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Characteristic Polynomials of Random Matrices and Noncolliding Diffusion Processes

Makoto Katori
Department of Physics, Chuo University

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Abstract

We consider the noncolliding Brownian motion (BM) with N particles starting from the eigenvalue distribution of Gaussian unitary ensemble (GUE) of $N \times N$ Hermitian random matrices with variance σ^2. We prove that this process is equivalent with the time shift $t \to t + \sigma^2$ of the noncolliding BM starting from the configuration in which all N particles are put at the origin. In order to demonstrate nontriviality of such equivalence for determinantal processes, we show that, even from its special consequence, determinantal expressions are derived for the ensemble averages of products of characteristic polynomials of random matrices in GUE. Another determinantal process, noncolliding squared Bessel process with index $\nu > -1$, is also studied in parallel with the noncolliding BM and corresponding results for characteristic polynomials are given for random matrices in the chiral GUE as well as in the Gaussian ensembles of class C and class D.
Keywords Characteristic polynomials of random matrices, Noncolliding diffusion processes, Determinantal processes, Brownian motions and squared Bessel processes

1 Introduction

We consider N-particle systems of the one-dimensional standard Brownian motions (BMs), $\boldsymbol{X}(t) = (X_1(t), X_2(t), \dots, X_N(t)), t \geq 0$, and of the squared Bessel processes (BESQ) with index $\nu > -1$, $\boldsymbol{X}^{(\nu)}(t) = (X_1^{(\nu)}(t), X_2^{(\nu)}(t), \dots, X_N^{(\nu)}(t)), t \geq 0$, both *conditioned never to collide with each other*, $N \in \mathbb{N} \equiv \{1, 2, 3, \dots\}$. The former process, which is called the *noncolliding BM* [27], solves the following set of stochastic differential equations (SDEs)

$$dX_j(t) = dB_j(t) + \sum_{1 \leq k \leq N, k \neq j} \frac{dt}{X_j(t) - X_k(t)}, \quad 1 \leq j \leq N, \quad t \geq 0, \tag{1.1}$$

with independent one-dimensional standard BMs $\{B_j(t)\}_{j=1}^N$, and the latter process, the *noncolliding BESQ* [30], does the following set of SDEs

$$\begin{aligned} dX_j^{(\nu)}(t) &= 2\sqrt{X_j^{(\nu)}(t)}d\widetilde{B}_j(t) + 2(\nu+1)dt \\ &\quad +4X_j^{(\nu)}(t) \sum_{1 \leq k \leq N, k \neq j} \frac{dt}{X_j^{(\nu)}(t) - X_k^{(\nu)}(t)}, \quad 1 \leq j \leq N, \quad t \geq 0, \end{aligned} \tag{1.2}$$

where $\{\widetilde{B}_j(t)\}_{j=1}^N$ are independent one-dimensional standard BMs different from $\{B_j(t)\}_{j=1}^N$ and, if $-1 < \nu < 0$, the reflection boundary condition is assumed at the origin. (See [16, 10, 35, 32, 42, 48, 26, 49, 46, 37, 44, 43, 38] for related interacting particle systems.) Let $\mathbb{R}$ be the collection of all real numbers and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, and consider subsets of the N-dimensional real space $\mathbb{R}^N$, $\mathbb{W}_N^{\mathrm{A}} = \{\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}$, and $\mathbb{W}_N^+ = \{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}_+^N : x_1 < \cdots < x_N\}$. The former is called the Weyl chambers of types A_{N-1}. We can prove that, provided $\boldsymbol{X}(0) \in \mathbb{W}_N^{\mathrm{A}}$ and $\boldsymbol{X}^{(\nu)}(0) \in \mathbb{W}_N^+$, then the SDEs (1.1) and (1.2) guarantee that with probability one $\boldsymbol{X}(t) \in \mathbb{W}_N^{\mathrm{A}}$, and $\boldsymbol{X}^{(\nu)}(t) \in \mathbb{W}_N^+$ for all $t > 0$. In both processes, at any positive time $t > 0$ there is no *multiple point* at which coincidence of particle positions $X_j(t) = X_k(t)$ or $X_j^{(\nu)}(t) = X_k^{(\nu)}(t)$ for $j \neq k$ occurs. It is the reason why these processes are called noncolliding diffusion processes [31]. We can consider them, however, starting from initial configurations with multiple points. An extreme example is the initial configuration in which all N particles are put at the origin. In order to describe configurations with multiple points we represent each particle configuration by a sum of delta measures in the form $\xi(\cdot) = \sum_{j=1}^N \delta_{x_j}(\cdot)$, where with given $y \in \mathbb{R}$, $\delta_y(\cdot)$ denotes the delta measure such that $\delta_y(x) = 1$ for $x = y$ and $\delta_y(x) = 0$ for $x \neq y$. Note that, by this definition, for $A \subset \mathbb{R}$, $\xi(A) = \int_A \xi(dx) = \sum_{1 \leq j \leq N : x_j \in A} 1 =$ the number of particles included in A. (The above mentioned example is then expressed by $\xi(\cdot) = N\delta_0(\cdot)$, which means that the origin is the multiple point with all N particles.) For a given total number of particles $N \in \mathbb{N}$, we write the configuration spaces as $\mathfrak{M}_N = \{\xi(\cdot) = \sum_{j=1}^N \delta_{x_j}(\cdot) : x_j \in \mathbb{R}, 1 \leq j \leq N\}$ and $\mathfrak{M}_N^+ = \{\xi(\cdot) = \sum_{j=1}^N \delta_{x_j}(\cdot) : x_j \in \mathbb{R}_+, 1 \leq j \leq N\}$. We consider the noncolliding BM and the noncolliding BESQ as $\mathfrak{M}_N$-valued and $\mathfrak{M}_N^+$-valued processes and write them as

$$\Xi(t, \cdot) = \sum_{j=1}^N \delta_{X_j(t)}(\cdot), \quad \Xi^{(\nu)}(t, \cdot) = \sum_{j=1}^N \delta_{X_j^{(\nu)}(t)}(\cdot), \quad t \geq 0,$$

respectively. The probability law of $\Xi(t, \cdot)$ starting from a fixed configuration $\xi \in \mathfrak{M}_N$ is denoted by $\mathbb{P}^\xi$ and that of $\Xi^{(\nu)}(t, \cdot)$ from $\xi \in \mathfrak{M}_N^+$ by $\mathbb{P}_\nu^\xi$, and the noncolliding diffusion processes specified by initial configurations are expressed by $(\Xi(t), t \in [0, \infty), \mathbb{P}^\xi)$ and $(\Xi^{(\nu)}(t), t \in [0, \infty), \mathbb{P}_\nu^\xi), \nu > -1$, respectively. We set $\mathfrak{M}_{N,0} = \{\xi \in \mathfrak{M}_N : \xi(\{x\}) \leq 1 \text{ for any } x \in \mathbb{R}\}$, and $\mathfrak{M}_{N,0}^+ = \{\xi \in \mathfrak{M}_N^+ : \xi(\{x\}) \leq 1 \text{ for any } x \in \mathbb{R}_+\}$, which denote collections of configurations without any multiple points.

In order to dynamically simulate the random matrix ensemble called the Gaussian unitary ensemble (GUE), Dyson considered the $N \times N$ Hermitian matrix-valued BM and showed that its eigenvalue process satisfies the SDEs given by (1.1) [14]. This eigenvalue process is called Dyson's BM model with parameter $\beta = 2$ or simply Dyson's model [47, 29]. The equivalence between Dyson's model and the noncolliding BM, $(\Xi(t), t \in [0, \infty), \mathbb{P}^\xi)$, implies that the random matrix theory [40, 15] is useful to classify and analyze noncolliding diffusion processes [25, 31]. In particular, if the initial configuration is given by $\xi = N\delta_0$, this equivalence concludes that, for any $t > 0$, the particle distribution of $\Xi(t)$ is equal to the eigenvalue distribution of $N \times N$ random matrices in the GUE with variance t. (Note that in the usual GUE the mean is set to be zero.) Here the probability density function (pdf) of the GUE eigenvalues with variance σ^2 is given by

$$\mu_{N,\sigma^2}(\xi) = \frac{\sigma^{-N^2}}{C_N} \exp\left(-\frac{|\boldsymbol{x}|^2}{2\sigma^2}\right) h_N(\boldsymbol{x})^2, \tag{1.3}$$

$\xi = \sum_{j=1}^N \delta_{x_j} \in \mathfrak{M}_N, x_1 \leq x_2 \leq \cdots \leq x_N$, where $C_N = (2\pi)^{N/2} \prod_{j=1}^N \Gamma(j)$ with the gamma function $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du$, $|\boldsymbol{x}|^2 = \sum_{j=1}^N x_j^2$, and

$$h_N(\boldsymbol{x}) = \prod_{1 \leq j < k \leq N} (x_k - x_j). \tag{1.4}$$

The expectation of a measurable function F of a random variable $\Xi \in \mathfrak{M}_N$ with respect to (1.3) is given by

$$\mathbf{E}_{N,\sigma^2}[F(\Xi)] = \int_{\mathbb{W}_N^{\mathrm{A}}} F(\xi)\mu_{N,\sigma^2}(\xi)d\boldsymbol{x} = \frac{1}{N!}\int_{\mathbb{R}^N} F(\xi)\mu_{N,\sigma^2}(\xi)d\boldsymbol{x} \tag{1.5}$$

with setting $\xi = \sum_{j=1}^N \delta_{x_j}, \boldsymbol{x} = (x_1, \dots, x_N)$, where $d\boldsymbol{x} = \prod_{j=1}^N dx_j$.

Assume $\nu \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ and let $M(t)$ be the $(N+\nu) \times N$ complex matrix-valued BM. Then the $N \times N$ matrix-valued process $L(t) = M(t)^\dagger M(t), t \geq 0$, where $M(t)^\dagger$ denotes the Hermitian conjugate of $M(t)$, is called the Laguerre process or the complex Wishart process [11]. The matrix $L(t)$ is Hermitian and positive definite, and König and O'Connell proved that the eigenvalue process of $L(t)$ satisfies the SDEs given by (1.2) [35]. Again by the random matrix theory [40, 15], this equivalence concludes that the particle distribution of $(\Xi^{(\nu)}(t), t \in [0,\infty), \mathbb{P}_\nu^{N\delta_0})$ at any given time $t > 0$ is equal to the distribution of squares of singular values of $(N+\nu) \times N$ complex random matrices in the chiral Gaussian unitary ensemble (chGUE) with index $\nu \in \mathbb{N}_0$ and variance t. Here the pdf of squares of singular values in chGUE with index $\nu \in \mathbb{N}_0$ and variance σ^2 is given by [51, 50]

$$\mu_{N,\sigma^2}^{(\nu)}(\xi) = \frac{\sigma^{-2N(N+\nu)}}{C_N^{(\nu)}} \prod_{j=1}^N (x_j^\nu e^{-x_j/2\sigma^2}) h_N(\boldsymbol{x})^2, \tag{1.6}$$

$\xi = \sum_{j=1}^N \delta_{x_j} \in \mathfrak{M}_N^+, 0 \leq x_1 \leq \cdots \leq x_N$, where $C_N^{(\nu)} = 2^{N(N+\nu)} \prod_{j=1}^N \Gamma(j)\Gamma(j+\nu)$. The expectation of a measurable function F of a random variable $\Xi \in \mathfrak{M}_N^+$ with respect to (1.6) is given by

$$\mathbf{E}_{N,\sigma^2}^{(\nu)}[F(\Xi)] = \int_{\mathbb{W}_N^+} F(\xi)\mu_{N,\sigma^2}^{(\nu)}(\xi)d\boldsymbol{x} = \frac{1}{N!}\int_{\mathbb{R}_+^N} F(\xi)\mu_{N,\sigma^2}^{(\nu)}(\xi)d\boldsymbol{x} \tag{1.7}$$

with setting $\xi = \sum_{j=1}^N \delta_{x_j}, \boldsymbol{x} = (x_1, \dots, x_N)$.

Let $\mathcal{H}(2N)$ be the space of $2N \times 2N$ Hermitian matrices and $\mathfrak{sp}(2N, \mathbb{C})$ and $\mathfrak{so}(2N, \mathbb{C})$ be the spaces of $2N \times 2N$ complex matrices representing the symplectic Lie algebra and the orthogonal Lie algebra, respectively. Altland and Zirnbauer introduced the Gaussian random matrix ensembles for the elements in $\mathcal{H}^{\mathrm{C}}(2N) = \mathfrak{sp} \cap \mathcal{H}(2N)$ and in $\mathcal{H}^{\mathrm{D}}(2N) = \mathfrak{so} \cap \mathcal{H}(2N)$, which are called the Gaussian ensembles of class C and class D, respectively. The eigenvalues of matrices both in the class C and class D ensembles are given by N pairs of positive and negative ones with the same absolute value. The pdfs of the squares of N positive eigenvalues are given by (1.6) with $\nu = 1/2$ for the class C ensemble and with $\nu = -1/2$ for the class D ensemble, when the variances are σ^2 [2, 3]. If we consider the $\mathcal{H}^{\mathrm{C}}(2N)$-valued BM and $\mathcal{H}^{\mathrm{D}}(2N)$-valued BM, the squares of each N positive eigenvalues satisfy the SDEs (1.2) with $\nu = 1/2$ and $\nu = -1/2$, respectively [25]. See also [10, 32, 42, 6, 36].

In the present paper, we consider the noncolliding BM whose initial configuration is distributed according to the pdf (1.3), denoted by $(\Xi(t), t \in [0,\infty), \mathbb{P}^{\mu_{N,\sigma^2}})$, and the noncolliding BESQ starting from the distribution (1.6) with not only $\nu \in \mathbb{N}_0$ but with $\nu > -1$ generally, denoted by $(\Xi^{(\nu)}(t), t \in [0,\infty), \mathbb{P}^{\mu_{N,\sigma^2}^{(\nu)}})$. We prove that for any $N \in \mathbb{N}, \sigma^2 > 0$ the following equalities are established,

$$\begin{aligned}
(\Xi(t), t \in [0,\infty), \mathbb{P}^{\mu_{N,\sigma^2}}) &\stackrel{\text{f.d.}}{=} (\Xi(t+\sigma^2), t \in [0,\infty), \mathbb{P}^{N\delta_0}), \\
(\Xi^{(\nu)}(t), t \in [0,\infty), \mathbb{P}_\nu^{\mu_{N,\sigma^2}^{(\nu)}}) &\stackrel{\text{f.d.}}{=} (\Xi^{(\nu)}(t+\sigma^2), t \in [0,\infty), \mathbb{P}_\nu^{N\delta_0}), \quad \nu > -1,
\end{aligned} \tag{1.8}$$

where $\stackrel{\text{f.d.}}{=}$ denotes the equivalence in finite dimensional distributions (see Theorem 2.1 and Remark 1). We would like to emphasize that these equalities are highly nontrivial and in order to demonstrate it we show in this paper that, even from very special consequence of (1.8), determinantal expressions of ensemble averages of products of characteristic polynomials of random matrices are derived. See [8, 17, 41, 9, 18, 19, 1, 7] for extensive study of characteristic polynomials of random matrices, especially in the connection with the Riemann zeta function [33, 34, 21].

We write the expectations of measurable function F of $N \times N$ random matrices $\{H\}$ in the GUE, of $(N+\nu)\times N$ random matrices $\{M\}$ in the chGUE with $\nu \in \mathbb{N}_0$, of $2N \times 2N$ random matrices $\{H^{\mathrm{C}}\}$ in the class C, and of $2N \times 2N$ random matrices $\{H^{\mathrm{D}}\}$ in the class D as $\langle F(H)\rangle_{\mathrm{GUE}(N,\sigma^2)}$, $\langle F(M)\rangle_{\mathrm{chGUE}(N,\nu,\sigma^2)}$, $\langle F(H^{\mathrm{C}})\rangle_{\mathrm{classC}(2N,\sigma^2)}$, and $\langle F(H^{\mathrm{D}})\rangle_{\mathrm{classD}(2N,\sigma^2)}$, respectively, where σ^2 denote the variances of these four kinds of Gaussian ensembles. Then for $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{C}^m$ the ensemble averages of m-product of characteristic polynomials of random matrices are defined as

$$M_{\mathrm{GUE}}(m, \boldsymbol{\alpha}; N, \sigma^2) \equiv \left\langle \prod_{n=1}^{m} \det(\alpha_n I_N - H) \right\rangle_{\mathrm{GUE}(N,\sigma^2)} = \left\langle \prod_{n=1}^{m}\prod_{j=1}^{N} (\alpha_n - \lambda_j) \right\rangle_{\mathrm{GUE}(N,\sigma^2)}, \tag{1.9}$$

$$\begin{aligned} M^{(\nu)}_{\mathrm{chGUE}}(m, \boldsymbol{\alpha}; N, \sigma^2) &\equiv \left\langle \prod_{n=1}^{m} \det(\alpha_n I_N - M^\dagger M) \right\rangle_{\mathrm{chGUE}(N,\nu,\sigma^2)} \\ &= \left\langle \prod_{n=1}^{m}\prod_{j=1}^{N} (\alpha_n - \kappa_j^2) \right\rangle_{\mathrm{chGUE}(N,\nu,\sigma^2)}, \quad \nu \in \mathbb{N}_0, \end{aligned} \tag{1.10}$$

and for $\sharp = \mathrm{C}$ and D

$$\begin{aligned} &M_{\mathrm{class}\sharp}(m, \boldsymbol{\alpha}; 2N, \sigma^2) \equiv \left\langle \prod_{n=1}^{m} \det(\alpha_n I_{2N} - H^\sharp) \right\rangle_{\mathrm{class}\sharp(2N,\sigma^2)} \\ &= \left\langle \prod_{n=1}^{m}\prod_{j=1}^{N} (\alpha_n - \varepsilon_j)(\alpha_n + \varepsilon_j) \right\rangle_{\mathrm{class}\sharp(2N,\sigma^2)} = \left\langle \prod_{n=1}^{m}\prod_{j=1}^{N} (\alpha_n^2 - \varepsilon_j^2) \right\rangle_{\mathrm{class}\sharp(2N,\sigma^2)}, \end{aligned} \tag{1.11}$$

where I_ℓ denotes the $\ell \times \ell$ unit matrix, $(\lambda_1, \dots, \lambda_N)$ are the eigenvalues of H, $(\kappa_1^2, \dots, \kappa_N^2)$ are the eigenvalues of $M^\dagger M$, $(\varepsilon_1, \dots, \varepsilon_N, -\varepsilon_1, \dots, -\varepsilon_N)$ are the eigenvalues forming "particle-hole pairing" of $H^\sharp$, $\sharp = \mathrm{C}$ or D. For (1.10) remark that each M in chGUE has such a singular value decomposition that $M = U^\dagger K V$, where $U \in \mathrm{U}(N+\nu), V \in \mathrm{U}(N)$,

$$K = \begin{pmatrix} \widehat{K} \\ O \end{pmatrix} \quad \text{with} \quad \widehat{K} = \mathrm{diag}\{\kappa_1, \kappa_2, \cdots, \kappa_N\}, \quad (\kappa_1, \dots, \kappa_N) \in \mathbb{W}_N^+$$

and O is the $\nu \times N$ zero matrix. The diagonal elements $(\kappa_1, \dots, \kappa_N)$ of $\widehat{K}$ are called the singular values of rectangular matrix M. Since $M^\dagger M = V^\dagger K^{\mathrm{T}} K V$, the eigenvalues of $M^\dagger M$ are squares of singular values $(\kappa_1^2, \dots, \kappa_N^2)$ [20].

We use the convention such that

$$\prod_{x \in \xi} f(x) = \exp\left\{ \int_{\mathbb{R}} \xi(dx) \log f(x) \right\} = \prod_{x \in \mathrm{supp}\ \xi} f(x)^{\xi(\{x\})}$$

for $\xi \in \mathfrak{M}_N$ and a function f on $\mathbb{R}$, where supp $\xi = \{x \in \mathbb{R} : \xi(\{x\}) > 0\}$. Then (1.9) is given by

$$M_{\mathrm{GUE}}(m, \boldsymbol{\alpha}; N, \sigma^2) = \mathbf{E}_{N,\sigma^2}\left[\prod_{n=1}^{m} \prod_{X \in \Xi} (\alpha_n - X) \right] \tag{1.12}$$

with (1.5). And if we define

$$M^{(\nu)}(m,\boldsymbol{\alpha};N,\sigma^2)=\mathbf{E}^{(\nu)}_{N,\sigma^2}\left[\prod_{n=1}^{m}\prod_{X\in\Xi}(\alpha_n-X)\right] \tag{1.13}$$

with (1.7) for $\nu>-1$, (1.10) and (1.11) are given as

$$M^{(\nu)}_{\text{chGUE}}(m,\boldsymbol{\alpha};N,\sigma^2)=M^{(\nu)}(m,\boldsymbol{\alpha};N,\sigma^2),\quad \nu\in\mathbb{N}_0, \tag{1.14}$$
$$M_{\text{classC}}(m,\boldsymbol{\alpha};2N,\sigma^2)=M^{(1/2)}(m,\boldsymbol{\alpha}^{\langle 2\rangle};N,\sigma^2), \tag{1.15}$$
$$M_{\text{classD}}(m,\boldsymbol{\alpha};2N,\sigma^2)=M^{(-1/2)}(m,\boldsymbol{\alpha}^{\langle 2\rangle};N,\sigma^2), \tag{1.16}$$

where for $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_m)\in\mathbb{C}^m$, $\boldsymbol{\alpha}^{\langle 2\rangle}\equiv(\alpha_1^2,\dots,\alpha_m^2)\in\mathbb{C}^m$.

We will show that, from the equalities (1.8), two sets of determinantal expressions are derived for (1.12), (1.14)-(1.16).

2 Preliminaries and Main Results

We define

$$p(t,y|x)=\begin{cases}\dfrac{1}{\sqrt{2\pi|t|}}\exp\left\{-\dfrac{(x-y)^2}{2t}\right\}, & t\in\mathbb{R}\setminus\{0\},\\ \delta(y-x), & t=0,\end{cases} \tag{2.1}$$

for $x,y\in\mathbb{C}$. For $z\in\mathbb{C},\nu>-1$, we define z^ν to be $\exp(\nu\log z)$, where the argument of z is given its principal value;

$$z^\nu=\exp\left[\nu\left\{\log|z|+\sqrt{-1}\arg(z)\right\}\right],\quad -\pi<\arg(z)\le\pi.$$

For $\nu>-1, y\in\mathbb{C}$, we set

$$p^{(\nu)}(t,y|x)=\begin{cases}\dfrac{1}{2|t|}\left(\dfrac{y}{x}\right)^{\nu/2}\exp\left(-\dfrac{x+y}{2t}\right)I_\nu\left(\dfrac{\sqrt{xy}}{t}\right), & t\in\mathbb{R}\setminus\{0\},x\in\mathbb{C}\setminus\{0\},\\ \dfrac{y^\nu}{(2|t|)^{\nu+1}\Gamma(\nu+1)}e^{-y/2t}, & t\in\mathbb{R}\setminus\{0\},x=0,\\ \delta(y-x), & t=0,x\in\mathbb{C},\end{cases} \tag{2.2}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind defined by [52, 4]

$$I_\nu(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)\Gamma(n+1+\nu)}\left(\frac{z}{2}\right)^{2n+\nu}.$$

The following equality holds,

$$\int_{\mathbb{R}}dy\,p(s,z|y)p(t,y|x)=p(s+t,z|x), \tag{2.3}$$
$$\int_{\mathbb{R}+}dy\,p^{(\nu)}(s,z|y)p^{(\nu)}(t,y|x)=p^{(\nu)}(s+t,z|x) \tag{2.4}$$

for $s,t\ge 0,x,z\in\mathbb{C}$. The former is confirmed just performing the Gaussian integral and the latter is proved by using Weber's second exponential integral of the Bessel functions [52] with appropriate analytic continuation [30]. When $t \geq 0, x, y \in \mathbb{R}$, $p(t, y|x)$ gives the transition probability density of the one-dimensional standard BM from x to y during time period t, and when $t \geq 0, x, y \in \mathbb{R}_+$, $p^{(\nu)}(t, y|x)$ gives that of the BESQ with index $\nu > -1$ from x to y during time period t (if $-1 < \nu < 0$, a reflection wall is put at the origin). The equalities (2.3) with $x, z \in \mathbb{R}$ and (2.4) with $x, z \in \mathbb{R}_+$ are the Chapman-Kolmogorov equations of these one-dimensional diffusion processes [45]. The extended versions of Chapmann-Kolmogorov equations, *i.e.*, (2.3) with $x \in \mathbb{C} \setminus \mathbb{R}$ and (2.4) with $x \in \mathbb{R} \setminus \mathbb{R}_+$ will have probability-theoretical interpretations related with martingales [24].

We introduce the Karlin-McGregor determinants [23]

$$
\begin{aligned}
f(t, \boldsymbol{y}|\boldsymbol{x}) &= \det_{1 \leq j,k \leq N} [p(t, y_j|x_k)], \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}, \\
f^{(\nu)}(t, \boldsymbol{y}|\boldsymbol{x}) &= \det_{1 \leq j,k \leq N} [p^{(\nu)}(t, y_j|x_k)], \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{+}, \quad \nu > -1,
\end{aligned} \tag{2.5}
$$

$t \geq 0$.

For given $\xi \in \mathfrak{M}_N, N \in \mathbb{N}$, we write $\xi(\cdot) = \sum_{j=1}^N \delta_{a_j}(\cdot)$ with $a_1 \leq a_2 \leq \cdots \leq a_N$. Then we set

$$
\xi_0(\cdot) \equiv 0 \quad \text{and} \quad \xi_n(\cdot) = \sum_{j=1}^{n} \delta_{a_j}(\cdot), \quad 1 \leq n \leq N.
$$

For $t > 0, x \in \mathbb{C}$, we define [5, 29]

$$
\phi_n^{(+)}(t, x; \xi) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathrm{C}(\xi_{n+1})} ds\, p(t, x|s) \frac{1}{\prod_{a \in \xi_{n+1}} (s-a)}, \tag{2.6}
$$

$$
\phi_n^{(-)}(t, x; \xi) = \int_{\mathbb{R}} ds\, p(-t, \mathrm{i}s|x) \prod_{a \in \xi_n} (\mathrm{i}s - a), \quad \xi \in \mathfrak{M}_N, \tag{2.7}
$$

and [12, 30]

$$
\phi_n^{(\nu,+)}(t, x; \xi) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathrm{C}(\xi_{n+1})} ds\, p^{(\nu)}(t, x|s) \frac{1}{\prod_{a \in \xi_{n+1}} (s-a)}, \tag{2.8}
$$

$$
\phi_n^{(\nu,-)}(t, x; \xi) = \int_{\mathbb{R}_-} ds\, p^{(\nu)}(-t, s|x) \prod_{a \in \xi_n} (s - a), \quad \xi \in \mathfrak{M}_N^{+}, \tag{2.9}
$$

$n = 0, 1, \ldots, N-1$, where $\mathrm{i} = \sqrt{-1}$, for $\zeta \in \mathfrak{M}_\ell, \ell \in \mathbb{N}$, $\mathrm{C}(\zeta)$ denotes a closed contour on the complex plane $\mathbb{C}$ encircling the points in supp ζ on the real line $\mathbb{R}$ once in the positive direction, and $\mathbb{R}_- = \{x \in \mathbb{R} : x \leq 0\}$. And for $t \geq 0$, let

$$
\begin{aligned}
h_N^{(\pm)}(t, \boldsymbol{y}; \xi) &= \det_{1 \leq j,k \leq N} [\phi_{j-1}^{(\pm)}(t, y_k; \xi)], \quad \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}, \quad \xi \in \mathfrak{M}_N, \\
h_N^{(\nu,\pm)}(t, \boldsymbol{y}; \xi) &= \det_{1 \leq j,k \leq N} [\phi_{j-1}^{(\nu,\pm)}(t, y_k; \xi)], \quad \boldsymbol{y} \in \mathbb{W}_N^{+}, \quad \xi \in \mathfrak{M}_N^{+}.
\end{aligned} \tag{2.10}
$$

Since $\phi_n^{(-)}(t, x; \xi)$ and $\phi_n^{(\nu,-)}(t, x; \xi)$ are monic polynomials of x of degree n,

$$
h_N^{(-)}(t, \boldsymbol{y}; \xi) = h_N^{(\nu,-)}(t, \boldsymbol{y}; \xi) = h_N(\boldsymbol{y}), \tag{2.11}
$$

which are independent of t and ξ. On the other hand, the following equalities are proved (Lemma 3.1 in [29] and Lemma 3.4 in [30]). For any $t \geq 0, \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}, \xi = \sum_{j=1}^N \delta_{x_j} \in \mathfrak{M}_N$ with $x_1 \leq x_2 \leq \cdots \leq x_N$,

$$
\frac{f(t, \boldsymbol{y}|\boldsymbol{x})}{h_N(\boldsymbol{x})} = h_N^{(+)}(t, \boldsymbol{y}; \xi), \tag{2.12}
$$

and for any $t \geq 0, \boldsymbol{y} \in \mathbb{W}_N^+, \xi = \sum_{j=1}^N \delta_{x_j} \in \mathfrak{M}_N^+$ with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N$,

$$\frac{f^{(\nu)}(t, \boldsymbol{y}|\boldsymbol{x})}{h_N(\boldsymbol{x})} = h_N^{(\nu,+)}(t, \boldsymbol{y}; \xi), \tag{2.13}$$

where, if some of the x_j's coincide, the LHS of (2.12) and (2.13) are interpreted by using l'Hôpital's rule.

For any $M \in \mathbb{N}$ and any increasing time-sequence $0 < t_1 < \cdots < t_M < \infty$, the multitime probability density of $(\Xi(t), t \in [0, \infty), \mathbb{P}^\xi)$ is given by

$$\begin{aligned} & p^\xi(t_1, \xi^{(1)}; \ldots; t_M, \xi^{(M)}) \\ & \quad = h_N^{(-)}(t_M, \boldsymbol{x}^{(M)}; \xi) \prod_{m=1}^{M-1} f(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)}|\boldsymbol{x}^{(m)}) h_N^{(+)}(t_1, \boldsymbol{x}^{(1)}, \xi) \end{aligned} \tag{2.14}$$

with $\xi = \sum_{j=1}^N \delta_{a_j} \in \mathfrak{M}_N$, $a_1 \leq a_2 \leq \cdots \leq a_N$ for the initial configuration and $\xi^{(m)} = \sum_{j=1}^N \delta_{x_j^{(m)}} \in \mathfrak{M}_{N,0}$, $\boldsymbol{x}^{(m)} = (x_1^{(m)}, \ldots, x_N^{(m)}) \in \mathbb{W}_N^{\mathrm{A}}$ for configurations at times $t_m, 1 \leq m \leq M$ [29]. Similarly, the multitime probability density of $(\Xi^{(\nu)}(t), t \in [0, \infty), \mathbb{P}_\nu^\xi)$ is given by

$$\begin{aligned} & p_\nu^\xi(t_1, \xi^{(1)}; \ldots; t_M, \xi^{(M)}) \\ & \quad = h_N^{(\nu,-)}(t_M, \boldsymbol{x}^{(M)}; \xi) \prod_{m=1}^{M-1} f^{(\nu)}(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)}|\boldsymbol{x}^{(m)}) h_N^{(\nu,+)}(t_1, \boldsymbol{x}^{(1)}; \xi) \end{aligned} \tag{2.15}$$

with $\xi = \sum_{j=1}^N \delta_{a_j} \in \mathfrak{M}_N^+$, $0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ for the initial configuration and $\xi^{(m)} = \sum_{j=1}^N \delta_{x_j^{(m)}} \in \mathfrak{M}_{N,0}^+$, $\boldsymbol{x}^{(m)} = (x_1^{(m)}, \ldots, x_N^{(m)}) \in \mathbb{W}_N^+$ for configurations at times $t_m, 1 \leq m \leq M$ [30]. By definitions (2.6) and (2.7), we can see that

$$\begin{aligned} \phi_n^{(+)}(t, x; N\delta_0) &= t^{-(n+1)/2} \frac{2^{-n/2}}{n!\sqrt{2\pi}} H_n\left(\frac{x}{\sqrt{2t}}\right) e^{-x^2/2t}, \\ \phi_n^{(-)}(t, x; N\delta_0) &= t^{n/2} 2^{-n/2} H_n\left(\frac{x}{\sqrt{2t}}\right), \quad 0 \leq n \leq N-1, \end{aligned} \tag{2.16}$$

where $H_n(x)$ is the Hermite polynomial of degree n,

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!} \\ &= 2^{n/2} \int_{\mathbb{R}} dy \, \frac{e^{-y^2/2}}{\sqrt{2\pi}} (\mathrm{i}y + \sqrt{2}x)^n = \frac{n!}{2\pi \mathrm{i}} \oint_{\mathrm{C}(\delta_0)} dz \, \frac{e^{2zx - z^2}}{z^{n+1}}, \end{aligned} \tag{2.17}$$

where $[r]$ denotes the largest integer that is not greater than $r \in R_+$, and by definitions (2.8) and (2.9),

$$\begin{aligned} \phi_n^{(\nu,+)}(t, x; N\delta_0) &= t^{-(n+1)} (-1)^n \frac{2^{-(n+\nu+1)}}{\Gamma(n+\nu+1)} \left(\frac{x}{t}\right)^\nu e^{-x/2t} L_n^\nu\left(\frac{x}{2t}\right), \\ \phi_n^{(\nu,-)}(t, x; N\delta_0) &= t^n (-1)^n 2^n n! L_n^\nu\left(\frac{x}{2t}\right), \end{aligned} \tag{2.18}$$

where $L_n^{(\nu)}(x)$ is the Laguerre polynomial of degree n with index ν,

$$
\begin{aligned}
L_n^\nu(x) &= \sum_{k=0}^{n}(-1)^k \frac{\Gamma(n+\nu+1)x^k}{\Gamma(k+\nu+1)(n-k)!k!} \\
&= \frac{1}{2\pi i}\oint_{C(\delta_0)} \frac{e^{-xu}(1+u)^{n+\nu}}{u^{n+1}} du.
\end{aligned}
\tag{2.19}
$$

Then we can prove the following.

Theorem 2.1 *For any $N, M \in \mathbb{N}$, any increasing time-sequence $0 < t_1 < \cdots < t_M < \infty$, and any $\sigma^2 > 0$,*

$$
\mathbf{E}_{N,\sigma^2}[p^{\Xi}(t_1,\xi^{(1)};\ldots;t_M,\xi^{(M)})] = p^{N\delta_0}(t_1+\sigma^2,\xi^{(1)};\ldots;t_M+\sigma^2,\xi^{(M)}) \tag{2.20}
$$

with $\xi^{(m)} \in \mathfrak{M}_{N,0}, 1 \leq m \leq M$, and

$$
\mathbf{E}^{(\nu)}_{N,\sigma^2}[p^{\Xi}_\nu(t_1,\xi^{(1)};\ldots;t_M,\xi^{(M)})] = p^{N\delta_0}_\nu(t_1+\sigma^2,\xi^{(1)};\ldots;t_M+\sigma^2,\xi^{(M)}) \tag{2.21}
$$

with $\xi^{(m)} \in \mathfrak{M}^+_{N,0}, 1 \leq m \leq M$.

Remark 1. When M-time probability density of a process is given for any $M \in \mathbb{N}$ and any time sequence $0 < t_1 < \cdots < t_M$, it is said that the finite dimensional distributions of the process is determined [45]. Eq.(2.20) (resp. Eq.(2.21)) means that the processes $(\Xi(t), t \in [0,\infty), \mathbb{P}^{\mu_{N,\sigma^2}})$ and $(\Xi(t+\sigma^2), t \in [0,\infty), \mathbb{P}^{N\delta_0})$ (resp. $(\Xi^{(\nu)}(t), t \in [0,\infty), \mathbb{P}_\nu^{\mu_{N,\sigma^2}})$ and $(\Xi^{(\nu)}(t+\sigma^2), t \in [0,\infty), \mathbb{P}_\nu^{N\delta_0})$) are equivalent in finite dimensional distributions for any $\sigma^2 > 0$, which is the fact expressed by (1.8).

For $\boldsymbol{x}^{(m)} = (x_1^{(m)}, \ldots, x_N^{(m)}) \in \mathbb{W}_N^{\rm A}$ or $\boldsymbol{x}^{(m)} \in \mathbb{W}_N^+$ with $\xi^{(m)}(\cdot) = \sum_{j=1}^N \delta_{x_j^{(m)}}(\cdot)$ and $N' \in \{1, 2, \ldots, N\}$, we put $\boldsymbol{x}_{N'}^{(m)} = (x_1^{(m)}, \ldots, x_{N'}^{(m)}) \in \mathbb{W}_{N'}^{\rm A}$ or $\boldsymbol{x}_{N'}^{(m)} \in \mathbb{W}_{N'}^+$, $1 \leq m \leq M$. For a sequence $(N_m)_{m=1}^M$ of positive integers less than or equal to N, we define the $(N_1, \ldots, N_M)$-*multitime correlation functions* of $(\Xi(t), t \in [0,\infty), \mathbb{P}^\xi)$ and $(\Xi^{(\nu)}(t), t \in [0,\infty), \mathbb{P}_\nu^\xi)$ by

$$
\begin{aligned}
&\rho^\xi(t_1, \boldsymbol{x}_{N_1}^{(1)}; \ldots; t_M, \boldsymbol{x}_{N_M}^{(M)}) \\
&= \int_{\prod_{m=1}^M \mathbb{R}^{N-N_m}} \prod_{m=1}^M \prod_{j=N_m+1}^N dx_j^{(m)} p^\xi(t_1,\xi^{(1)};\ldots;t_M,\xi^{(M)}) \prod_{m=1}^M \frac{1}{(N-N_m)!}, \\
&\rho_\nu^\xi(t_1, \boldsymbol{x}_{N_1}^{(1)}; \ldots; t_M, \boldsymbol{x}_{N_M}^{(M)}) \\
&= \int_{\prod_{m=1}^M \mathbb{R}_+^{N-N_m}} \prod_{m=1}^M \prod_{j=N_m+1}^N dx_j^{(m)} p_\nu^\xi(t_1,\xi^{(1)};\ldots;t_M,\xi^{(M)}) \prod_{m=1}^M \frac{1}{(N-N_m)!},
\end{aligned}
\tag{2.22}
$$

respectively. In the previous papers we have shown that for any fixed initial configuration ξ the noncolliding BM and the noncolliding BESQ with finite numbers of particles are *determinantal processes* in the sense that any multitime correlation function is given by a determinant [29, 30]

$$
\begin{aligned}
\rho^\xi(t_1, \boldsymbol{x}_{N_1}^{(1)}; \ldots; t_M, \boldsymbol{x}_{N_M}^{(M)}) &= \det_{\substack{1 \leq j \leq N_m, 1 \leq k \leq N_n \\ 1 \leq m,n \leq M}} [\mathbb{K}^\xi(t_m, x_j^{(m)}; t_n, x_k^{(n)})] \\
\rho_\nu^\xi(t_1, \boldsymbol{x}_{N_1}^{(1)}; \ldots; t_M, \boldsymbol{x}_{N_M}^{(M)}) &= \det_{\substack{1 \leq j \leq N_m, 1 \leq k \leq N_n \\ 1 \leq m,n \leq M}} [\mathbb{K}_\nu^\xi(t_m, x_j^{(m)}; t_n, x_k^{(n)})], \quad \nu > -1.
\end{aligned}
\tag{2.23}
$$

Here the correlation kernels are given by

$$
\begin{aligned}
\mathbb{K}^{\xi}(s,x;t,y) &= \sum_{n=0}^{N-1} \phi_n^{(+)}(s,x;\xi)\phi_n^{(-)}(t,y;\xi) - \mathbf{1}(s>t)p(s-t,x|y) \\
&= \int_{\mathbb{R}} \xi(dx') \int_{\mathbb{R}} dy' p(s,x|x')\Phi(\xi;x',\mathrm{i}y')p(-t,\mathrm{i}y'|y) \\
&\quad -\mathbf{1}(s>t)p(s-t,x|y), \quad \xi \in \mathfrak{M}_{N,0}, \qquad (2.24)\\
\mathbb{K}^{\xi}_{\nu}(s,x;t,y) &= \sum_{n=0}^{N-1} \phi_n^{(\nu,+)}(s,x;\xi)\phi_n^{(\nu,-)}(t,y;\xi) - \mathbf{1}(s>t)p^{(\nu)}(s-t,x|y) \\
&= \int_{\mathbb{R}_+} \xi(dx') \int_{\mathbb{R}_-} dy' p^{(\nu)}(s,x|x')\Phi(\xi;x',y')p^{(\nu)}(-t,y'|y) \\
&\quad -\mathbf{1}(s>t)p^{(\nu)}(s-t,x|y), \quad \xi \in \mathfrak{M}^{+}_{N,0}, \quad \nu>-1, \qquad (2.25)
\end{aligned}
$$

where

$$
\Phi(\xi,x,z) = \prod_{a\in\xi\cap\{x\}^{\mathrm{c}}} \frac{z-a}{x-a}, \tag{2.26}
$$

and $\mathbf{1}(\omega)=1$ if ω is satisfied and $\mathbf{1}(\omega)=0$ otherwise (Proposition 2.1 in [29] and Theorem 2.1 in [30]). The function $\Phi(\xi,x,z)$ is an entire function of $z\in\mathbb{C}$ expressed by the Weierstrass canonical product with genus 0, whose zeros are given by supp $\xi\cap\{x\}^{\mathrm{c}}$ [39, 28].

As direct consequences of Theorem 2.1, we have the following equalities for multitime correlation functions;

$$
\mathbf{E}_{N,\sigma^2}[\rho^{\Xi}(t_1,\boldsymbol{x}^{(1)}_{N_1};\dots;t_M,\boldsymbol{x}^{(M)}_{N_M})] = \rho^{N\delta_0}(t_1+\sigma^2,\boldsymbol{x}^{(1)}_{N_1};\dots;t_M+\sigma^2,\boldsymbol{x}^{(M)}_{N_M}) \tag{2.27}
$$

with $x^{(m)}_{N_m}\in\mathbb{W}^{\mathrm{A}}_{N_m}, 1\le m\le M$, and

$$
\mathbf{E}^{(\nu)}_{N,\sigma^2}[\rho^{\Xi}_{\nu}(t_1,\boldsymbol{x}^{(1)}_{N_1};\dots;t_M,\boldsymbol{x}^{(M)}_{N_M})] = \rho^{N\delta_0}_{\nu}(t_1+\sigma^2,\boldsymbol{x}^{(1)}_{N_1};\dots;t_M+\sigma^2,\boldsymbol{x}^{(M)}_{N_M}) \tag{2.28}
$$

with $x^{(m)}_{N_m}\in\mathbb{W}^{+}_{N_m}, 1\le m\le M$, for any $M\in\mathbb{N}, 0<t_1<\cdots<t_M<\infty$. Therefore, infinite systems of equalities between determinants of correlation kernels are obtained as a corollary of Theorem 2.1.

Corollary 2.2 *For any $N,M\in\mathbb{N}, 0<t_1<\cdots<t_M<\infty, N_m\in\{1,2,\dots,N\}, 1\le m\le M$, $\sigma^2>0$,*

$$
\begin{aligned}
&\mathbf{E}_{N,\sigma^2}\left[\det_{\substack{1\le j\le N_m,1\le k\le N_n\\ 1\le m,n\le M}} [\mathbb{K}^{\Xi}(t_m,x^{(m)}_j;t_n,x^{(n)}_k)]\right] \\
&\quad = \det_{\substack{1\le j\le N_m,1\le k\le N_n\\ 1\le m,n\le M}} [\mathbb{K}^{N\delta_0}(t_m+\sigma^2,x^{(m)}_j;t_n+\sigma^2,x^{(n)}_k)],
\end{aligned} \tag{2.29}
$$

$$
\begin{aligned}
&\mathbf{E}^{(\nu)}_{N,\sigma^2}\left[\det_{\substack{1\le j\le N_m,1\le k\le N_n\\ 1\le m,n\le M}} [\mathbb{K}^{\Xi}_{\nu}(t_m,x^{(m)}_j;t_n,x^{(n)}_k)]\right] \\
&\quad = \det_{\substack{1\le j\le N_m,1\le k\le N_n\\ 1\le m,n\le M}} [\mathbb{K}^{N\delta_0}_{\nu}(t_m+\sigma^2,x^{(m)}_j;t_n+\sigma^2,x^{(n)}_k)].
\end{aligned} \tag{2.30}
$$

In particular, as $M = 1, N_1 = L \leq N$,

$$\mathbf{E}_{N,\sigma^2}\left[\det_{1\leq j,k\leq L}[\mathbb{K}^{\Xi}(t,x_j;t,x_k)]\right] = \det_{1\leq j,k\leq L}[\mathbb{K}^{N\delta_0}(t+\sigma^2,x_j;t+\sigma^2,x_k)], \tag{2.31}$$

$$\mathbf{E}^{(\nu)}_{N,\sigma^2}\left[\det_{1\leq j,k\leq L}[\mathbb{K}^{\Xi}_{\nu}(t,x_j;t,x_k)]\right] = \det_{1\leq j,k\leq L}[\mathbb{K}^{N\delta_0}_{\nu}(t+\sigma^2,x_j;t+\sigma^2,x_k)] \tag{2.32}$$

hold for any $t > 0, \sigma^2 > 0$.

The proof of Theorem 2.1 is given in Sect. 3.1.

The main purpose of the present paper is to show that the equalities in Corollary 2.2 are nontrivial even in the special cases given by (2.31) and (2.32), and from them the determinantal expressions for the ensemble averages of $2n$-products of characteristic polynomials of random matrices are derived for any $n \in \mathbb{N}$. We show two sets of determinantal expressions. The first one is given by the following theorem.

Theorem 2.3 *For any $N, n \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_{2n}) \in \mathbb{C}^{2n}$, $\sigma^2 > 0$,*

$$M_{\rm GUE}(2n,\boldsymbol{\alpha};N,\sigma^2) = \frac{\gamma_{N,2n}\sigma^{n(2N+n)}}{h_n(\alpha_1,\cdots,\alpha_n)h_n(\alpha_{n+1},\cdots,\alpha_{2n})} \times \det_{1\leq j,k\leq n}\left[\frac{1}{\alpha_j-\alpha_{n+k}}\left|\begin{array}{cc} H_{N+n}(\alpha_j/\sqrt{2\sigma^2}) & H_{N+n}(\alpha_{n+k}/\sqrt{2\sigma^2}) \\ H_{N+n-1}(\alpha_j/\sqrt{2\sigma^2}) & H_{N+n-1}(\alpha_{n+k}/\sqrt{2\sigma^2})\end{array}\right|\right] \tag{2.33}$$

with

$$\gamma_{N,2n} = 2^{-n(2N+2n-1)/2}\prod_{\ell=2}^{n}\frac{(N+n-\ell)!}{(N+n-1)!}, \tag{2.34}$$

and for $\nu > -1$

$$M^{(\nu)}(2n,\boldsymbol{\alpha};N,\sigma^2) = \frac{\gamma^{(\nu)}_{N,2n}(2\sigma^2)^{n(2N+n)}}{h_n(\alpha_1,\cdots,\alpha_n)h_n(\alpha_{n+1},\cdots,\alpha_{2n})} \times \det_{1\leq j,k\leq n}\left[\frac{1}{\alpha_j-\alpha_{n+k}}\left|\begin{array}{cc} L^{\nu}_{N+n}(\alpha_j/2\sigma^2) & L^{\nu}_{N+n}(\alpha_{n+k}/2\sigma^2) \\ L^{\nu}_{N+n-1}(\alpha_j/2\sigma^2) & L^{\nu}_{N+n-1}(\alpha_{n+k}/2\sigma^2)\end{array}\right|\right], \tag{2.35}$$

with

$$\gamma^{(\nu)}_{N,2n} = (-1)^n\left(\frac{(N+n)!}{\Gamma(N+n+\nu)}\right)^{n-1}\prod_{\ell=1}^{n-1}\Gamma(N+\nu+\ell)\prod_{m=1}^{n+1}(N+m-1)!. \tag{2.36}$$

By setting $\nu \in \mathbb{N}_0, \nu = 1/2$ and $\nu = -1/2$ in (2.35), the determinantal expressions are given for $M^{(\nu)}_{\rm chGUE}, M_{\rm classC}$ and $M_{\rm classD}$ through (1.14)-(1.16).

Proof is given in Sect. 3.2. The above expressions can be simplified by using the following identity, which was given by Ishikawa et al.[22]. For $n \geq 2, \boldsymbol{x} = (x_1, \ldots, x_n), \boldsymbol{y} = (y_1, \ldots, y_n), \boldsymbol{a} =$

$(a_1,\ldots,a_n), \boldsymbol{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n$,

$$
\begin{aligned}
&\det_{1\le j,k\le n}\left[\frac{1}{y_k-x_j}\left|\begin{array}{cc}1 & a_j\\ 1 & b_k\end{array}\right|\right]\\
&= \frac{(-1)^{n(n-1)/2}}{\prod_{j=1}^n\prod_{k=1}^n(y_k-x_j)}\left|\begin{array}{cccccccc}
1 & x_1 & \cdots & x_1^{n-1} & a_1 & a_1x_1 & \cdots a_1x_1^{n-1}\\
1 & x_2 & \cdots & x_2^{n-1} & a_2 & a_2x_2 & \cdots a_2x_2^{n-1}\\
 & & \cdots & & & & \cdots\\
1 & x_n & \cdots & x_n^{n-1} & a_n & a_nx_n & \cdots a_nx_n^{n-1}\\
1 & y_1 & \cdots & y_1^{n-1} & b_1 & b_1y_1 & \cdots b_1y_1^{n-1}\\
1 & y_2 & \cdots & y_2^{n-1} & b_2 & b_2y_2 & \cdots b_2y_2^{n-1}\\
 & & \cdots & & & & \cdots\\
1 & y_n & \cdots & y_n^{n-1} & b_n & b_ny_n & \cdots b_ny_n^{n-1}
\end{array}\right|. \qquad (2.37)
\end{aligned}
$$

Remark 2. For $n\in\mathbb{N}, p,q\in\mathbb{N}_0$ satisfying $p+q=n$, and $\boldsymbol{x},\boldsymbol{a}\in\mathbb{C}^n$, denote by $V^{p,q}(\boldsymbol{x};\boldsymbol{a})$ the $n\times n$ matrix with j-th row

$$(1, x_j,\cdots,x_j^{p-1}, a_j, a_jx_j,\cdots,a_jx_j^{q-1}).$$

If $q=0$, then $p=n$ and $V^{n,0}(\boldsymbol{x};\boldsymbol{a})=V^{n,0}(\boldsymbol{x})=(x_j^{k-1})_{1\le j,k\le n}$ is the Vandermonde matrix and its determinant $\det V^{n,0}(\boldsymbol{x})$ is equal to the product of differences of n variables, $h_n(\boldsymbol{x})$, given by (1.4). As a generalization of the Cauchy determinant

$$\det_{1\le j,k\le n}\left(\frac{1}{x_j+y_k}\right)=\frac{h_n(\boldsymbol{x})h_n(\boldsymbol{y})}{\prod_{j=1}^n\prod_{k=1}^n(x_j+y_k)},$$

$\boldsymbol{x},\boldsymbol{y}\in\mathbb{C}^n$, Ishikawa et al. [22] proved the following equalities involving the generalized Vandermonde determinants $V^{p,q}$. Let $n\in\mathbb{N}$, $p,q\in\mathbb{N}_0$. For $\boldsymbol{x}=(x_1,\cdots,x_n), \boldsymbol{y}=(y_1,\cdots,y_n), \boldsymbol{a}=(a_1,\cdots,a_n), \boldsymbol{b}=(b_1,\cdots,b_n)\in\mathbb{C}^n$ and $\boldsymbol{z}=(z_1,\cdots,z_{p+q}), \mathrm{C}=(c_1,\cdots,c_{p+q})\in\mathbb{C}^{p+q}$

$$
\begin{aligned}
&\det_{1\le j,k\le n}\left[\frac{1}{y_k-x_j}\det V^{p+1,q+1}(x_j,y_k,\boldsymbol{z};a_j,b_k,\mathrm{C})\right]\\
&= \frac{(-1)^{n(n-1)/2}}{\prod_{j=1}^n\prod_{k=1}^n(y_k-x_j)}\det V^{p,q}(\boldsymbol{z},\mathrm{C})^{n-1}\det V^{n+p,n+q}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};\boldsymbol{a},\boldsymbol{b},\mathrm{C}). \qquad (2.38)
\end{aligned}
$$

When $p=q=0$, $\det V^{0,0}(\boldsymbol{z},\mathrm{C})=1$ and

$$\det V^{1,1}(x_j,y_k;a_j,b_k)=\left|\begin{array}{cc}1 & a_j\\ 1 & b_k\end{array}\right|.$$

Then as a special case of (2.38), (2.37) is obtained.

For $\sigma^2>0$ define

$$\widehat{H}_\ell(\alpha;\sigma^2)=\left(\frac{\sigma^2}{2}\right)^{\ell/2}H_\ell\left(\frac{\alpha}{\sqrt{2\sigma^2}}\right), \quad \alpha\in\mathbb{R}, \qquad (2.39)$$

$$\widehat{L}_\ell^\nu(\alpha;\sigma^2)=(-2\sigma^2)^\ell\ell!L_\ell^\nu\left(\frac{\alpha}{2\sigma^2}\right), \quad \alpha\in\mathbb{R}_+, \qquad (2.40)$$

$\ell\in\mathbb{N}_0$, which are both monic polynomials of α with order ℓ. By using the identity (2.37) and recurrence relations of Hermite polynomials and Laguerre polynomials, we can prove the following second set of determinantal expressions.

Theorem 2.4 *For any* $N, n \in \mathbb{N}, \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_{2n}) \in \mathbb{C}^{2n}$, $\sigma^2 > 0$,

$$M_{\rm GUE}(2n, \boldsymbol{\alpha}; N, \sigma^2) = \frac{1}{h_{2n}(\boldsymbol{\alpha})} \det_{1 \le j,k \le 2n} \left[\widehat{H}_{N+j-1}(\alpha_k; \sigma^2) \right], \tag{2.41}$$

$$M^{(\nu)}_{\rm chGUE}(2n, \boldsymbol{\alpha}; N, \sigma^2) = \frac{1}{h_{2n}(\boldsymbol{\alpha})} \det_{1 \le j,k \le 2n} \left[\widehat{L}^{\nu}_{N+j-1}(\alpha_k; \sigma^2) \right], \quad \nu \in \mathbb{N}_0, \tag{2.42}$$

$$\begin{aligned} M_{\rm classC}(2n, \boldsymbol{\alpha}; 2N, \sigma^2) &= \frac{1}{h_{2n}(\boldsymbol{\alpha}^{\langle 2 \rangle})} \det_{1 \le j,k \le 2n} \left[\widehat{L}^{1/2}_{N+j-1}(\alpha_k^2; \sigma^2) \right] \\ &= \frac{1}{h_{2n}(\boldsymbol{\alpha}^{\langle 2 \rangle}) \prod_{j=1}^{2n} \alpha_j} \det_{1 \le j,k \le 2n} \left[\widehat{H}_{2N+2j-1}(\alpha_k; \sigma^2) \right], \end{aligned} \tag{2.43}$$

and

$$\begin{aligned} M_{\rm classD}(2n, \boldsymbol{\alpha}; 2N, \sigma^2) &= \frac{1}{h_{2n}(\boldsymbol{\alpha}^{\langle 2 \rangle})} \det_{1 \le j,k \le 2n} \left[\widehat{L}^{-1/2}_{N+j-1}(\alpha_k^2; \sigma^2) \right] \\ &= \frac{1}{h_{2n}(\boldsymbol{\alpha}^{\langle 2 \rangle})} \det_{1 \le j,k \le 2n} \left[\widehat{H}_{2(N+j-1)}(\alpha_k; \sigma^2) \right]. \end{aligned} \tag{2.44}$$

The determinantal expressions (2.41)-(2.44) can be obtained from the general formula given by Brézin and Hikami as Eq.(14) in [8]. (See also [17, 41, 18, 19] and Sect.22.4 in [40].) Since our new expressions (2.33) and (2.35) are independently derived in the present paper, if we combine the present result and that of Brézin and Hikami [8], the special case of identity (2.37) of Ishikawa et al. [22] is concluded.

Remark 3. In this paper, we derive the ensemble averages of products of characteristic polynomials of random matrices from the equalities (2.31) and (2.32). As stated in Corollary 2.2, these equalities are special cases with $M = 1$ of the systems of equalities (2.29) and (2.30). It will be an interesting problem to clarify all the information involved in (2.29) and (2.30) (see [13]).

3 Proofs of Theorems

3.1 Proof of Theorem 2.1

By (2.16) and the fact that $H_n(x/2)$ is a monic polynomial of order $n \in \mathbb{N}_0$ and thus

$$\det_{1 \le j,k \le N} [H_{j-1}(x_k/2)] = h_N(\boldsymbol{x}),$$

we obtain the equality

$$h_N^{(+)}(\sigma^2, \boldsymbol{x}; N\delta_0) = \frac{\mu_{N,\sigma^2}(\xi)}{h_N(\xi)}, \quad \xi = \sum_{j=1}^{N} \delta_{x_j} \in \mathfrak{M}_{N,0}, \boldsymbol{x} \in \mathbb{W}_N^{\rm A}. \tag{3.1}$$

Combining this with (2.12) gives the equality

$$f(t_1, \boldsymbol{x}^{(1)} | \boldsymbol{x}) h_N^{(+)}(\sigma^2, \boldsymbol{x}; N\delta_0) = h_N^{(+)}(t_1, \boldsymbol{x}^{(1)}; \xi) \mu_{N,\sigma^2}(\xi) \tag{3.2}$$

for any $t_1 > 0, \xi = \sum_{j=1}^N \delta_{x_j} \in \mathfrak{M}_N, \boldsymbol{x}^{(1)} \in \mathbb{W}_N^{\mathrm{A}}$. Then (2.14) with $\xi = N\delta_0$ gives

$$\begin{aligned}
&p^{N\delta_0}(\sigma^2, \xi; t_1 + \sigma^2, \xi^{(1)}; \dots; t_M + \sigma^2, \xi^{(M)}) \\
&= h_N^{(-)}(t_M + \sigma^2, \boldsymbol{x}^{(M)}; N\delta_0) \prod_{m=1}^{M-1} f(t_{m+1} - t_m; \boldsymbol{x}^{(m+1)} | \boldsymbol{x}^{(m)}) f(t_1, \boldsymbol{x}^{(1)} | \boldsymbol{x}) h_N^{(+)}(\sigma^2, \boldsymbol{x}; N\delta_0) \\
&= h_N^{(-)}(t_M, \boldsymbol{x}^{(M)}; \xi) \prod_{m=1}^{M-1} f(t_{m+1} - t_m; \boldsymbol{x}^{(m+1)} | \boldsymbol{x}^{(m)}) h_N^{(+)}(t_1, \boldsymbol{x}^{(1)}; \xi) \mu_{N,\sigma^2}(\xi) \\
&= \mu_{N,\sigma^2}(\xi) p^{\xi}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}),
\end{aligned} \tag{3.3}$$

where (2.11) was used. Integrating the both sides with respect to ξ over $\mathfrak{M}_N$ according to (1.5), (2.20) is obtained. Similarly, the equality

$$f^{(\nu)}(t_1, \boldsymbol{x}^{(1)} | \boldsymbol{x}) h_N^{(\nu,+)}(\sigma^2, \boldsymbol{x}; N\delta_0) = h_N^{(\nu,+)}(t_1, \boldsymbol{x}^{(1)}; \xi) \mu_{N,\sigma^2}^{(\nu)}(\xi) \tag{3.4}$$

is established for any $t_1 > 0, \xi = \sum_{j=1}^N \delta_{x_j} \in \mathfrak{M}_N^+, \xi^{(1)} \in \mathbb{W}_N^+$, and then we have

$$\mu_{N,\sigma^2}^{(\nu)}(\xi) p_\nu^{\xi}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}) = p_\nu^{N\delta_0}(\sigma^2, \xi; t_1 + \sigma^2, \xi^{(1)}; \dots; t_M + \sigma^2, \xi^{(M)}). \tag{3.5}$$

Integrating the both sides with respect to ξ over $\mathfrak{M}_N^+$ according to (1.7), (2.21) is obtained. ▮

3.2 Proof of Theorem 2.3

First we derive the expression (2.33) from (2.31). We observe that the LHS of (2.31) is written as follows.

Lemma 3.1 *For any $\sigma^2 > 0, t > 0, 1 \le L \le N, \boldsymbol{x}_L = (x_1, \dots, x_L) \in \mathbb{W}_L^{\mathrm{A}}$,*

$$\begin{aligned}
&\mathbf{E}_{N,\sigma^2}\left[\det_{1 \le j,k \le L} [\mathbb{K}^{\Xi}(t, x_j; t, x_k)]\right] \\
&= \int_{\mathbb{R}^L} d\boldsymbol{u} \int_{\mathbb{R}^L} d\boldsymbol{w}\, e^{-|\boldsymbol{u}|^2/2\sigma^2} \prod_{j=1}^L p(t, x_j | u_j) p(-t, \mathrm{i}w_j | x_j) \frac{\sigma^{-L(2N-L)}}{(2\pi)^{L/2} \prod_{\ell=1}^L (N-\ell)!} \\
&\quad \times h_L(\boldsymbol{u}) h_L(\mathrm{i}\boldsymbol{w}) M_{\mathrm{GUE}}(2L, (u_1, \dots, u_L, \mathrm{i}w_1, \dots, \mathrm{i}w_L); N - L, \sigma^2).
\end{aligned} \tag{3.6}$$

Proof Assume that $\xi = \sum_{j=1}^N \delta_{a_j}$ with $\boldsymbol{a} = (a_1, \dots, a_N) \in \mathbb{W}_N^{\mathrm{A}}$. Then

$$\begin{aligned}
&\rho(t, \boldsymbol{x}_L) = \det_{1 \le j,k \le L} [\mathbb{K}^{\xi}(t, x_j; t, x_k)] \\
&= \det_{1 \le j,k \le L} \left[\sum_{q=1}^N p(t, x_j | a_q) \int_{\mathbb{R}} dw\, p(-t, \mathrm{i}w | x_k) \prod_{1 \le \ell \le N, \ell \ne q} \frac{\mathrm{i}w - a_\ell}{a_q - a_\ell}\right] \\
&= L! \sum_{1 \le q_1 < \dots < q_N \le N} \int_{\mathbb{R}^L} d\boldsymbol{w} \prod_{j=1}^L \left\{ p(-t, \mathrm{i}w_j | x_j) \prod_{1 \le \ell_j \le N, \ell_j \ne q_j} \frac{\mathrm{i}w_j - a_{\ell_j}}{a_{q_j} - a_{\ell_j}} \right\} \det_{1 \le j,k \le L} [p(t, x_j | a_{q_k})],
\end{aligned}$$

where multilinearlity and antisymmetry property of determinant have been used. Let $\mathbf{I}_N = \{1, 2, \dots, N\}, \mathbf{I}_L = \{1, 2, \dots, L\}$. For a given ordered set of indices $\mathbf{q}_L = \{q_1, q_2, \dots, q_L\}, 1 \le$

$q_1 < \cdots < q_L \leq N$, we see

$$
\begin{aligned}
& h_N(\boldsymbol{a})^2 \prod_{j=1}^{L} \prod_{1 \leq \ell_j \leq N, \ell_j \neq q_j} \frac{\mathrm{i}w_j - a_{\ell_j}}{a_{q_j} - a_{\ell_j}} \\
& = (-1)^{L(L-1)/2} h_{N-L}((a_j)_{j \in \mathbf{I}_N \setminus \mathbf{q}_L}) \\
& \quad \times \prod_{k \in \mathbf{I}_L} \prod_{j \in \mathbf{I}_N \setminus \mathbf{q}_L} (a_{q_k} - a_j)(\mathrm{i}w_k - a_j) \times \prod_{k \in \mathbf{I}_L} \prod_{j \in \mathbf{I}_L, j \neq k} (\mathrm{i}w_k - a_{p_j}).
\end{aligned}
$$

We also note that
$$
\frac{\sigma^{-N^2}}{N! C_N} = \frac{\sigma^{-(N-L)^2}}{(N-L)! C_{N-L}} \times \frac{\sigma^{-L(2N-N')}}{(2\pi)^{L/2} \prod_{n=0}^{L-1} (N-n)!}.
$$

Then the LHS of (3.6) becomes

$$
\begin{aligned}
& \int_{\mathbb{R}^N} d\boldsymbol{a}\, \frac{\sigma^{-N^2}}{N! C_N} e^{-|\boldsymbol{a}|^2/2\sigma} h_N(\boldsymbol{a})^2 \rho^{\xi}(t, \boldsymbol{x}_L) \\
= \; & \frac{(-1)^{L(L-1)/2} \sigma^{-L(2N-L)}}{(2\pi)^{L/2} \prod_{n=0}^{L-1} (N-n)!} L! \sum_{1 \leq q_1 < \cdots < q_L \leq N} \int_{\mathbb{R}^L} d\boldsymbol{w} \prod_{j=1}^{L} p(-t, \mathrm{i}w_j | x_j) \prod_{k=1}^{L} \int_{\mathbb{R}} da_{q_k}\, e^{-a_{q_k}^2/2\sigma^2} \\
\times \; & \prod_{\ell \in \mathbf{I}_L} \prod_{m \in \mathbf{I}_L, m \neq \ell} (\mathrm{i}w_\ell - a_{q_m}) \det_{1 \leq j,k \leq L} [p(t, x_j | a_{q_k})] \mathbf{E}_{N-L, \sigma^2} \left[\prod_{k \in \mathbf{I}_L} \prod_{j \in \mathbf{I}_L \setminus \mathbf{q}_L} (a_{q_k} - a_j)(\mathrm{i}w_k - a_j) \right] \\
= \; & \frac{(-1)^{L(L-1)/2} \sigma^{-L(2N-L)}}{(2\pi)^{L/2} \prod_{n=1}^{L} (N-n)!} \int_{\mathbb{R}^L} d\boldsymbol{w} \prod_{j=1}^{L} p(-t, \mathrm{i}w_j | x_j) \int_{\mathbb{R}^L} d\boldsymbol{v}\, e^{-|\boldsymbol{v}|^2/2\sigma^2} \prod_{k=1}^{L} \prod_{1 \leq j \leq L, j \neq k} (\mathrm{i}w_k - v_j) \\
\times \; & \det_{1 \leq j,k \leq L} [p(t, x_j | v_k)] M_{\mathrm{GUE}}(2L, (v_1, \dots, v_L, \mathrm{i}w_1, \dots, \mathrm{i}w_L); N-L, \sigma^2),
\end{aligned}
$$

where we have replaced the integral variables $(a_{q_1}, \dots, a_{q_L})$ by $(v_1, \dots, v_L) \equiv \boldsymbol{v}$. By definition

$$
\det_{1 \leq j,k \leq L} [p(t, x_j | v_k)] = \sum_{\tau \in \mathcal{S}_L} \mathrm{sgn}(\tau) \prod_{j=1}^{L} p(t, x_j | v_{\tau(j)}),
$$

where $\mathcal{S}_L$ denotes the collection of all permutation of $(1, 2, \dots, L)$. For each $\tau \in \mathcal{S}_L$, set $v_{\tau(j)} = u_j, 1 \leq j \leq L$, that is, $v_j = u_{\tau^{-1}(j)}, 1 \leq j \leq L$. Then the above equals

$$
\begin{aligned}
& \frac{(-1)^{L(L-1)/2} \sigma^{-L(2N-L)}}{(2\pi)^{L/2} \prod_{n=1}^{L} (N-n)!} \int_{\mathbb{R}^L} d\boldsymbol{u} \int_{\mathbb{R}^L} d\boldsymbol{w}\, e^{-|\boldsymbol{u}|^2/2\sigma^2} \prod_{j=1}^{L} p(t, x_j | u_j) p(-t, \mathrm{i}w_j | x_j) \\
& \times \sum_{\tau \in \mathcal{S}_L} \mathrm{sgn}(\tau) \prod_{k=1}^{L} \prod_{1 \leq j \leq L, j \neq k} (\mathrm{i}w_k - u_{\tau^{-1}(j)}) \\
& \quad \times M_{\mathrm{GUE}}(2L, (u_{\tau^{-1}(1)}, \dots, u_{\tau^{-1}(L)}, \mathrm{i}w_1, \dots, \mathrm{i}w_L); N-L, \sigma^2).
\end{aligned}
$$

By definition (1.9), $M_{\mathrm{GUE}}(m, \boldsymbol{\alpha}; N, \sigma^2)$ is symmetric in $\boldsymbol{\alpha}$, and thus

$$
\begin{aligned}
& M_{\mathrm{GUE}}(2L, (u_{\tau^{-1}(1)}, \dots, u_{\tau^{-1}(L)}, \mathrm{i}w_1, \dots, \mathrm{i}w_L); N-L, \sigma^2) \\
& = M_{\mathrm{GUE}}(2L, (u_1, \dots, u_L, \mathrm{i}w_1, \dots, \mathrm{i}w_L); N-L, \sigma^2).
\end{aligned}
$$

We can confirm that for any $n \geq 2, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$\sum_{\kappa \in \mathcal{S}_n} \mathrm{sgn}(\kappa) \prod_{k=1}^{n} \prod_{1 \leq j \leq n, j \neq k} (x_k - y_{\kappa(j)}) = (-1)^{[n/2]} h_n(\boldsymbol{x}) h_n(\boldsymbol{y}).$$

We can see that $[L/2] + L(L-1)/2$ is even for any $L \in \mathbb{N}$. Then (3.6) is obtained. ∎

Then we consider the RHS of (2.31). By the first expression of (2.24) and the fact (2.16), we have the Hermite kernel [40, 15]

$$\mathbb{K}^{N\delta_0}(t+\sigma^2, x; t+\sigma^2, y) = \frac{e^{-x^2/2(t+\sigma^2)}}{\sqrt{2\pi(t+\sigma^2)}} \sum_{n=0}^{N-1} \frac{1}{2^n n!} H_n\left(\frac{x}{\sqrt{2(t+\sigma^2)}}\right) H_n\left(\frac{y}{\sqrt{2(t+\sigma^2)}}\right), \quad (3.7)$$

$t > 0, \sigma^2 > 0, x, y \in \mathbb{R}$. By the extended version of Chapman-Kolmogorov equation (2.3), the following integral formulas are derived.

Lemma 3.2 *For $t > 0, \sigma^2 > 0, n \in \mathbb{N}_0, x, y \in \mathbb{R}$*

$$\int_{\mathbb{R}} du\, p(t, x|u) H_n\left(\frac{u}{\sqrt{2\sigma^2}}\right) e^{-u^2/2\sigma^2} = \left(\frac{\sigma^2}{t+\sigma^2}\right)^{(n+1)/2} H_n\left(\frac{x}{\sqrt{2(t+\sigma^2)}}\right) e^{-x^2/2(t+\sigma^2)}, \quad (3.8)$$

$$\int_{\mathbb{R}} dv\, H_n\left(\frac{\mathrm{i}v}{\sqrt{2\sigma^2}}\right) p(-t, \mathrm{i}v|x) = \left(\frac{t+\sigma^2}{\sigma^2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2(t+\sigma^2)}}\right), \quad (3.9)$$

$$\begin{aligned}\mathbb{K}^{N\delta_0}(t+\sigma^2, x; t+\sigma^2, y) &= \frac{1}{2^N (N-1)! \sqrt{\pi}} \int_{\mathbb{R}} du \int_{\mathbb{R}} dv\, e^{-u^2/2\sigma^2} p(t, x|u) p(-t, \mathrm{i}v|y) \\ &\times \frac{1}{u - \mathrm{i}v} \begin{vmatrix} H_N(u/\sqrt{2\sigma^2}) & H_N(\mathrm{i}v/\sqrt{2\sigma^2}) \\ H_{N-1}(u/\sqrt{2\sigma^2}) & H_{N-1}(\mathrm{i}v/\sqrt{2\sigma^2}) \end{vmatrix}. \end{aligned} \quad (3.10)$$

Proof Since the equality

$$\int_{\mathbb{R}} du\, p(t, x|u) p(\sigma^2, u|s) = p(t+\sigma^2, x|s)$$

holds for $t > 0, \sigma^2 > 0, x \in \mathbb{R}, s \in \mathbb{C}$, (2.6) gives

$$\int_{R} du\, p(t, x|u) \phi_n^{(+)}(\sigma^2, u; \xi) = \phi_n^{(+)}(t+\sigma^2, x; \xi),$$

$n \in \mathbb{N}_0, t > 0, \sigma^2 > 0, x \in \mathbb{R}$. If we set $\xi = N\delta_0$ and use the fact (2.16), we obtain (3.8). Similarly, by the equality

$$\begin{aligned}\int_{\mathbb{R}} dv\, p(-\sigma^2, \mathrm{i}s|\mathrm{i}v) p(-t, \mathrm{i}v|x) &= \int_{\mathbb{R}} dv\, p(\sigma^2, s|v) p(t, v|-\mathrm{i}x) \\ &= p(t+\sigma^2, s|-\mathrm{i}x) = p(-(t+\sigma^2), \mathrm{i}s|x),\end{aligned}$$

$t > 0, \sigma^2 > 0, x \in \mathbb{R}$, and by the definition (2.7), we see

$$\int_{\mathbb{R}} dv\, \phi_n^{(-)}(\sigma^2, \mathrm{i}v; \xi) p(-t, \mathrm{i}v|x) = \phi_n^{(-)}(t+\sigma^2, x; \xi).$$

If we set $\xi = N\delta_0$ and use the fact (2.16), we obtain (3.9). Inserting (3.8) and (3.9) into the RHS of (3.7), we have

$$\begin{aligned}\mathbb{K}^{N\delta_0}(t+\sigma^2, x; t+\sigma^2, t) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} du \int_{\mathbb{R}} dv\, e^{-u^2/2\sigma^2} p(t, x|u) p(-t, \mathrm{i}v|y) \\ &\quad \times \sum_{n=0}^{N-1} \frac{1}{2^n n!} H_n\left(\frac{u}{\sqrt{2\sigma^2}}\right) H_n\left(\frac{\mathrm{i}v}{\sqrt{2\sigma^2}}\right).\end{aligned}$$

Here we use the Christoffel-Darboux formula (see, for example, [4]),

$$\sum_{n=0}^{N-1} \frac{1}{2^n n!} H_n(x) H_n(y) = \frac{1}{2^N (N-1)!(x-y)} \begin{vmatrix} H_N(x) & H_N(y) \\ H_{N-1}(x) & H_{N-1}(y) \end{vmatrix},$$

and then (3.10) is obtained. ∎

The RHS of (2.31) is thus written as follows.

Lemma 3.3 *For any $\sigma^2 > 0, t > 0, 1 \le L \le N, \boldsymbol{x}_L = (x_1, \dots, x_L) \in \mathbb{W}_L^{\mathrm{A}}$,*

$$\begin{aligned}&\det_{1\le j,k\le L} [\mathbb{K}^{N\delta_0}(t+\sigma^2, x_j; t+\sigma^2, x_k)] \\ &= \int_{\mathbb{R}^L} d\boldsymbol{u} \int_{\mathbb{R}^L} d\boldsymbol{w}\, e^{-|\boldsymbol{u}|^2/2\sigma^2} \prod_{j=1}^{L} p(t, x_j|u_j) p(-t, \mathrm{i}w_j|x_j) \\ &\quad \times \frac{1}{(2^N (N-1)!\sqrt{\pi})^L} \det_{1\le j,k\le L} \left[\frac{1}{u_j - \mathrm{i}w_k} \begin{vmatrix} H_N(u_j/\sqrt{2\sigma^2}) & H_N(\mathrm{i}w_k/\sqrt{2\sigma^2}) \\ H_{N-1}(u_j/\sqrt{2\sigma^2}) & H_{N-1}(\mathrm{i}w_k/\sqrt{2\sigma^2}) \end{vmatrix} \right].\end{aligned} \tag{3.11}$$

Proof By definition of determinant, the LHS of (3.11) is given by

$$\begin{aligned}&\frac{1}{(2^N(N-1)!\sqrt{\pi})^L} \sum_{\tau\in\mathcal{S}_L} \mathrm{sgn}(\tau) \int_{\mathbb{R}^L} d\boldsymbol{u} \int_{\mathbb{R}^L} d\boldsymbol{v}\, e^{-|\boldsymbol{u}|^2/2\sigma^2} \\ &\times \prod_{j=1}^{L} \left\{ p(t, x_j|u_j) \frac{1}{u_j - \mathrm{i}v_j} \begin{vmatrix} H_N(u_j/\sqrt{2\sigma^2}) & H_N(\mathrm{i}v_j/\sqrt{2\sigma^2}) \\ H_{N-1}(u_j/\sqrt{2\sigma^2}) & H_{N-1}(\mathrm{i}v_j/\sqrt{2\sigma^2}) \end{vmatrix} p(-t, \mathrm{i}v_j|x_{\tau(j)}) \right\}.\end{aligned}$$

Note that for each permutation $\tau \in \mathcal{S}_L$

$$\prod_{j=1}^{L} p(-t, \mathrm{i}v_j|x_{\tau(j)}) = \prod_{j=1}^{L} p(-t, \mathrm{i}v_{\tau^{-1}(j)}|x_j).$$

We set $v_{\tau^{-1}(j)} = w_j, 1 \le j \le N$, that is, $v_j = w_{\tau(j)}, 1 \le j \le N$. Then the above equals

$$\begin{aligned}&\frac{1}{2^{NL}((N-1)!)^L \pi^{L/2}} \int_{\mathbb{R}^L} d\boldsymbol{u} \int_{\mathbb{R}^L} d\boldsymbol{w}\, e^{-|\boldsymbol{u}|^2/2\sigma^2} \prod_{j=1}^{L} \{p(t, x_j|u_j) p(-t, \mathrm{i}w_j|x_j)\} \\ &\times \sum_{\tau\in\mathcal{S}_L} \mathrm{sgn}(\tau) \prod_{j=1}^{L} \left\{ \frac{1}{u_j - \mathrm{i}w_{\tau(j)}} \begin{vmatrix} H_N(u_j/\sqrt{2\sigma^2}) & H_N(\mathrm{i}w_{\tau(j)}/\sqrt{2\sigma^2}) \\ H_{N-1}(u_j/\sqrt{2\sigma^2}) & H_{N-1}(\mathrm{i}w_{\tau(j)}/\sqrt{2\sigma^2}) \end{vmatrix} \right\},\end{aligned}$$

which is the RHS of (3.11). ∎

Proof of Eq.(2.33) in Theorem 2.3 Since the equality (2.31) in Corollary 2.2, holds for any $t > 0, \sigma^2 > 0$, Lemmas 3.1 and 3.3, imply that the integrand of multiple Gaussian integral in the RHS of (3.6) is equal to that in the RHS of (3.11). By replacing L and $N-L$ by n and N, respectively, and $(u_1, \dots, u_L, iw_1, \dots, iw_L)$ by $(\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{2n}) = \alpha \in \mathbb{C}^{2n}$, we obtain (2.33). ∎

Next we prove (2.35). The LHS of (2.32) is written as follows.

Lemma 3.4 *For any $\sigma^2 > 0, t > 0, 1 \le L \le N, \boldsymbol{x}_L = (x_1, \dots, x_L) \in \mathbb{W}_L^+$,*

$$\begin{aligned}
&\mathbf{E}_{N,\sigma^2}^{(\nu)} \left[\det_{1 \le j,k \le L} [\mathbb{K}_\nu^{\Xi}(t, x_j; t, x_k)] \right] \\
&= \int_{\mathbb{R}_+^L} d\boldsymbol{u} \int_{\mathbb{R}_-^L} d\boldsymbol{w} \prod_{j=1}^L u_j^\nu e^{-u_j/2\sigma^2} p^{(\nu)}(t, x_j|u_j) p^{(\nu)}(-t, w_j|x_j) \frac{(2\sigma^2)^{-L(2N-L+\nu)}}{\prod_{\ell=1}^L \Gamma(N+\nu+1-\ell)(N-\ell)!} \\
&\quad \times h_L(\boldsymbol{u}) h_L(\boldsymbol{w}) M^{(\nu)}(2L, (u_1, \dots, u_L, w_1, \dots, w_L); N-L, \sigma^2). \qquad (3.12)
\end{aligned}$$

Since we can prove this lemma in the similar way to Lemma 3.1, we omit the proof.

Then we consider the RHS of (2.32). By the first expression of (2.25) and the fact (2.18), we have the Laguerre kernel [15]

$$\begin{aligned}
&\mathbb{K}_\nu^{N\delta_0}(t+\sigma^2, x; t+\sigma^2, y) \\
&\quad = \frac{x^\nu e^{-x/2(t+\sigma^2)}}{\{2(t+\sigma^2)\}^{\nu+1}} \sum_{n=0}^{N-1} \frac{n!}{\Gamma(n+\nu+1)} L_n^\nu \left(\frac{x}{2(t+\sigma^2)} \right) L_n^\nu \left(\frac{y}{2(t+\sigma^2)} \right), \qquad (3.13)
\end{aligned}$$

$t > 0, \sigma^2 > 0, x, y \in \mathbb{R}_+$. By the extended version of Chapman-Kolmogorov equation (2.4), the following integral formulas are derived.

Lemma 3.5 *For $t > 0, \sigma^2 > 0, n \in \mathbb{N}_0, x, y \in \mathbb{R}_+$*

$$\int_{\mathbb{R}_+} du\, p^{(\nu)}(t, x|u) L_n^\nu \left(\frac{u}{2\sigma^2} \right) u^\nu e^{-u/2\sigma^2} = \left(\frac{\sigma^2}{t+\sigma^2} \right)^{n+\nu+1} L_n^\nu \left(\frac{x}{2(t+\sigma^2)} \right) x^\nu e^{-x/2(t+\sigma^2)}, \qquad (3.14)$$

$$\int_{\mathbb{R}_-} dv\, L_n^\nu \left(\frac{v}{2\sigma^2} \right) p^{(\nu)}(-t, v|x) = \left(\frac{t+\sigma^2}{\sigma^2} \right)^n L_n^\nu \left(\frac{x}{2(t+\sigma^2)} \right), \qquad (3.15)$$

$$\begin{aligned}
&\mathbb{K}_\nu^{N\delta_0}(t+\sigma^2, x; t+\sigma^2, y) \\
&\quad = -\frac{N!}{(2\sigma^2)^\nu \Gamma(N+\nu)} \int_{\mathbb{R}_+} du \int_{\mathbb{R}_-} dv\, u^\nu e^{-u/2\sigma^2} p^{(\nu)}(t, x|u) p^{(\nu)}(-t, v|y) \\
&\qquad \times \frac{1}{u-v} \begin{vmatrix} L_N^\nu(u/2\sigma^2) & L_N^\nu(v/2\sigma^2) \\ L_{N-1}^\nu(u/2\sigma^2) & L_{N-1}^\nu(v/2\sigma^2) \end{vmatrix}. \qquad (3.16)
\end{aligned}$$

Proof Since the equality

$$\int_{\mathbb{R}_+} du\, p^{(\nu)}(t, x|u) p^{(\nu)}(\sigma^2, u|s) = p^{(\nu)}(t+\sigma^2, x|s)$$

holds for $t > 0, \sigma^2 > 0, x \in \mathbb{R}_+, s \in \mathbb{C}$, (2.8) gives

$$\int_{R_+} du\, p^{(\nu)}(t, x|u)\phi_n^{(\nu,+)}(\sigma^2, u; \xi) = \phi_n^{(\nu,+)}(t + \sigma^2, x; \xi),$$

$n \in \mathbb{N}_0, t > 0, \sigma^2 > 0, x \in \mathbb{R}_+$. If we set $\xi = N\delta_0$ and use (2.18), we obtain (3.14). Similarly, by the equality

$$\begin{aligned}\int_{\mathbb{R}_-} dv\, p^{(\nu)}(-\sigma^2, -w|v)p^{(\nu)}(-t, v|x) &= \int_{\mathbb{R}_+} du\, p^{(\nu)}(\sigma^2, w|u)p^{(\nu)}(t, u| - x) \\ &= p^{(\nu)}(t + \sigma^2, w| - x) = p^{(\nu)}(-(t + \sigma^2), -w|x),\end{aligned}$$

$t > 0, \sigma^2 > 0, x, w \in \mathbb{R}_+$, and by the definition (2.9), we see

$$\int_{\mathbb{R}_-} dv\, \phi_n^{(\nu,-)}(\sigma^2, v; \xi)p^{(\nu)}(-t, v|x) = \phi_n^{(\nu,-)}(t + \sigma^2, x; \xi).$$

If we set $\xi = N\delta_0$ and use (2.18), we obtain (3.15). Inserting (3.14) and (3.15) into the RHS of (3.13), we have

$$\begin{aligned}\mathbb{K}_\nu^{N\delta_0}(t + \sigma^2, x; t + \sigma^2, t) &= \frac{1}{(2\sigma^2)^{\nu+1}} \int_{\mathbb{R}_+} du \int_{\mathbb{R}_-} dv\, u^\nu e^{-u/2\sigma^2} p^{(\nu)}(t, x|u)p^{(\nu)}(-t, v|y) \\ &\quad \times \sum_{n=0}^{N-1} \frac{n!}{\Gamma(n + \nu + 1)} L_n^\nu\left(\frac{u}{2\sigma^2}\right) L_n^\nu\left(\frac{v}{2\sigma^2}\right).\end{aligned}$$

Here we use the Christoffel-Darboux formula [4]

$$\sum_{n=0}^{N-1} \frac{n!}{\Gamma(n + \nu + 1)} L_n^\nu(x)L_n^\nu(y) = -\frac{N!}{\Gamma(N + \nu)(x - y)} \begin{vmatrix} L_N^\nu(x) & L_N^\nu(y) \\ L_{N-1}^\nu(x) & L_{N-1}^\nu(y) \end{vmatrix}.$$

Then (3.16) is obtained. ∎

The RHS of (2.32) is thus written as follows.

Lemma 3.6 *For any $\sigma^2 > 0, t > 0, 1 \le L \le N, \boldsymbol{x}_L = (x_1, \dots, x_L) \in \mathbb{W}_L^+$,*

$$\begin{aligned}&\det_{1\le j,k\le L}[\mathbb{K}_\nu^{N\delta_0}(t + \sigma^2, x_j; t + \sigma^2, x_k)] \\ &= \int_{\mathbb{R}_+^L} d\boldsymbol{u} \int_{\mathbb{R}_-^L} d\boldsymbol{w} \prod_{j=1}^L u_j^\nu e^{-u_j/2\sigma^2} p^{(\nu)}(t, x_j|u_j)p^{(\nu)}(-t, w_j|x_j) \\ &\quad \times \left(-\frac{N!}{(2\sigma^2)^\nu \Gamma(N + \nu)}\right)^L \det_{1\le j,k\le L}\left[\frac{1}{u_j - w_k} \begin{vmatrix} L_N^\nu(u_j/2\sigma^2) & L_N^\nu(w_k/2\sigma^2) \\ L_{N-1}^\nu(u_j/2\sigma^2) & L_{N-1}^\nu(w_k/2\sigma^2) \end{vmatrix}\right].\end{aligned} \tag{3.17}$$

Proof of Eq.(2.35) in Theorem 2.3 Since the equality (2.32) in Corollary 2.2, holds for any $t > 0, \sigma^2 > 0$, Lemmas 3.4 and 3.6, imply that the integrand of multiple Gaussian integral in the RHS of (3.12) is equal to that in the RHS of (3.17). By replacing L and $N - L$ by n and N, respectively, and $(u_1, \dots, u_L, w_1, \dots, w_L)$ by $(\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{2n}) = \alpha \in \mathbb{C}^{2n}$, we obtain (2.35). ∎

3.3 Proof of Theorem 2.4

First we prove that (2.41) is equal to (2.33). Since

$$\begin{aligned}
&\frac{1}{\alpha_j - \alpha_{n+k}} \begin{vmatrix} H_{N+n}(\alpha_j/\sqrt{2\sigma^2}) & H_{N+n}(\alpha_{n+k}/\sqrt{2\sigma^2}) \\ H_{N+n-1}(\alpha_j/\sqrt{2\sigma^2}) & H_{N+n-1}(\alpha_{n+k}/\sqrt{2\sigma^2}) \end{vmatrix} \\
&\quad = H_{N+n-1}\left(\frac{\alpha_j}{\sqrt{2\sigma^2}}\right) H_{N+n-1}\left(\frac{\alpha_{n+k}}{\sqrt{2\sigma^2}}\right) \\
&\quad \times \frac{1}{\alpha_{n+k} - \alpha_j} \begin{vmatrix} 1 & H_{N+n}(\alpha_j/\sqrt{2\sigma^2})/H_{N+n-1}(\alpha_j/\sqrt{2\sigma^2}) \\ 1 & H_{N+n}(\alpha_{n+k}/\sqrt{2\sigma^2})/H_{N+n-1}(\alpha_{n+k}/\sqrt{2\sigma^2}) \end{vmatrix},
\end{aligned}$$

we can apply the identity (2.37) and obtain the following,

$$\begin{aligned}
&\det_{1 \le j,k \le n} \left[\frac{1}{\alpha_j - \alpha_{n+k}} \begin{vmatrix} H_{N+n}(\alpha_j/\sqrt{2\sigma^2}) & H_{N+n}(\alpha_{n+k}/\sqrt{2\sigma^2}) \\ H_{N+n-1}(\alpha_j/\sqrt{2\sigma^2}) & H_{N+n-1}(\alpha_{n+k}/\sqrt{2\sigma^2}) \end{vmatrix} \right] \\
&= \prod_{j=1}^{n} \left\{ H_{N+n-1}\left(\frac{\alpha_j}{\sqrt{2\sigma^2}}\right) H_{N+n-1}\left(\frac{\alpha_{n+k}}{\sqrt{2\sigma^2}}\right) \right\} \frac{(-1)^{n(n-1)/2}}{\prod_{j=1}^{n}\prod_{k=1}^{n}(\alpha_{n+k} - \alpha_n)} \\
&\quad \times V^{n,n}\left((\alpha_1, \dots, \alpha_n), (\alpha_{n+1}, \dots, \alpha_{2n}); \left(\frac{H_{N+n}(\frac{\alpha_j}{\sqrt{2\sigma^2}})}{H_{N+n-1}(\frac{\alpha_j}{\sqrt{2\sigma^2}})} \right)_{j=1}^{n}, \left(\frac{H_{N+n}(\frac{\alpha_{n+j}}{\sqrt{2\sigma^2}})}{H_{N+n-1}(\frac{\alpha_{n+j}}{\sqrt{2\sigma^2}})} \right)_{j=1}^{n} \right) \\
&= \frac{(-1)^{n(n-1)/2}}{\prod_{j=1}^{n}\prod_{k=1}^{n}(\alpha_{n+k} - \alpha_n)} \det[A_{N+n-1} \quad A_{N+n}],
\end{aligned}$$

where A_ℓ is the $2n \times n$ rectangular matrix whose (j,k)-element is given by $\alpha_j^{k-1} H_\ell(\alpha_j/\sqrt{2\sigma^2})$, $1 \le j \le 2n, 1 \le k \le n$. Here we use the recurrence relation of Hermite polynomials

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{3.18}$$

Then we find that

$$\begin{aligned}
\alpha^{k-1} H_{N+n-1}\left(\frac{\alpha}{\sqrt{2\sigma^2}}\right) &= (2\sigma^2)^{(k-1)/2} \prod_{\ell=1}^{k-1} (N+n-\ell) H_{N+n-k}\left(\frac{\alpha}{\sqrt{2\sigma^2}}\right) \\
&\quad + \text{linear combination of } \left\{ H_\ell(\alpha/\sqrt{2\sigma^2}) : N+n-k < \ell < N+n+k-1 \right\}, \\
\alpha^{k-1} H_{N+n}\left(\frac{\alpha}{\sqrt{2\sigma^2}}\right) &= \left(\frac{\sigma^2}{2}\right)^{(k-1)/2} H_{N+n+k-1}\left(\frac{\alpha}{\sqrt{2\sigma^2}}\right) \\
&\quad + \text{linear combination of } \left\{ H_\ell(\alpha/\sqrt{2\sigma^2}) : N+n-k < \ell < N+n+k-1 \right\},
\end{aligned}$$

for $k = 2, 3, \dots$. Therefore, by the multilinearlity of determinant,

$$\det[A_{N+n-1} \quad A_{N+n}] = \sigma^{n(n-1)} \prod_{\ell=2}^{n} \frac{(N+n-1)!}{(N+n-\ell)!} \det[B\,C],$$

where B and C are $2n \times n$ rectangular matrices whose (j,k)-elements are given by

$$H_{N+n-k}\left(\frac{\alpha_j}{\sqrt{2\sigma^2}}\right) \quad \text{and} \quad H_{N+n+k-1}\left(\frac{\alpha_j}{\sqrt{2\sigma^2}}\right),$$

respectively. Since $\det[B\,C] = (-1)^{[n/2]} \det_{1\le j,k\le 2n}[H_{N+k-1}(\alpha_j/\sqrt{2\sigma^2})]$, $(-1)^{[n/2]+n(n-1)/2} = 1$ for any $n \in \mathbb{N}$, and

$$h_n(\alpha_1,\ldots,\alpha_n)h_n(\alpha_{n+1},\ldots,\alpha_{2n})\prod_{j=1}^{n}\prod_{k=1}^{n}(\alpha_{n+k}-\alpha_j) = h_{2n}(\alpha_1,\ldots,\alpha_{2n}), \tag{3.19}$$

proof of the equivalence of (2.33) and (2.41) is completed.

Next we prove that (2.35) is equal to

$$\frac{\Gamma(N+\nu+n)}{(N+n)!}\frac{1}{h_{2n}(\boldsymbol{\alpha})}\det_{1\le j,k\le 2n}\left[\widehat{L}^{\nu}_{N+j-1}(\alpha_k;\sigma^2)\right] \tag{3.20}$$

for any $\nu > -1$. By applying the identity (2.37), we have

$$\begin{aligned}
&\det_{1\le j,k\le n}\left[\frac{1}{\alpha_j-\alpha_{n+k}}\ \middle|\ \begin{array}{cc} L^{\nu}_{N+n}(\alpha_j/2\sigma^2) & L^{\nu}_{N+n}(\alpha_{n+k}/2\sigma^2) \\ L^{\nu}_{N+n-1}(\alpha_j/2\sigma^2) & L^{\nu}_{N+n-1}(\alpha_{n+k}/2\sigma^2)\end{array}\right] \\
&= \frac{(-1)^{n(n-1)/2}}{\prod_{j=1}^{n}\prod_{k=1}^{n}(\alpha_{n+k}-\alpha_n)}\det[\widetilde{A}_{N+n-1}\quad \widetilde{A}_{N+n}],
\end{aligned}$$

where $\widetilde{A}_\ell$ is the $2n\times n$ rectangular matrix whose (j,k)-element is given by $\alpha_j^{k-1}L^{\nu}_{\ell}(\alpha_j/2\sigma^2)$, $1\le j\le 2n, 1\le k\le n$. Here we use the recurrence relation of Laguerre polynomials

$$(n+1)L^{\nu}_{n+1}(x) = (-x+2n+\nu+1)L^{\nu}_{n}(x) - (n+\nu)L^{\nu}_{n-1}(x). \tag{3.21}$$

Then we find that

$$\begin{aligned}
\alpha^{k-1}L^{\nu}_{N+n-1}\left(\frac{\alpha}{2\sigma^2}\right) &= (-2\sigma^2)^{k-1}\prod_{\ell=1}^{k-1}(N+n+\nu-\ell)L^{\nu}_{N+n-k}\left(\frac{\alpha}{2\sigma^2}\right) \\
&\quad+\text{linear combination of }\left\{L^{\nu}_{\ell}(\alpha/2\sigma^2) : N+n-k<\ell<N+n+k-1\right\}, \\
\alpha^{k-1}L^{\nu}_{N+n}\left(\frac{\alpha}{2\sigma^2}\right) &= (-2\sigma^2)^{k-1}\prod_{\ell=1}^{k-1}(N+n+\ell)L^{\nu}_{N+n+k-1}\left(\frac{\alpha}{2\sigma^2}\right) \\
&\quad+\text{linear combination of }\left\{L^{\nu}_{\ell}(\alpha/2\sigma^2) : N+n-k<\ell<N+n+k-1\right\},
\end{aligned}$$

for $k = 2, 3, \ldots$. Therefore, by the multilinearlity of determinant,

$$\det[\widetilde{A}_{N+n-1}\quad \widetilde{A}_{N+n}] = (2\sigma^2)^{n(n-1)}\left(\frac{\Gamma(N+n+\nu)}{(N+n)!}\right)^{n-1}\prod_{\ell=1}^{n-1}\frac{(N+n+\ell)!}{\Gamma(N+\nu+\ell)}\det[\widetilde{B}\,\widetilde{C}],$$

where $\widetilde{B}$ and $\widetilde{C}$ are $2n\times n$ rectangular matrices whose (j,k)-elements are given by

$$L^{\nu}_{N+n-k}\left(\frac{\alpha_j}{2\sigma^2}\right) \quad\text{and}\quad L^{\nu}_{N+n+k-1}\left(\frac{\alpha_j}{2\sigma^2}\right),$$

respectively. Since $\det[\widetilde{B}\,\widetilde{C}] = (-1)^{[n/2]}\det_{1\le j,k\le 2n}[L^{\nu}_{N+k-1}(\alpha_j/2\sigma^2)]$, and (3.19), the equivalence of (2.35) and (3.20) is proved for $\nu > -1$. For $\nu \in \mathbb{N}_0$, (2.42) is immediately obtained. The first expressions in (2.43) and (2.44) are obtained by setting $\nu = 1/2$ and $\nu = -1/2$ in (3.20) and by applying the relations (1.15) and (1.16). For the second expressions in (2.43) and (2.44), we use the following relations between the Hermite polynomials and Laguerre polynomials with $\nu = \pm 1/2$,

$$xL_n^{1/2}\left(\frac{x^2}{2}\right) = \frac{(-1)^n}{2^{2n+1/2}n!}H_{2n+1}\left(\frac{x}{\sqrt{2}}\right), \tag{3.22}$$

$$L_n^{-1/2}\left(\frac{x^2}{2}\right) = \frac{(-1)^n}{2^{2n}n!}H_{2n}\left(\frac{x}{\sqrt{2}}\right). \tag{3.23}$$

Then the proof of Theorem 2.4 is completed. ∎

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Department of Physics, Chuo University
Kasuga, Bunkyo-ku, Tokyo 112-8551,
Japan
e-mail: katori@phys.chuo-u.ac.jp