, the result which was proved for rotating (non-stratified) compressible fluid in [24].

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